STOCHASTIC CLAIMS RESERVING WHEN PAST CLAIM NUMBERS ARE KNOWN

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Abstract

This paper addresses the problem of estimating future claim payments when two run-off triangles are available: one of the number of claims, the other of total amounts. Each single claim can have partial payments included in the total for several development periods. The method does not require additional information, such as measures of exposure and claims inflation. The approach adopted is to model the mean claim amount as a function of operational time, using generalized linear models. Techniques are described for fitting and comparing a number of models of this type, and for predicting the total of future claims from the best fitting model. Formal statistical tests are used for comparing models. It is shown how the root-mean-square (RMS) error of prediction can be calculated, making due allowance for modelling error and random variation in both the number and amounts of future payments. Models are formulated to make explicit allowance for claims inflation and partial payments. Assumptions are minimal, and diagnostic techniques are described for checking their validity in each application. Numerical examples are given.

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1. INTRODUCTION

Background, Notation, and Overview

This paper complements a previous work by the author [10]. That paper, like many others on stochastic claims reserving in property/casualty insurance, deals with methods applicable when the past run-off of the number of claims is not known (a common situation for actuaries in the U.K.). This paper addresses the problem of claims reserving when at least two run-off triangles are available: one of the number of claims, the other of claim amounts. These primary triangles may be:

(a) the number of claims closed, and the total of all payments on all claims closed (partial payments assigned to the development period of settlement);

(b) the total number of payments, including partial payments, and the usual paid claims triangle (with each partial payment assigned to the development period in which it was made); or

(c) the number of claims closed, and the usual paid claims triangle (with each partial payment assigned to the development period in which it was made).

Of these possibilities, (a) and (b) are the simplest to model, and are considered first. They are equivalent to each other as far as the modelling and prediction methods proposed in this paper are concerned. Later, it is shown how basically the same methods can be applied in situation (c), which is more common in practice.

If w is used to label origin (i.e., accident, report, or policy) years, and d to label development periods, the two run-off triangles can be denoted Y_{wd} and N_{wd}, respectively, where w runs from 1 to W, and d runs from 0 to $T-1$. This notation is used for the incremental, rather than the cumulative, run-off. For example, in case (a), N_{wd} is the number of claims closed in development period d of origin year w, and Y_{wd} is the total of payments on these claims made in development period d and previous development periods. In case (c), N_{wd} is the number of claims closed as in (a), but Y_{wd} is the total amount of all partial payments made in development period d of origin year w, including partial payments on claims not yet closed. In both cases (a) and (c), N_{wd} should exclude claims settled with no payment if possible. Such claims do not contribute to Y_{wd}, so their inclusion in N_{wd} introduces an undesirable element of additional random variation.

As in [10], the methods described here do not involve an assumption that the run-off pattern has been the same for all origin years; indeed, the shape of the run-off may be different for each origin year. Similarly, there is no assumption that the claim size distribution is the same for all development periods. It is common for larger claims to take longer to settle so that the mean claim size increases with d. Higher moments of the distribution may also depend on d. The methods allow projection as far into the future as is necessary, not limited by the extent of the data. The data triangles may have missing values. This does not cause any problems provided the total number of data points is sufficient to fit an adequate model. The occasional negative values which occur in real data can also be handled without special treatment.

The approach used in [10] is to derive a model for the known data Y_{wd} from more basic models for the unknown quantities N_{wd} and the individual claim amounts X_{wd} (also unknown). The assumptions of the models for N_{wd} and X_{wd} are then checked indirectly by applying diagnostic tests to the resulting model for Y_{wd}. If satisfactory, the model that is fitted to the Y_{wd} is used to project into the future. The logical progression of this approach to situations where N_{wd} is known would be to formulate models for N_{wd} and X_{wd} separately (as before), but then to test each of these models directly from the data. This should allow good models to be found for each of these components. These models could then be used to project N_{wd} and X_{wd} separately, and the projections combined into projections for the total payments Y_{wd}. However, the calculation of standard errors for predictions of Y_{wd} obtained in this way is complex. Hayne [3] deals with the case when, for each origin year, the distribution of future claim amounts X_{wd} does not depend on the development period d. The intention in this paper is to remove this restriction (as, for example, when larger claims tend to take longer to settle than smaller claims). In this case, the calculation of standard errors for the predictions would be extremely complex using real-development time, because the precise time of settlement of each future claim (hence, the appropriate claim size distribution) is uncertain. The problem is simplified in this paper by making use of the concept of *operational time*. This concept seems to have been used first in claims reserving by Reid [7] and later taken up by Taylor [8, 9], but a fresh approach, including a number of innovations, is proposed in this paper.

Operational time, τ, is defined as the proportion of all claims closed to date. Thus, for each origin year, operational time starts at 0, and increases ultimately to 1. If the individual claim amounts X can be modelled as a function of operational time τ rather than development time d, then there is no need for a separate model of the number of claims. This is because the dependence of the number of claims on operational time is known exactly: it follows trivially from the definition of operational time. Projections of future payments Y can therefore be obtained from the model of claim size $X_{w\tau}$ alone, and the problem is an order of magnitude simpler than when X_{wd} and N_{wd} are both projected.

The data N_{wd} is used at three points in the operational time approach:

- to estimate the ultimate number M_w of claims for each origin year w (obviously, numbers of claims reported are also useful for this estimation, if available);
- to calculate a triangle of operational times (for use as the explanatory variable in the claim size model); and
- to calculate the observed mean claim sizes Y_{wd}/N_{wd} (for use as the dependent variable in the claim size model).

There are often substantive reasons for expecting the size of individual claims to depend more on operational time than on development time. The main reason is that changes in claim handling procedures may affect the actual delay to settlement but should not affect the size of claims. The plausibility of such arguments need not be left entirely to judgment. It is possible to use the figures themselves to verify this basic hypothesis of operational time methods. This is shown in Appendix B.

Summary of Later Sections

Sections 2 through 6 deal with circumstances (a) or (b); that is, when the claim counts triangle gives the number of individual components of each element of the claim amounts triangle. Section 7 describes special procedures and enhancements to the method of earlier sections which may be necessary for case (c). All sections conclude with a numerical example. The data for the examples have been taken from Berquist and Sherman [1], and are reproduced in Appendix A. The data are actually of type (c), so the methods of Sections 2 through 6 are not wholly appropriate. They are applied purely for illustrative purposes. Section 7 also contains an analysis of the data used by Taylor [9]. They are also of type (c), and are given in Appendix A.

Section 2 gives a complete account of the method applicable in cases (a) or (b), under several simplifying assumptions. The assumptions are unrealistic but are made initially in order to simplify the presentation. Sections 3 through 5 show how the assumptions can be relaxed. The assumptions used in Section 2 are that:

1. The expected claim size at each operational time τ is the same for all origin years, after allowing for claims inflation. In other words, the mean claim amount in real terms is a function of τ but not w. It can therefore be denoted m_τ. (In the presence of inflation, the mean claim amount will depend on w also. See Section 4.)

2. The coefficient of variation of individual claim amounts is the same for all operational times τ, that is:

$$\mathrm{Var}\,(X_\tau) = \varphi^2 \cdot m_\tau^2 , \tag{1.1}$$

   where X_τ is the size of an individual claim at operational time τ, and φ is the coefficient of variation.

3. The data Y_{wd} have been adjusted for inflation so the triangle is in constant money terms.

4. The ultimate number of claims M_w is fully known (that is, there is no uncertainty) for each origin year w.

Assumption 1 is the only condition that must hold in order to predict future claim payments using methods proposed in this paper. Even Assumption 1 needs not be an assumption in the sense that its validity can be checked using the data themselves. (This is the subject of Appendix B.) Section 3 describes how Assumption 2 can be tested and relaxed if necessary. Assumption 3 cannot often be valid in practice because the rate of claims inflation is usually unknown. Section 4 shows how the rate of inflation can be estimated and removed from the data at the same time as fitting the claim size model, rendering preadjustment unnecessary. Assumption 4 only holds in practice if the origin years are report years. Often with accident or policy years, there will be considerable uncertainty in the estimates of ultimate numbers M_w. Section 5 describes how this uncertainty can be taken into account.

The main point of Sections 2 through 4 is to discover how the mean and the variance of an individual claim X_τ depend on the operational time τ. When this has been achieved, since we know the operational time τ of every future claim (from the definition of operational time), we can find the expected value and the variance of every future claim. This, in turn, can be used to find the expected value and the variance of the total of all future claims.

A broad outline of how predictions can be made from a fitted operational time model was provided in the previous paragraph. (Here the term "model" refers to the mathematical representation of the relationship between operational time and the mean and variance of X_τ.) The details of prediction are given in Appendices D, E, and F. These are more complex than would be expected from the comments above: first, because of parameter uncertainty (that is, the fitted model will not be exactly right); second, because uncertainty in the ultimate numbers M_w implies uncertainty in the operational time of each future claim; and, third, because of uncertainty about future claims inflation. Section 6 shows how uncertain future claims inflation can be included in the predictions obtained from an operational time model. This is necessary to comply with standard reserving practices.

All the models proposed in this paper are *generalized linear models*. Such models can be fitted using an algorithm known as Fisher's scoring method. This is the algorithm used in the well-known statistical package GLIM, which was used for all the numerical examples in this paper. Fisher's scoring method maximizes the so-called *quasi-likelihood*, or equivalently, minimizes the *deviance*. The deviance can be regarded as a generalization of the weighted sum of squared differences between observed and fitted values. The weights are determined from the assumed variances of the observations. The generalization is that the variance of each observation may be a function of its mean, which, of course, is not known. The purpose of fitting the model is to estimate the mean. Fisher's scoring method sometimes gives conventional maximum likelihood estimates. In other cases, it gives estimates which have all the desirable properties of maximum likelihood estimates (asymptotically unbiased, efficient, and Normal) although they may not actually be maximum likelihood estimates. An approximate variance/covariance matrix for the parameter estimates is also produced by the algorithm. Further details are not given here as they are well documented elsewhere: the theory in McCullagh and Nelder [5], briefly in Hogg and Klugman [4], and practical aspects in the GLIM manual [6]. The application of GLIM in actuarial work has previously been advocated by Brown [2].

2. SIMPLIFIED SCENARIO

Assumptions

Throughout Section 2, the four assumptions listed in Section 1 are made. These assumptions are not thought to be realistic, but are made at this stage to simplify the presentation. Assumptions 2, 3, and 4 are relaxed in later sections.

Transformation of the Data

In order to model the dependence of claim size on operational time, the original data triangles Y_{wd} and N_{wd} must first be transformed into a triangle τ_{wd} of operational times, and a triangle S_{wd} of observed mean claim amounts. In the subsequent modelling, τ will be the explanatory variable, and S will be the dependent variable.

Operational time, τ, which has previously been defined as the proportion of claims closed, is an alternative to development time, d. This definition gives the value of operational time *between* claim settlements. In this paper, the value of operational time *at* each claim settlement is defined to be the mean of the values immediately before and after settlement. So, for example, if there are M claims for a certain origin year, the operational time of settlement of the N^{th} claim is given by $\tau = (N - \frac{1}{2})/M$. The values of operational time for each claim settlement are $(\frac{1}{2})/M, (\frac{3}{2})/M, \ldots, (M - \frac{1}{2})/M$. These values are shown as crosses in Figure 1, which illustrates a typical relationship between operational time and true development time. The mean operational time of the N_{wd} claims in development period d of origin year w can be calculated as:

$$\tau_{wd} = (N_{w,1} + N_{w,2} + \ldots + N_{s,d-1} + \tfrac{1}{2} \cdot N_{w,d})/M_w . \tag{2.1}$$

Note that only half of N_{wd} is included in the numerator in order to give the *mean* operational time for these claims.

The sample mean size $S_{w\tau}$ of the N_{wd} claims from origin year w observed at mean operational time τ can be calculated as $S_{w\tau} = Y_{wd}/N_{wd}$. As $S_{w\tau}$ is a sample mean, its expected value is equal to the mean of the underlying population:

$$\mathrm{E}(S_{w\tau}) = m_\tau . \tag{2.2}$$

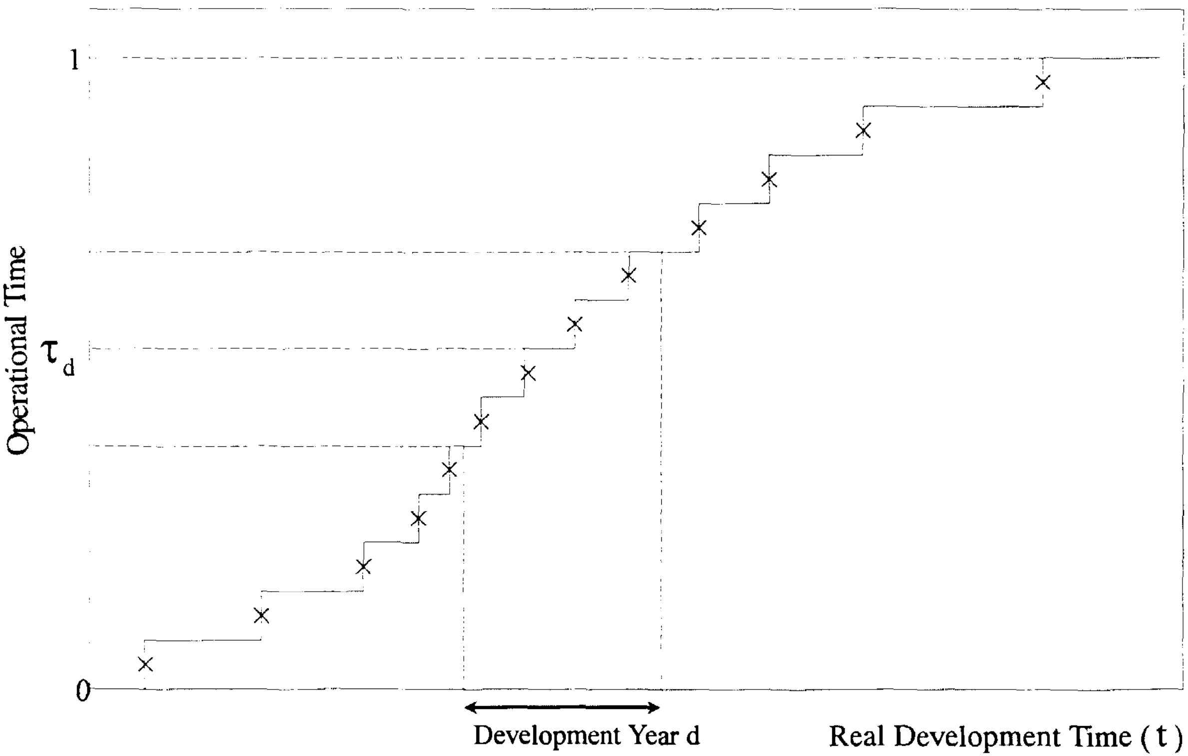
The variance of $S_{w\tau}$ is the population variance divided by the sample size. Using the population variance of Assumption 2 (Equation 1.1) gives:

$$\mathrm{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_\tau^2/N_{wd} \tag{2.3}$$

Equations 2.2 and 2.3 are actually approximations in general because the N_{wd} claims do not have exactly the same mean and variance. Equation 2.2 is exact if m_τ is linear in τ, and equation 2.3 is exact if m_τ^2 is linear in τ. Both are good approximations if m_τ does not vary greatly within each development period.

FIGURE 1

TYPICAL RELATIONSHIP BETWEEN OPERATIONAL TIME AND REAL DEVELOPMENT TIME



The form of Equation 2.3 for the variance of $S_{w\tau}$ can be:

- tested (as described below) and if not true, modified (as described in Section 3), and
- used to test and compare alternatives for the systematic component of the model; that is, the dependence of m_τ on τ (also described below).

It is not necessary to have any further knowledge about the distribution of the data S in order to fit models of "generalized linear" type for the m_τ; the variance alone is sufficient.

Equation 2.1 defines a relationship between τ, w, and d for the observed data. Given any two, the third can be found. By virtue of this known relationship, N_{wd} can alternatively be expressed as $N_{w\tau}$, and this is done for the remainder of the paper.

Models for the Mean Claim Size

An expression is needed to describe how expected severity varies as a function of the length of time a claim is open. A number of possible relationships between m_τ and τ are considered:

1. $m_\tau = \exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \ln(\tau))$
2. $m_\tau = \exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2)$
3. $m_\tau = (\beta_0 + \beta_1 \cdot \tau)^2$
4. $m_\tau = 1/(\beta_0 + \beta_1/\tau)$.

All these models are of generalized linear form; that is, the mean m_τ of the data $S_{w\tau}$ is some function of a known linear form of the unknown parameters β:

$$h(m_\tau) = \boldsymbol{x}_\tau \cdot \beta,$$

where

$h(m_\tau)$ is a known function,

x_τ is a known vector, and

$\beta = (\beta_0, \beta_1, \beta_2)$.

Table 1 gives $h(m_\tau)$ and x_τ for each of the models.

TABLE 1

Model	m_τ	$h(m_\tau)$	x_τ
1	$\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \ln(\tau))$	$\ln(m)$	$(1, \tau, \ln(\tau))$
2	$\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2)$	$\ln(m)$	$(1, \tau, \tau^2)$
3	$(\beta_0 + \beta_1 \cdot \tau)^2$	$\sqrt{m}$	$(1, \tau)$
4	$1/(\beta_0 + \beta_1/\tau)$	$1/m$	$(1, 1/\tau)$

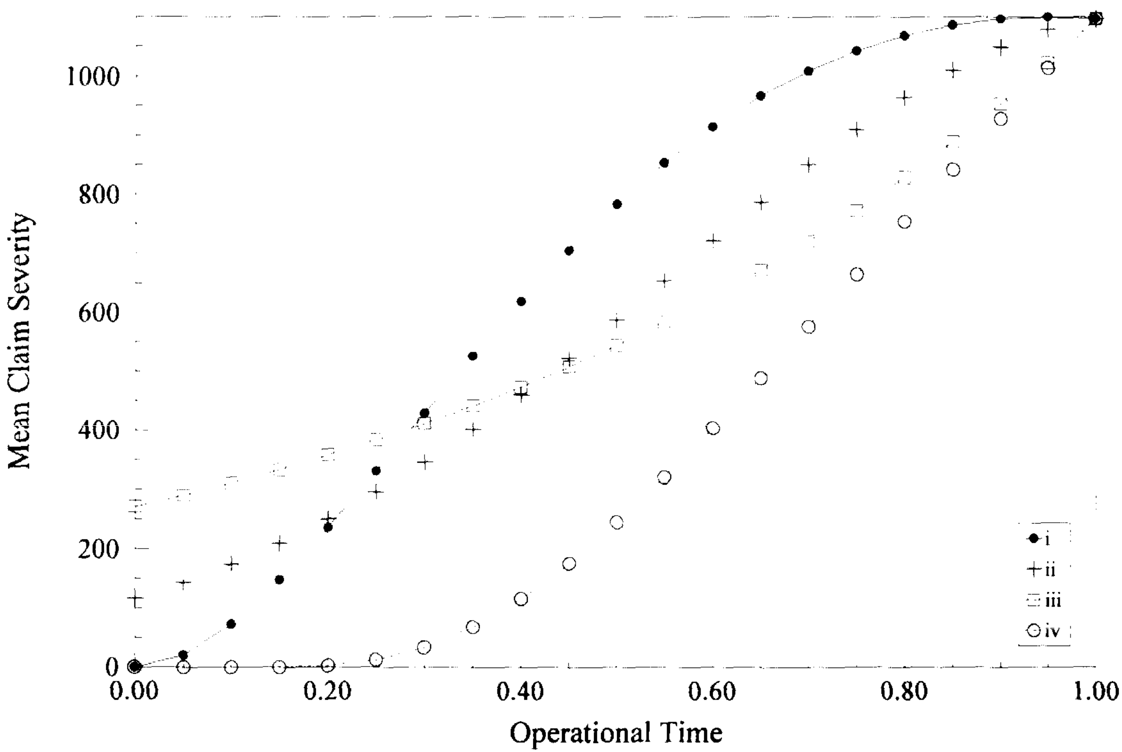
Appendix C shows how models for the mean claim size as a function of operational time can be interpreted in terms of real development time. Such models often correspond to simple relationships between the mean claim size and the distribution function of the delay. A graph of m_τ for each of the models is given in Figure 2. Although Figure 2 shows typical shapes, each model embodies a family of curves, and different shapes can be obtained within each family by varying the β-parameters. Of course, many other generalized linear models for m_τ could be formulated. All such models can be fitted, tested, and projected using the methods described below. The four models considered here have been chosen arbitrarily, for illustrative purposes.

Testing the Variance Assumption

As all the proposed models for m_τ are of generalized linear form, they can be fitted efficiently, given a second moment assumption, using Fisher's scoring method. First, it is necessary to test the proposed second moment assumption (Equation 2.3). This can be done by fitting a model (referred to as "Model 0") which makes minimal assumptions about the form of m_τ.

FIGURE 2

TYPICAL SHAPES OF VARIOUS MODELS RELATING MEAN SEVERITY TO OPERATIONAL TIME



A suitably minimal assumption is that the expected claim size m_τ is a piecewise exponential function of τ; that is, $\ln(m_\tau)$ is a piecewise linear function of τ. The important point is that this form of model is very flexible. Any reasonable function m_τ can be well approximated in this way if the intervals are sufficiently small. It is probably sufficient to take a number of sub-intervals of equal width, the number being equal to the observed number, T, of development periods. A subscript, j, is used to label these sub-intervals of the observed operational time range.

Model 0 can be expressed as:

$$m_\tau = \exp\,(\beta_0 + \sum_j \beta_j \cdot \tau_j)\,, \tag{2.4}$$

where each τ_j is the amount of τ lying in each of the sub-intervals of the operational time scale such that $\tau = \sum_j \cdot \tau_j$. This gives a continuous piecewise linear function in the exponent of Equation 2.4. The β_j are the slopes of the line segments. See Figure 3. An example of this piecewise exponential form for m_τ is shown in Figure 4. No assumption is made about the relationship between the β_j values as j varies from zero to T.

In terms of $h(m)$ and $\boldsymbol{x}_\tau$, Model 0 is:

$$h(m) = \ln(m)$$

$$\boldsymbol{x}_\tau = (1, \tau_1, \tau_2, \tau_3, \ldots, \tau_T)\,.$$

If all the sub-intervals of operational time have the same width u, then $\boldsymbol{x}_\tau$ is of the form:

$$\boldsymbol{x}_\tau = (1, u, u, \ldots, u, \tau_k, 0, 0, 0),$$

where $(k-1)u < \tau < k \cdot u$ and τ_k is the fractional part in this sub-interval. Of course, k and τ_k may differ for each data point $S_{w\tau}$, but since τ is known for each data point, $\boldsymbol{x}_\tau$ can be determined and the model fitted to estimate the parameters β_j for $j = 1$ to T.

FIGURE 3

TYPICAL MODEL ZERO (ON LOG SCALE)

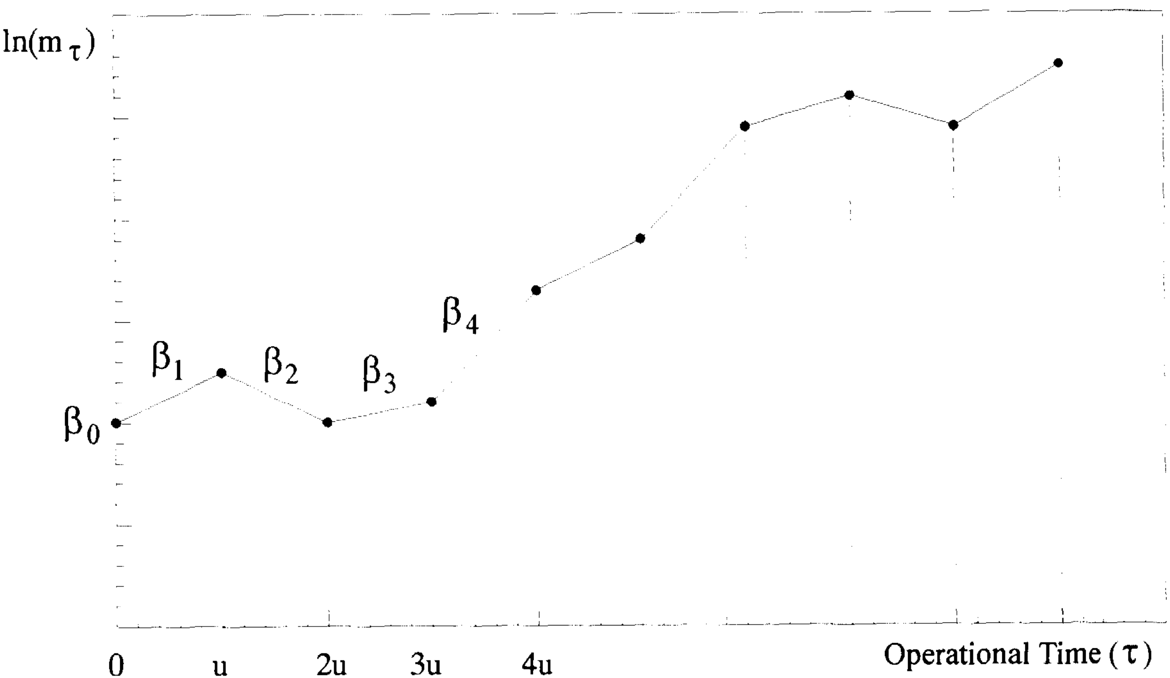
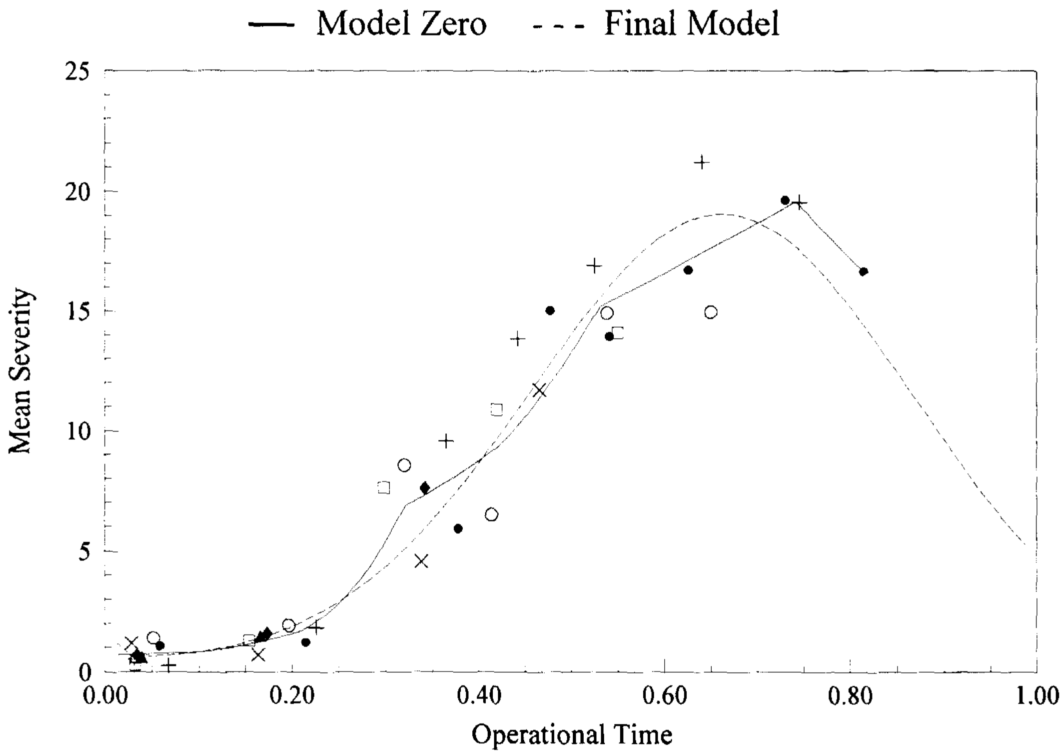


FIGURE 4

FITTED MEAN SEVERITIES FOR BERQUIST AND SHERMAN DATA



Consider the quantities:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{N_{w\tau}}/m_\tau \,. \tag{2.5}$$

If the variance of S is indeed as specified by Equation 2.3 then these quantities have $E(R_{w\tau}) = 0$ and $\mathrm{Var}\,(R_{w\tau}) = \varphi^2$. After fitting Model 0 (by Fisher's scoring method), the $R_{w\tau}$ can be estimated by using the fitted values for the m_τ (these estimated $R_{w\tau}$ are the *standardized residuals*).

The variance assumption can be tested by plotting the $R_{w\tau}$ against τ. The variance should be constant; that is, it should not depend on τ. In such a case, φ^2 can be estimated as follows:

$$\varphi_0^2 = \Big(\sum_{w\tau} R_{w\tau}^2/(n - T - 1)\Big), \tag{2.6}$$

where

n is the total number of points in the triangle, and

$T + 1$ is the number of β-parameters.

If the residual plot shows heteroscedasticity (that is, the variance appears to depend on τ), then the variance assumption (Assumption 2 of Section 1) should be modified. (See Section 3.)

Testing Models for the Mean Claim Size

Model 0 is so flexible that we can be fairly confident it will provide a good fit. The quality of fit of other models can therefore be assessed by comparison with the fit of Model 0. When the variance assumption has been validated, any other model for m_τ of generalized linear form can be formally tested as follows. After fitting by Fisher's scoring method, the standardized residuals can be calculated from:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{N_{w\tau}}/m_\tau \quad \text{(as for Model 0)}.$$

From these, another estimate of φ^2 is given by:

$$\varphi_1^2 = (\sum_{w\tau} R_{w\tau}^2)/(n-p), \tag{2.7}$$

where

n is the total number of points in the triangle, and

p is the number of parameters in the model (the βs), either two or three for each model listed in this section.

The following statistic can then be calculated:

$$F = [\varphi_1^2 \cdot (n-p)/(T+1-p) - \varphi_0^2 \cdot (n-T-1)/(T+1-p)]/\varphi_0^2\,, \tag{2.8}$$

where φ_0^2 is the estimate of φ^2 obtained from Model 0.

This should be compared against the theoretical F-distribution with $(T+1-p)$ and $(n-T-1)$ degrees of freedom. If the F-statistic is too large, then the current model for m_τ cannot be accepted. In such a case, the lack of fit may well be apparent from the plot of residuals against τ. For some values of τ, the mean may appear to be significantly different from zero. If the F-statistic could reasonably have come from the theoretical F-distribution, then the fitted means m_τ obtained using Model 0 do not vary significantly from the form assumed in the current model. Therefore, the current model can be accepted. Several of the models proposed in this section may give reasonably small F-statistics. If so, tables will indicate which F-statistic corresponds to the largest tail probability, but it may be safer to use a more general model, of which all acceptable models are special cases.

Estimates of φ^2 (hence F-tests) alternatively may be based on the minimized deviance rather than the sum of squares of the standardized residuals. This is more satisfactory in view of the likely skewness of the data. The deviance is less sensitive to the incidence of large claims than the residual sum of squares, so it will be more stable. The deviance is:

$$Q = 2 \cdot \sum_{w\tau} N_{w\tau} \cdot [-\ln(S_{w\tau}/m_\tau) + (S_{w\tau} - m_\tau)/m_\tau], \tag{2.9}$$

from which:

$$\varphi_0^2 = Q_0/(n - T - 1),$$

$$\varphi_1^2 = Q_1/(n - p),$$

hence an F-statistic from Equation 2.8.

The choice between using the residual sum of squares or the deviance to construct F-statistics arises because in neither case is the distribution truly the F-distribution. With an infinite number of data points, and models which were restricted cases of Model 0, both alternatives would have the true F-distribution. Neither of these conditions is satisfied, but the F-statistic based on the minimized deviance provides an effective, pragmatic technique for testing and comparing models. It is of no practical consequence that precise probability levels cannot be assigned to the F-statistics. Further details on the relevant theory are given by McCullagh and Nelder [5].

Prediction

If a simple model is found with an acceptably small F-statistic (not much greater than one), then it can be used for predicting future payments. The expected value of each future claim is obtained by evaluating the fitted mean m_τ at the operational time τ as defined earlier. Similarly, the variance of each future claim is obtained by evaluating Equation 1.1, using the fitted mean m_τ, and the estimate of φ^2 given by the minimized deviance as described in the preceding paragraphs. Assuming the amounts of future claims are stochastically independent, the mean and variance of the total can be obtained as the sum of the figures for the individual claims. The resulting variance must then be augmented to allow for estimation error in the fitted means m_τ. Details are given in Appendix D.

Numerical Example

The data used in the examples are the medical malpractice data published in Berquist and Sherman [1]. They are given in Appendix A. To satisfy Assumption 3 of Section 1, the Y_{wd} values of Appendix A were brought up to 1976 terms using an assumed inflation rate of 15% (the rate used by Berquist and Sherman) before calculating the sample means $S_{w\tau} = Y_{wd}/N_{wd}$. The triangle of operational times was calculated using Equation 2.1 and is given in Table A.4. A plot of the sample means $S_{w\tau}$ against operational time τ is given in Figure 5.

Model 0, which has nine parameters (one intercept parameter, and a slope parameter for each of eight subintervals of the observed range [0.0, 0.85] of operational time) gave a minimum deviance of 1,803. The plot of standardized residuals against τ is shown in Figure 6. This shows clear evidence of heteroscedasticity. The spread of the points decreases as τ increases. This indicates that the variance assumption (Assumption 2, Equations 1.1 and 2.3) is false. Consequently, all results obtained using this variance assumption are invalid. The minimized deviance, the number of residual degrees of freedom, and the F-statistic for each mean claim size model are given in Table 2. The number of degrees of freedom, *df*, is the number of data points less the number of model parameters. It appears in the denominator of Equation 2.7.

TABLE 2

Model	Deviance	*df*	F
1	3,417	33	4.0
2	2,685	33	2.2
3	3,521	34	3.7
4	4,521	34	5.8

It is stressed that the clear falseness of the variance assumption renders the above figures meaningless. They are presented here merely to illustrate orders of magnitude, and to show how the F-statistic relates to the deviance. The example is continued in Section 3.

FIGURE 5

OBSERVED MEAN SEVERITY AGAINST MEAN OPERATIONAL TIME
(DATA FROM BERQUIST AND SHERMAN)

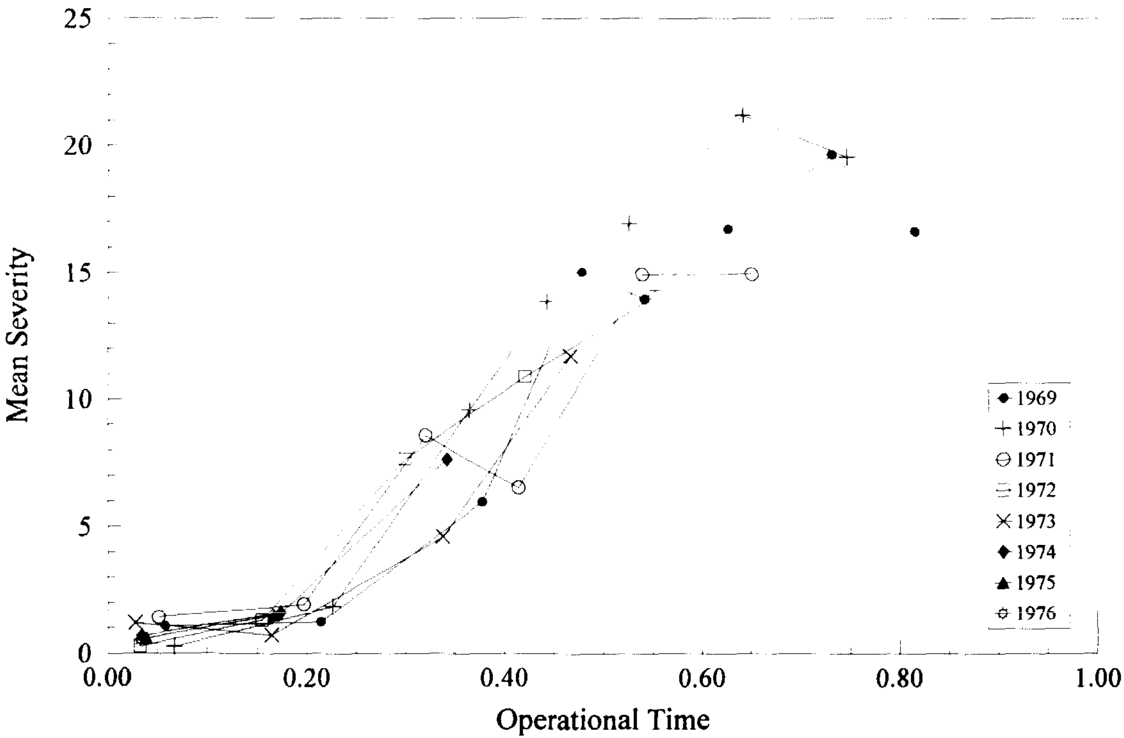
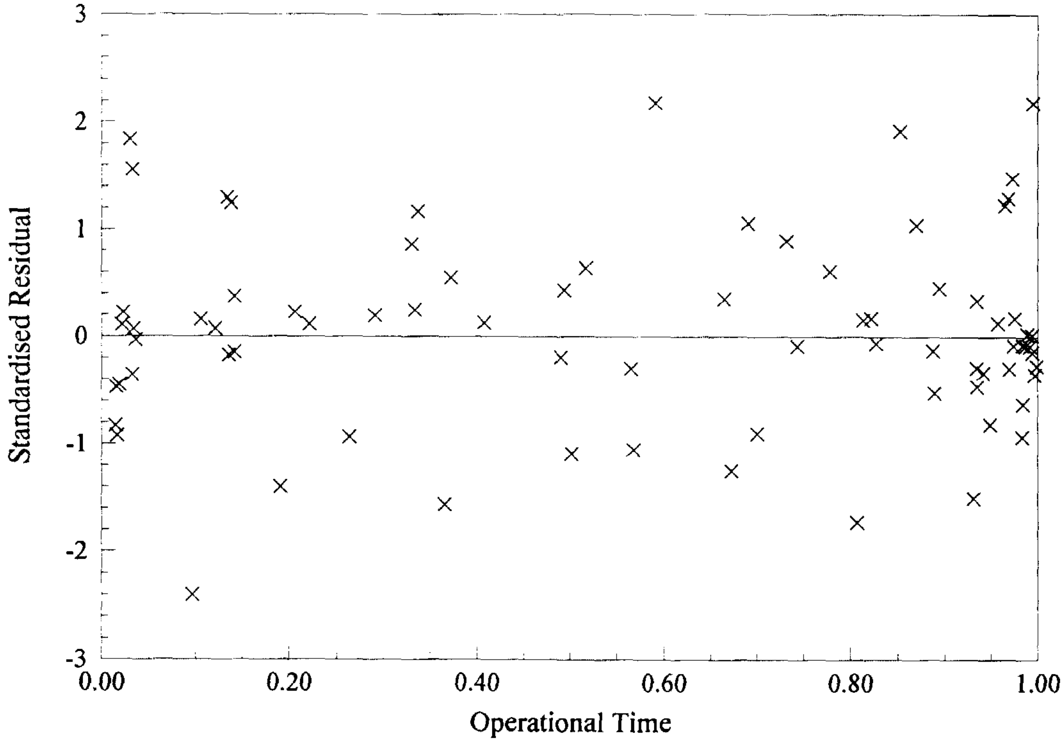


FIGURE 6

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 2$



3. RELAXING THE VARIANCE ASSUMPTION

Theory

If the initial variance assumption (Assumption 2, Equations 1.1 and 2.3) is found to be incorrect when tested as described in Section 2, then an alternative must be tried. The coefficient of variation of individual claims may depend on the mean claim size m_τ. Since this is usually an increasing function of τ, the nature of the dependence should be apparent from the plot of standardized residuals against τ for Model 0. For example, if this plot suggests that the variance is decreasing as τ increases, then the coefficient of variation decreases as the mean m_τ increases. Such a case can probably be modelled adequately by replacing Assumption 2 from Section 1 with:

$$\mathrm{Var}\,(X_\tau) = \varphi^2 \cdot m_\tau^\alpha \text{ , for some } \alpha < 2. \tag{3.1}$$

In terms of the sample mean $S_{w\tau}$, this is:

$$\mathrm{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_\tau^\alpha / N_{w\tau} \,. \tag{3.2}$$

Model 0 can be refitted on this basis (details are given below) and the standardized residuals examined to determine whether α needs to be further adjusted. The standardized residuals are given by:

$$R_{w\tau} = (S_{w\tau} - m_\tau) \cdot \sqrt{(N_{w\tau}/m_\tau^\alpha)} \,. \tag{3.3}$$

Similarly, if after fitting Model 0 using the initial assumption the standardized residuals fan out, then the model should be refitted with $\alpha > 2$.

When the variance has been satisfactorily modelled in this way for Model 0, the other models can be fitted using the variance function defined by Equation 3.1, with α taking the value found using Model 0.

The deviance to be minimized when α is not equal to one or two is given by:

$$Q = 2 \cdot \sum_{u\tau} N_{u\tau} \cdot [S_{u\tau} \cdot (S_{u\tau}^{1-\alpha} - m_\tau^{1-\alpha})/(1-\alpha) - (S_{u\tau}^{2-\alpha} - m_\tau^{2-\alpha})/(2-\alpha)\,]. \tag{3.4}$$

This is the quantity which is used to calculate F-statistics for testing and comparing the different models for the mean claim size m_τ. (See Section 2.)

Numerical Example

The example of Section 2 has been rerun using an index $\alpha = 1.5$ in the variance function, instead of $\alpha = 2$. The minimized deviance for Model 0 (which has nine parameters) is now 2,404. The plot of standardized residuals against operational time τ is given in Figure 7. It shows no evidence of heteroscedasticity, so the variance assumption (Equations 3.1 and 3.2 with $\alpha = 1.5$) is acceptable, and the results of modelling under this assumption are valid. The minimized deviance and the F-statistic for each of the models of Section 2 are listed in Table 3.

TABLE 3

Model	Deviance	*df*	*F*
1	5,829	33	6.41
2	3,567	33	2.18
3	5,053	34	4.25
4	6,568	34	6.68

Table 3 shows that none of these models fit the data very well. For a model to be acceptable, the F-statistic must be much closer to one.

An F-value of 1.22 is achieved by the following four-parameter model, which is a generalization of Models 1 and 2:

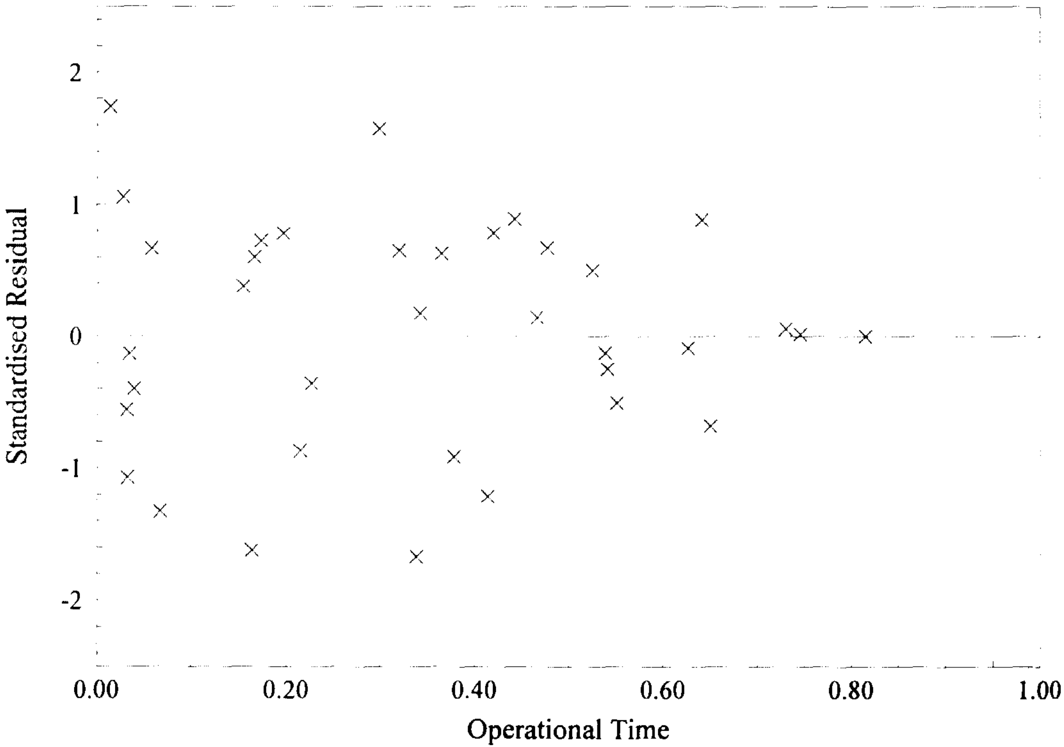
$$h(m) = \ln(m)$$

$$x_\tau = (1, \tau, \tau^2, \ln(\tau)) .$$

The estimated parameters (with their standard errors) are:

FIGURE 7

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 1.5$



β_0	-3.90	(1.08)	
β_1	18.3	(2.87)	
β_2	-12.8	(2.29)	(coefficient of τ^2)
β_3	-0.87	(0.33)	(coefficient of $\ln(\tau)$) .

This model has five fewer parameters than model zero, so the F-statistic has five and 27 degrees of freedom. Statistical tables indicate a greater than one-in-three chance of an F-value as large as 1.22, if the model is true. In other words, the variation of the fitted values m_τ, obtained using Model 0, around the curve obtained under the present model could well be purely random. So, the present model gives a good representation of the underlying pattern in the data. This is confirmed by Figure 4, which shows the fitted values of m_τ under both models. The difference between the two curves is insignificant compared to the random variation in the data.

Figure 4 also shows that the fitted curve for m_τ decreases for τ greater than about 0.66. This decrease exists in the data (Figure 5), but there are no data for operational times greater than 0.82. It is reasonable to question whether a decreasing curve for m_τ should be projected beyond this value. It is shown in Section 7 that the decrease in the observed mean claim amounts is caused largely by the presence of partial payments. In the terms of the primary triangles in Section 1, the data is actually type (c), not (a) or (b). It is analyzed here as if it were type (a) or (b) purely to illustrate the method. In practice, one should be very wary of projecting a decreasing curve for m_τ beyond the observed range of operational times, in either case (a) or (b).

The fitted model also has a minimum at $\tau = 0.05$ and m_τ tending to infinity as τ tends to zero. Although unrealistic, this is not important because projections are required only for operational times greater than 0.064 (the present operational time for the latest year of origin, 1976).

Table 4 gives the following quantities for each origin year: estimates of expected total of future payments, approximate standard errors of these estimates, estimates of standard deviations of total future payments, and approximate root-mean-square (RMS) errors of prediction.

Columns 1 and 3 have been calculated by totalling the estimated mean and variance for all future claims, as described in Section 2. Column 2 is the standard error of Column 1 arising from uncertainty in the estimated β-parameters of the mean m_τ. It has been calculated using the formulae derived in Appendix D. Column 4 is the combination of Columns 3 and 4, calculated as the square root of the sum of their squares. This is appropriate because the uncertainty represented by the standard errors in Column 2 is independent of the uncertainty represented by Column 3. Column 2 arises from random variation in past claims, whereas Column 3 arises from random variation in future claims (as described in Appendix D).

TABLE 4

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Root-Mean-Square Error
1969	3,350	1,209	959	1,543
1970	6,260	1,875	1,382	2,329
1971	14,835	3,422	2,239	4,089
1972	25,177	4,497	2,999	5,405
1973	35,842	5,120	3,607	6,263
1974	40,098	4,642	3,779	5,985
1975	47,265	4,921	4,032	6,362
1976	59,001	5,989	4,461	7,467
All	231,828	31,270	8,960	32,528

The final row (labelled "All") is for all origin years combined. The predicted total of future payments for all origin years combined is \$232 million. This is simply the sum of the figures in Column 1. The uncertainty represented by Column 2 is highly correlated between origin years, because the same set of parameter estimates (β_j, given earlier in this section) is used for all origin years. Therefore, the standard error representing this source of uncertainty for all years combined (31,270) is only slightly less than the sum of the standard errors for each origin year (the sum of Column 2 is 31,675). Full details of this calculation are given in Appendix D. In contrast, the uncertainty represented by Column 3 is stochastically independent between origin years, because the future claims for each origin year are mutually disjoint sets. Therefore, the standard deviation for all years combined (8,960), is simply the square-root of the sum of the squares of the figures in Column 3. The RMS error of prediction for all origin years combined can be calculated in the same way as for a single origin year; i.e., $32{,}528 = \sqrt{31{,}270^2 + 8{,}960^2}$, because the first component represents uncertainty arising from random variation in past claims, and the second component represents uncertainty arising from random variation in future claims. A reasonably safe reserve (for all origin years combined) can be calculated by adding one RMS error ($32.5 million) to the best estimate ($231.8 million) to give $264 million. However, since the data were adjusted to remove claims inflation, this is in 1976 terms. Section 6 shows how future claims inflation can be included in the predictions. Also, the assumed past inflation rate of 15% may not be correct, and no allowance has been made for the uncertainty in the estimates of ultimate claim numbers, M_w, used in the calculations. These two matters are dealt with in Sections 4 and 5, respectively.

4. SIMULTANEOUS ESTIMATION OF INFLATION

Basic Assumptions

This section describes techniques that can be applied to data Y_{wd} that has not been adjusted for inflation. For some of the models specified in Section 2, the force of claims inflation can be estimated from the data at the same time as estimating the other parameters.

The sample mean payment amounts $S_{w\tau}$ are now calculated as $S_{w\tau} = Y_{wd}/N_{wd}$ using the unadjusted Y_{wd}. The expected value of $S_{w\tau}$ will now depend on the origin year w (as well as τ) because of inflation, so is denoted $m_{w\tau}$. However, by assumption 1 of Section 1, the mean in real terms is the same for all origin years. Thus, $m_{w\tau}$ is of the form:

$$m_{w\tau} = \exp\left[(w + d/P) \cdot i\right] \cdot m_\tau \tag{4.1}$$

for some m_τ which are the same for all origin years. Here, i represents the annual force of claims inflation; P represents the number of development periods per year; and $w + d/P$ is, therefore, the calendar time (in years) of each data point.

The initial variance assumption is that the coefficient of variation of individual claims is constant, which implies:

$$\text{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_{w\tau}^2 / N_{w\tau}\,.$$

As before, this can be generalized, if necessary, to:

$$\text{Var}\,(S_{w\tau}) = \varphi^2 \cdot m_{w\tau}^{\alpha} / N_{w\tau}\,. \tag{4.2}$$

Models for the Mean Claim Size

Given a model for m_τ, Equation 4.1 yields a model for the mean $m_{w\tau}$ of the data $S_{w\tau}$. If i is to be treated as a parameter to be estimated, then the model for m_τ must have $\ln(m_\tau)$ linear in the unknown parameters in order for $m_{w\tau}$ to be of generalized linear form:

$h(m_{w\tau}) = \boldsymbol{x}_{w\tau} \cdot \beta$, for some known vector $\boldsymbol{x}_{w\tau}$.

Thus, of the models for m_τ proposed in Section 2, only Models 0, 1, and 2 can be fitted directly using Fisher's scoring method. These all have $h(m) = \ln(m)$.

If $\beta = (i, \beta_0, \beta_1, \ldots)$ where the β_j are the same as in Section 2, then the vector $\boldsymbol{x}_{w\tau}$ is given by:

Model 0: $\boldsymbol{x}_{w\tau} = (w + d/P, u, \ldots u, \tau_j, 0, \ldots 0)$

Model 1: $\boldsymbol{x}_{w\tau} = (w + d/P, 1, \tau, \ln(\tau))$

Model 2: $\boldsymbol{x}_{w\tau} = (w + d/P, 1, \tau, \tau^2)$.

$\boldsymbol{x}_{w\tau}$ is known for each data point as w, d, and τ are known.

Model fitting and testing can proceed with these models exactly as described in Sections 2 and 3, except that the number of parameters in each model has increased by one. If T is the number of operational time intervals used in Model 0, the number of parameters of the model is now $T+2$. The number of residual degrees of freedom is therefore $n-T-2$. This should replace $n-T-1$ in Equations 2.6 and 2.8. Similarly, the number of parameters p of Models 1 and 2 is now one greater than previously.

The question remains of how to fit models such as Models 3 and 4 which do not have $h(m)=\ln(m)$, when the rate of claims inflation is not known. The following procedure can be employed. First, fit Models 0, 1, and 2 as described above (generalizing the variance assumption if necessary). If none of these models gives an acceptable fit when compared to Model 0 (using F-tests as in Sections 2 and 3), then use the force of inflation estimated using Model 0 to adjust the data Y_{wd} into constant money terms. All models can then be fitted to the inflation adjusted data as described in Sections 2 and 3, and the best model determined. If the best model is one such as Models 1 or 2, then the version fitted to the unadjusted data should be used.

Although $x_{w\tau}$ can be determined from the data for each cell of the triangle, it is not fully known for cells corresponding to the future. The relationship between τ, d, and w for the future depends on the rate at which claims will be settled, which is uncertain. Having fitted a model, the formulae of Appendix D apply only to the factor m_τ of Equation 4.1 so the predictions are in constant prices. A further stage of estimation is necessary before claims inflation can be incorporated in projections. This approach is illustrated in Section 6.

Note that the methods described here assume that past claims inflation has been at a constant rate. In cases where this is considered to be a poor approximation, the data Y_{wd} should be preadjusted to remove any non-constant elements of claims inflation believed to be present.

Numerical Example

The methods of Section 4 are illustrated by repeating the example of Sections 2 and 3, this time with the sample means $S_{w\tau}$ calculated from the unadjusted data Y_{wd} and N_{wd} from Appendix A. As in Section 2, the residual plot from Model 0 with $\alpha = 2$ shows that this value is incorrect, and as in Section 3, the value $\alpha = 1.5$ is found to be acceptable.

Table 5 gives the minimized deviance and the F-statistic for comparing each of Models 1 and 2 to Model 0. These F-statistics each have six and 26 degrees of freedom. The table also gives the estimated force of claims inflation (and its standard error) obtained from each of the models.

TABLE 5

Model	Deviance	*df*	F	Inflation
0	1,961	26	—	0.132 (0.035)
1	4,896	32	6.49	0.141 (0.047)
2	2,865	32	2.00	0.138 (0.036)

From statistical tables, there is only about a one-in-10 chance that an F-variate with six and 26 degrees of freedom is as large as 2. Therefore, neither Model 1 nor 2 adequately represents the data. This implies that all results obtained from these models are invalid, including the estimates of the force of claims inflation given above.

However, the model used in Section 3 still gives a good fit when applied to the unadjusted data with an additional parameter for inflation. The minimized deviance is 2,402, which gives an F-value of 1.17 on five and 26 degrees of freedom. The estimated parameters (with their standard errors) are:

i	0.135	(0.034)	
β_0	-3.71	(1.06)	
β_1	17.8	(2.80)	
β_2	-12.5	(2.20)	(coefficient of τ^2)
β_3	-0.80	(0.33)	(coefficient of $\ln(\tau)$) .

The figure 0.135 for the force of claims inflation corresponds to a 14.5% annual rate.

The final results in 1976 terms are:

TABLE 6

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Root-Mean-Square Error
1969	3,450	1,169	898	1,475
1970	6,397	1,800	1,287	2,213
1971	15,034	3,261	2,071	3,863
1972	25,360	4,271	2,761	5,086
1973	35,962	4,873	3,312	5,892
1974	40,132	4,464	3,464	5,651
1975	47,279	4,796	3,696	6,055
1976	59,015	5,876	4,089	7,158
All	232,630	29,988	8,229	31,096

Adding one standard error to the best estimate gives a reserve for all origin years combined of $264 million, in 1976 terms. Although these results hardly differ from those obtained in Section 3, more confidence can be placed in them now, because the inflation rate has been estimated from the data themselves, and not based on any prior assumptions. However, no allowance has yet been made for the uncertainty in the ultimate numbers of claims M_w.

5. ALLOWING FOR UNCERTAINTY IN ULTIMATE NUMBER OF CLAIMS

Theory

In previous sections, it has been assumed that the ultimate number of claims, M_w, is accurately known for each origin year w (Assumption 4, Section 1). This assumption is realized in practice only if the origin years are report years. For any other definition of origin year, there will be an unknown number (possibly zero) of IBNR claims. This number has to be estimated in order to arrive at an estimate of M_w. It is shown in this section how the uncertainty in the M_w estimates can be taken into account in calculating standard errors of the final results. First, the source of the M_w estimates is briefly considered.

For reserving purposes, the origin years must usually be either accident years or policy years. In such cases, in addition to the triangle of the number of settled claims, N_{wd}, it may also be possible to obtain a triangle of the number of reported claims. Such a triangle will often give more information about the ultimate number of claims M_w than does N_{wd}, because claims are reported before being settled. However, the reported claims triangle will include those claims eventually settled with no payment, whereas N_{wd} and M_w should not. These matters should be considered carefully when estimating the ultimate number M_w of non-zero claims. Whether reported claims, settled claims, or both, are available for estimating M_w, the stochastic method previously developed by the author [10] can be used. As well as estimates of M_w, this method gives standard errors of the estimates, v_w. The following paragraphs describe how these standard errors can be used.

The quantities M_w are used at two points in the methods described in previous sections: in calculating the triangle of operational times τ (Equation 2.1) and in calculating predictions from the fitted model (Appendix D).

In the following paragraphs, the effect of variability in M_w is considered for each of these in turn.

If M_w is overestimated for a particular origin year w, then the operational times for that origin year will all be underestimated by a certain factor. (M_w appears in the denominator of Equation 2.1.) The observed average claim amounts $S_{w\tau}$ for that origin year, therefore, will be for later operational times than those calculated and will tend to overestimate the true mean claim amount for the calculated operational times. Conversely, if M_w is underestimated, then the mean claim sizes will also be underestimated. However, estimation of the mean claim size m_τ is done by fitting a model to the data for all origin years simultaneously. Provided the estimates M_w are unbiased and not highly correlated, the effects will tend to cancel out across origin years. There will be more variability in the data $S_{w\tau}$ across origin years w than there would otherwise be, but this variability is already taken into account through the estimate of the scale parameter φ^2. The additional effect on the variability of the final results is therefore minimal and can reasonably be ignored.

Experience with a number of data sets has confirmed these comments. Ultimate counts M_w may be estimated by a variety of methods, but the parameter estimates of the fitted model for m_τ are invariably very similar whichever set of estimates M_w is used to calculate the operational times. Usually, it is only the last few origin years that have much uncertainty in the ultimate number M_w, and these origin years contribute only a few data points (τ, S) for the modelling. Therefore, the results of the modelling are relatively insensitive to the choice of estimates M_w.

Having estimated the parameters β_j of a model relating mean claim size m_τ to operational time, the method described in Appendix D has been used in previous sections to project the fitted model and to calculate the mean-square-error of the projections. An estimate $\hat{\mu}$ of the expected total of future payments for a single origin year is calculated by summing the fitted mean m_τ over the operational times τ of each expected future claim. The values of τ for this summation are given in Equation D.1.

In Appendix E it is shown that, whatever the fitted model for m_τ, each increment of one in the estimate $\hat{M}$ will cause the estimate $\hat{\mu}$ to increase by approximately $[\tau_0 \cdot m_0 + \hat{\mu}/\hat{M}]$, where τ_0 is the operational time reached for the origin year, and m_0 is the fitted mean value corresponding to τ_0. This implies that the additional uncertainty in $\hat{\mu}$ caused by the uncertainty in M is represented by a standard error u given by:

$$u = [\tau_0 \cdot m_0 + \hat{\mu}/\hat{M}] \cdot v\,, \tag{5.1}$$

where v is the standard error of the estimate $\hat{M}$.

Numerical Example

Continuing with the example of Section 4, Table 7 gives the quantities in Equation 5.1 for each origin year.

TABLE 7

Year	$\hat{\mu}$	τ_0	m_0	$\hat{M}$	v	u
1969	3,450	0.85	12.67	2,664	70	845
1970	6,397	0.79	15.94	2,896	102	1,505
1971	15,034	0.70	18.65	4,065	148	2,484
1972	25,360	0.62	18.41	4,771	215	3,580
1973	35,962	0.53	15.27	5,280	314	4,671
1974	40,132	0.41	8.87	4,837	461	5,481
1975	47,279	0.25	3.03	5,169	690	6,843
1976	59,015	0.06	0.66	6,257	1,097	10,393

$\hat{\mu}$ is the best estimate of future payments as given in Section 4. $\hat{M}$ and v come directly from Appendix A. τ_0 is the row total N_0 of the number-of-claims-settled triangle divided by M. m_0 is calculated from τ_0 using the fitted model, which is:

$$m_\tau = \exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2 + \beta_3 \cdot \ln(\tau)) ,$$

with

$$\beta_0 = -3.71, \ \beta_1 = 17.8, \ \beta_2 = -12.5, \ \beta_3 = -0.80.$$

u is given from the other quantities using Equation 5.1.

Table 8 gives the final results. Columns 1, 2, and 3 are as in Table 6 and Column 4 holds the new component of uncertainty.

TABLE 8

Year	(1) Total Expected Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Additional Uncertainty (Number of Claims)	(5) Root-Mean-Square Error
1969	3,450	1,169	898	845	1,700
1970	6,397	1,800	1,287	1,505	2,676
1971	15,034	3,261	2,071	2,484	4,593
1972	25,360	4,271	2,761	3,580	6,220
1973	35,962	4,873	3,312	4,671	7,519
1974	40,132	4,464	3,464	5,481	7,872
1975	47,279	4,796	3,696	6,843	9,137
1976	59,015	5,876	4,089	10,393	12,620
All	232,630	29,988	8,229	15,122	34,578

The uncertainty in Column 4 for all years combined has been calculated on the assumption that the estimates M_{w} are mutually independent. It is the square root of the sum of the squares of the separate origin year figures. If non-zero covariances for the M_{w} were known, they could easily be brought into the calculation.

The three components of error (Columns 2, 3, and 4) are always mutually independent (to a good approximation), so the overall RMS error (Column 5) is simply the square root of the sum of the squares of these three columns.

Allowing for uncertainty in the number of claims outstanding has resulted in an increase in the overall standard error from \$31.1 million to \$34.6 million. The reserve based on best estimate plus one standard error has changed from \$264 to \$267 million, an increase of 1.01%.

To demonstrate the validity of Equation 5.1, the variation of $\hat{\mu}$ with $\hat{M}$ has been investigated empirically. In Table 9, the first column gives the theoretical rate of change of $\hat{\mu}$ with $\hat{M}$ for each origin year; that is, the quantity in square brackets in Equation 5.1. The remaining columns show the actual changes in $\hat{\mu}$ per unit change in M, when M is changed by the amount shown at the head of each column. A dash indicates that a result could not be calculated because the changed value for M was less than the number of claims paid to date, N_0.

For example, in Table 9, the figure 16.56 in the fifth column for 1972 was obtained as follows: The best estimate 4,771 of the ultimate number of claims M was increased by 100 to 4,871. Since the number N_0 of claims to date is 2,938, this implies 1,933 claims remaining. The fitted model m_τ of Section 5 was summed over the 1,933 different values $\tau = 2{,}938.5/4{,}871$ to $\tau = 4{,}870.5/4{,}871$. This gave the result 27,016. This is 1,656 greater than the best estimate of 25,360; and, since M was increased by 100, the mean rate of change is 16.56.

TABLE 9

		Change in M							
Year	(1) Theoretical Value	(2) -10	(3) 10	(4) -100	(5) 100	(6) -500	(7) 500	(8) -1,000	(9) 1,000
1969	12.08	12.00	12.15	11.29	12.79	—	14.82	—	15.85
1970	14.75	14.70	14.80	14.22	15.20	11.36	16.26	—	16.48
1971	16.78	16.78	16.79	16.67	16.86	15.80	16.91	13.64	16.59
1972	16.65	16.66	16.64	16.73	16.56	16.89	16.15	16.55	15.56
1973	14.88	14.89	14.86	15.00	14.75	15.51	14.27	16.06	13.74
1974	11.89	11.90	11.88	11.98	11.81	12.38	11.51	13.02	11.22
1975	9.92	9.92	9.92	9.93	9.90	10.01	9.85	10.14	9.80
1976	9.47	9.47	9.47	9.47	9.47	9.47	9.47	9.48	9.47

These results show that, for all origin years, the rate of change is almost constant within the range $M \pm v$, and is close to the theoretical value, so Equation 5.1 is a good approximation.

6. FUTURE INFLATION

Theory

Previous sections have been concerned with finding a model m_τ of the mean claim amount in constant money terms, and using the fitted model to calculate predictions in constant money terms. This section is concerned with the inclusion of future claims inflation in predictions, with due allowance for the inevitable uncertainty. This is necessary if the predictions are to be used as a basis for setting reserves, because reserves are conventionally in current money terms (not discounted).

Uncertainty in future claims inflation arises from two sources:

1. uncertainty in the future rate of claims inflation, and
2. uncertainty in the timing of the run-off of future payments.

Appendix F shows how both these elements of uncertainty can be taken into account simultaneously. Obviously, if the run-off of future claim payments is expected to take many years, moderate uncertainty in the future rate of claims inflation may lead to substantial uncertainty in current price predictions, because of the exponential effect of inflation.

Numerical Example

To illustrate the method of Appendix F, future inflation is introduced into the predictions obtained in Section 5. An exponential run-off of the remaining claim settlements over development time is used for all origin years. The time scale of the run-off can be estimated by examining the triangle of operational times (Table A.4).

Since the origin years are accident years, the mean delay to settlement is approximately d years for claims closed in development year d (except $d=0$, for which the mean delay is about 0.33 years). The triangle of operational times indicates a "half life" of just over three years. The 95% confidence range for the half life is judged to be 2.8 to 3.6 years. This corresponds to a best estimate of 3.2, and a coefficient of variation of about 0.06. In the notation of Appendix F: $U_\varphi = 0.06^2 = 0.0036$.

From Equation F.13, the best estimate of the parameter β of the exponential distribution is:

$$\beta = 3.2/\ln(2) = 4.6 \text{ years.}$$

Using Equation F.14, the remaining real delay t corresponding to future operational time τ is estimated (in years) to be:

$$t = H(\tau) = -4.6 \cdot \ln[\,(1-\tau)/(1-\tau_0)\,] \,.$$

For this example, it is assumed that the estimate of the average force of future claims inflation (from mid-1976 onwards) is 0.1 with a standard error of 0.02. Thus, it is expected that inflation will be less in the future than in the past, but the 95% confidence range of 0.06 to 0.14 contains the best estimate 0.135 of the average past force of inflation (Section 4). In the notation of Appendix F, $i = 0.1$ and $U_i = 0.02^2$. Equation F.7 gives 0.021^2 for the variance U_θ due to both uncertainty in the future force of inflation and uncertainty in the future time scale. This is only slightly greater than U_i, indicating that the second element of uncertainty is relatively minor.

The current price predictions for each origin year are given in Table 10.

TABLE 10

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Additional Uncertainty (Inflation)	(4) Standard Deviation	(5) Additional Uncertainty (Number of Claims)	(6) Root-Mean-Square Error
1969	5,531	2,056	735	1,306	900	2,699
1970	9,934	3,202	1,230	1,801	1,629	4,203
1971	22,794	6,027	2,657	2,819	2,767	7,680
1972	38,233	8,374	4,345	3,735	4,160	10,966
1973	54,798	10,254	6,299	4,530	5,791	14,103
1974	63,436	10,329	7,752	4,917	7,702	15,821
1975	79,899	12,211	10,903	5,580	11,197	20,603
1976	109,297	16,480	17,054	6,658	19,209	31,236
All	383,922	68,658	50,976	12,124	24,812	89,861

Columns 1, 2, 3, 4, and 5 are as in Table 8, except that each expected claim amount m_τ has been inflated using the factor $\exp(i \cdot H(\tau))$ before finding the total for each origin year. The new Column 3 gives the additional element of uncertainty calculated from Equation F.11. The figure for all years combined (50,976) is simply the sum of the figures for the separate origin years (this comes from repeating the argument given in Appendix D using the variance-covariance matrix of Appendix F). Intuitively, it is clear that this new component of uncertainty will be highly correlated between origin years, because the projections for all origin years are based on the same estimate of future inflation. If we overestimate future inflation, then we overestimate the reserve for all origin years simultaneously. The apparent *perfect* correlation (additivity of Column 3) is an approximation resulting from the use of first order Taylor series for these standard errors (Appendices D and F). The use of Taylor series approximations does not induce apparent perfect correlation in Column 2 (68,658 is less than the column total of 68,933) because these standard errors represent uncertainty in more than one parameter estimate (the βs) and the estimation errors are not perfectly mutually correlated.

Column 6 gives the overall mean-square-error calculated as the square root of the sum of the squares of Columns 2, 3, 4, and 5.

It is interesting to look at the delays and inflation factors of the last claims as given by the estimated function $H(\tau)$. The expected ultimate number of claims is 6,257 for origin year 1976. The operational times of the last three claims for this origin year are therefore 0.99960, 0.99976, and 0.99992. The expected delays from accident to settlement of these claims (calculated from $t = -4.6 \cdot \ln(1 - \tau)$) are 36.0, 38.3, and 43.4 years, respectively. Using the estimated force of inflation $i = 0.1$, the estimated inflation factors are 36.6, 46.1, and 76.7. These factors are obviously very sensitive to the estimate i, which is why the extra element of uncertainty can be substantial.

In practice, the function $H(\tau)$ would be estimated more carefully than in this example, making use of any additional information on likely delays. The estimated time scale parameter (4.6 in the example) need not be the same for all origin years. The method detailed in [10] can be applied to the number of claims triangle to obtain an estimate and a standard error (hence a value for U_{φ}) for each origin year. A Gamma, rather than an exponential run-off can be used to construct $H(\tau)$ if necessary, but this is unlikely to make much difference to the results except for the last one or two origin years. The use of a Gamma run-off is illustrated in the example of Section 7.

7. PARTIAL PAYMENTS

Theory

Paid claims run-off triangles are usually of type (c) (see Section 1) in practice. That is, the counts triangle N_{wd} is the number of claims closed in each development year d of each origin year w, but the paid amounts triangle Y_{wd} is the total of all payments made in development period d of origin year w. Each Y_{wd} includes partial payments on claims settled at some later development period, as well as the settlement payments counted in N_{wd}. The following paragraphs describe special procedures that may be necessary when the partial-payment component of Y_{wd} is substantial.

Dropping the subscripts w and d temporarily, each Y has two components:

$$Y = Y_1 + Y_2 ,$$

where

Y_1 is the total of payments made on claims closed, and

Y_2 is the total of payments made on claims not closed.

N_1 will denote the number of settlement payments; that is, the number of individual payments making up Y_1. Similarly, N_2 will denote the number of prepayments on claims not yet closed: the number of individual payments in Y_2. These quantities are not all known; the data consists only of Y and N_1, for each w, d combination. The mean claim amount which can be calculated from the data is:

$$S = (Y_1 + Y_2)/N_1 . \qquad (7.1)$$

Since the expected values of Y_1 and Y_2 may follow two different patterns as operational time τ varies from 0 to 1, the form of the expected value of S (as a function of τ) is likely to be more complex than when the component Y_2 is not included (situation (a) from Section 1). Furthermore, random variation of S around its expected value will be negatively correlated with random variation in N_1. This is explained further in the following sections.

Both N_1 and N_2 are subject to random variation. Initially it is assumed that they are stochastically independent. This will be discussed further below. If N_1 is higher than expected (it just happens that a large number of claims reach the settlement stage at about the same time), Y_1 will be correspondingly high, as it is the total of the N_1 settlement payments. But, N_2 (hence Y_2) will not be affected, so S will tend to be lower than expected. Conversely, a low value for N_1 will tend to give a high value for S, because Y_1 will be proportionately low, but Y_2 will not.

There is an argument which suggests that N_1 and N_2 may be positively correlated. This would limit the negative association between N_1 and S described above, and would eliminate it completely if the expected value of N_2, given N_1, were proportional to N_1. The argument is that both N_1 and N_2 may be affected in the same direction by a common cause; namely, increased activity by the insurance company on claim payment procedures, regardless of whether the payments are settlements or prepayments. This is discussed further in Appendix G. There are also arguments which suggest that N_1 and N_2 may be negatively correlated. This would substantially increase the negative association between N_1 and S described above. First, if the number of claims closed in a certain development period is unusually large, the number of claims left outstanding at the end of the period will be correspondingly small, so the number of partial payments on such claims will also tend to be small. Second, if many claims are ready for settlement at about the same time, the demand on resources made by these settlements may reduce the resources available to deal with prepayments on outstanding claims.

Previous sections have been concerned with modelling the expected value m_τ of the sample means calculated from $S = Y_1/N_1$. (In the terms of Section 1, this is situation (a).) It is shown in Appendix G that, under certain assumptions, the effect of including partial payments Y_2 in the numerator of S is approximately the same as increasing the mean m_τ by a factor of $\exp(c \cdot R_{wd})$, where c is a constant, and R_{wd} is the ratio of the number of claims outstanding in development period d to the number settled during development period d. That is, if L_{wd} is the number of claims outstanding, then $R_{wd} = L_{wd}/N_{wd}$. The coefficient c represents the expected partial payment per outstanding claim (including those with no partial payments), as a proportion of the mean size of settlement payments. For example, if the average number of partial payments in any development period is one for every five outstanding claims, and the mean size of these partial payments is half the mean size of settlement payments, then $c = 0.2 \times 0.5$.

If the model for m_τ has a linear form for $\ln(m_\tau)$, then the factor $\exp(c \cdot R_{wd})$ simply introduces a further term to the linear exponent. If R_{wd} is known, c can be estimated in the same way as the other parameters of the model. For example, in the models of Section 4, the parameter vector becomes:

$$\beta = (c, i, \beta_0, \beta_1, \ldots)$$

and the vector $\boldsymbol{x}_{w\tau}$ of known explanatory variables becomes:

Model 0: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, u \ldots u, \tau_j, 0 \ldots 0)$

Model 1: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, 1, \tau, \ln(\tau))$

Model 2: $\boldsymbol{x}_{w\tau} = (R_{wd}, w + d/P, 1, \tau, \tau^2)$.

Other models (such as Models 3 and 4) can be fitted by first dividing each observation S by $\exp(c \cdot R_{wd})$, using the value of c estimated from Model 0. This procedure is similar to the preadjustment for inflation described in Section 4.

Appendix G also shows that if m_τ represents the mean with the factor $\exp(c \cdot R_{wd})$ included, then Equation 2.3 for the variance of S becomes approximately:

$$\mathrm{Var}(S_{w\tau}) = \varphi^2 \cdot m_\tau^2 / (\exp(c \cdot R_{wd}) \cdot N_{wd}) . \qquad (7.2)$$

Therefore, to fit the models above, the factor $N_{w\tau}$ must be replaced by $\exp(c \cdot R_{wd}) \cdot N_{wd}$ in the deviance (Equation 2.9). An initial estimate of c is required for this purpose. If the estimate of c obtained by fitting Model 0 differs significantly from the initial estimate, then other results should be disregarded and the model fitting should be repeated using the new estimate of c in the deviance.

A number of assumptions are made in Appendix G, leading to the results quoted above. There is no need to consider too carefully how realistic these assumptions are in each application. The purpose of the mathematics in Appendix G is to find a broad model of which can be tested against the data. Standard statistical techniques, such as residual plots and F-tests can be used to determine whether or not the models adequately represent any particular data set. In a similar vein, although the mathematics of Appendix G deal only with the case $\alpha = 2$ in the variance function, other values of α can be used in fitting the models of this section exactly as described in Section 3, if the data indicate that this is necessary.

If the coefficient c is found to be significant, then forecasting is not as simple as in the pure operational time models of Sections 2 and 3. In order to include the partial payment effect in the forecasts, values of $R = L/N$ must be projected for future operational times so that $x_{w\tau}$ is known. In some cases, R can be modelled as a function of operational time. Projections can then be obtained as described in earlier sections, with just one change. The expression at Equation D.2 for the variance of an origin year total (standard deviation columns in the results tables) is replaced by:

$$\sigma^2 = \varphi^2 \cdot \sum_{\tau} (m_\tau^\alpha / \exp(c \cdot R_\tau)). \qquad (7.3)$$

This follows from Equation 7.2. The quantity $\varphi^2 \cdot m_\tau^\alpha / \exp(c \cdot R_\tau)$ is the variance of the total of payments made over a single increment $1/M$ in operational time. m_τ is the expected value. More generally, separate projections of R can be made for each origin year. The situation is much the same as for claims inflation.

If the reported number of claims triangle is not known, then L, the number of claims outstanding, is not known. In such a case, L can be defined as the number of claims not yet settled (whether reported or not). If the proportion not yet reported is approximately constant over development time, then L is increased by some constant factor under this alternative definition, so the parameter c will be decreased by the same factor. If the run-off of claims closed is approximately exponential from some point onwards, then $R = L/N$ is approximately constant. This can simplify projections. If the run-off is exponential from operational time 0 onwards, then the parameter c will probably be insignificant because $c \cdot R$ will be subsumed into the parameter β_0 of m_τ. In such a case, the models of Sections 2, 3, and 4 can be used even if partial payments are substantial.

Numerical Example

To illustrate, the methods described above are applied to the data from Berquist and Sherman [1] used in previous examples. These methods are more appropriate for this data set than the methods of earlier sections because the claim amounts triangle includes partial payments. In the terminology of Section 1, the data set is type (c).

The variable R_{wd} was calculated using the numbers L_{wd} of claims outstanding given in Table A.5. Table 11 gives the minimized deviance and F-statistics for the models described in this section. The F-statistic for Model 0 compares Model 0 to the less restrictive model of Appendix B. It has eight and 17 degrees of freedom. The F-statistics for Models 1 and 2 compare each of these models to Model 0. They have six and 25 degrees of freedom. The models were fitted using the variance function given in Equation 7.2, with a prior estimate of 0.1 for c and an index $\alpha = 1.5$, instead of 2. This gave satisfactory residual plots. Figure 8 shows the standardized residuals from Model 0 plotted against operational time. The same residuals are plotted against R_{wd} in Figure 9 (for the reasons given in Appendix G).

FIGURE 8

RESIDUAL PLOT FOR MODEL ZERO

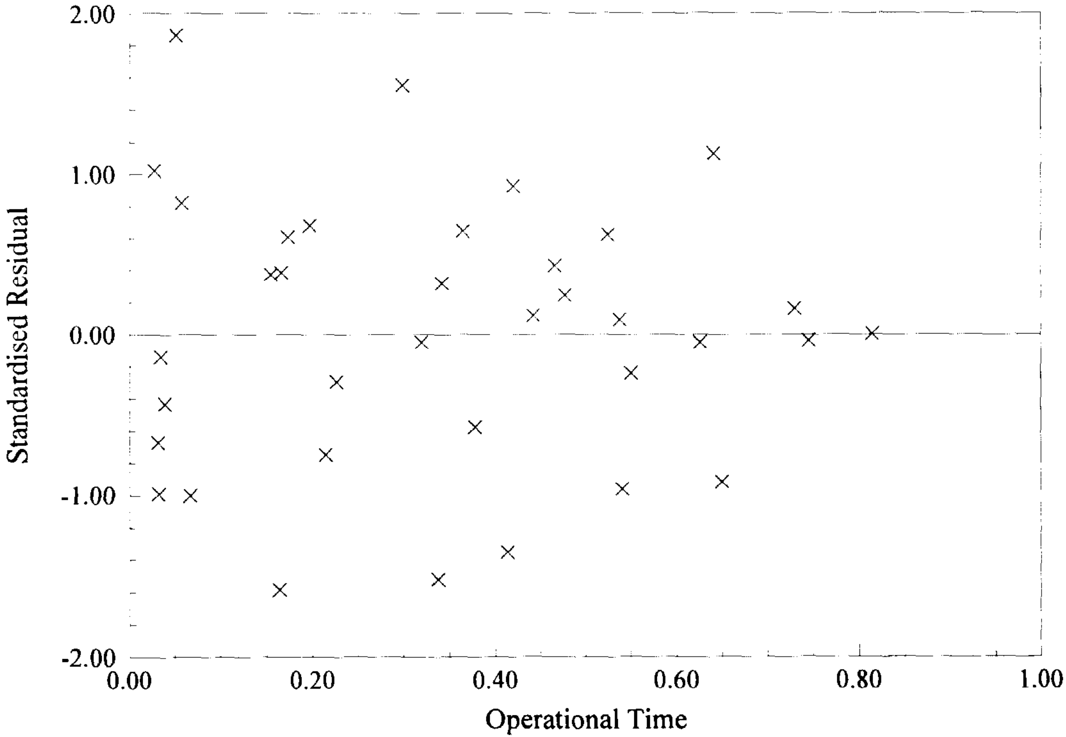


FIGURE 9

RESIDUAL PLOT FOR MODEL ZERO

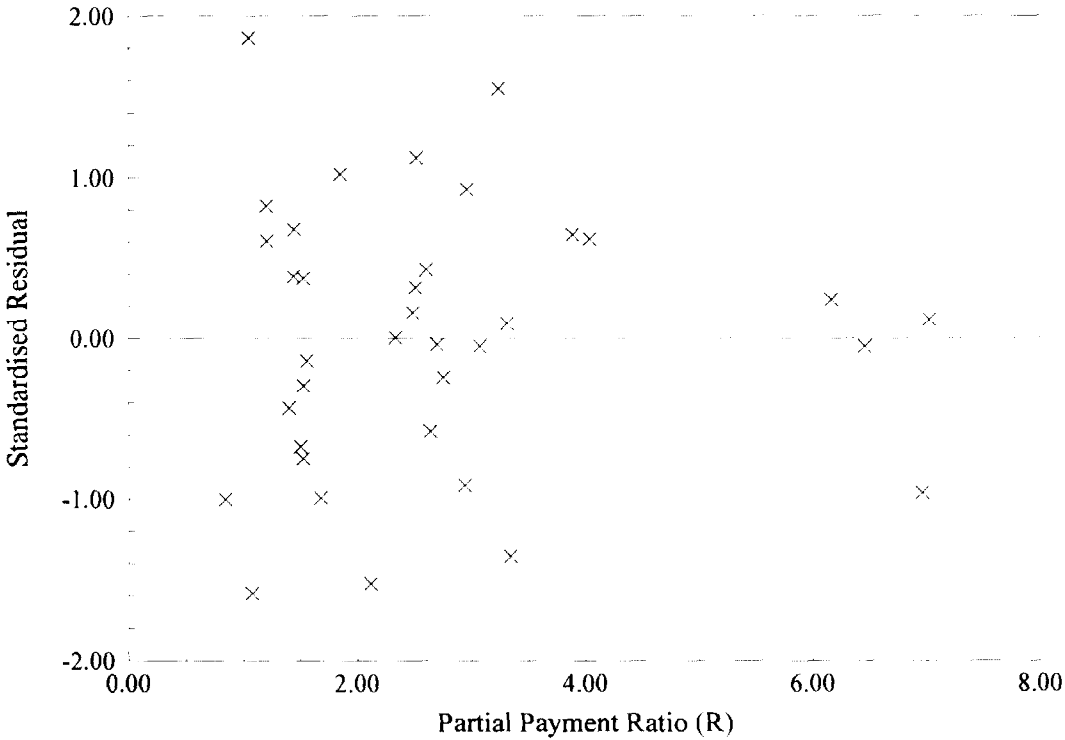


TABLE 11

Model	Deviance	df	F
0	2,143	25	0.29
1	3,909	31	3.43
2	2,773	31	1.23

Model 2 appears to fit reasonably well. A direct comparison of the deviance with that obtained in Table 5 is not valid, because the prior weights have been changed but, compared to Model 0, the fit is considerably better than in Table 5. The parameter estimates (and standard errors) for Model 2 are:

c	0.127	(0.044)	
i	0.176	(0.036)	
β_0	-1.04	(0.21)	
β_1	9.56	(1.16)	
β_2	-6.24	(1.26)	(coefficient of τ^2) .

The magnitude of c is consistent with its theoretical interpretation and is not significantly different from the prior estimate of 0.1 used in the weights.

In the examples of previous sections, the fitted models m_τ have always been decreasing for large values of τ (see Exhibit 4, for example). In Section 3, this decrease was attributed to the presence of partial payments in the data. In the present example, the partial payments have explicitly been taken into account by including the factor $\exp(c \cdot R)$ in the model. Figure 10 shows that R tends to decrease from about $\tau = 0.5$ onwards, so the partial payment factor $\exp(c \cdot R)$ also decreases. However, the other factor of the fitted model, $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2)$, also decreases for large values of τ. This is shown in Figure 11. The next paragraph describes how to test whether this remaining decrease is genuine or is due to estimation error.

FIGURE 10

FITTED MODEL FOR PARTIAL PAYMENT RATIO

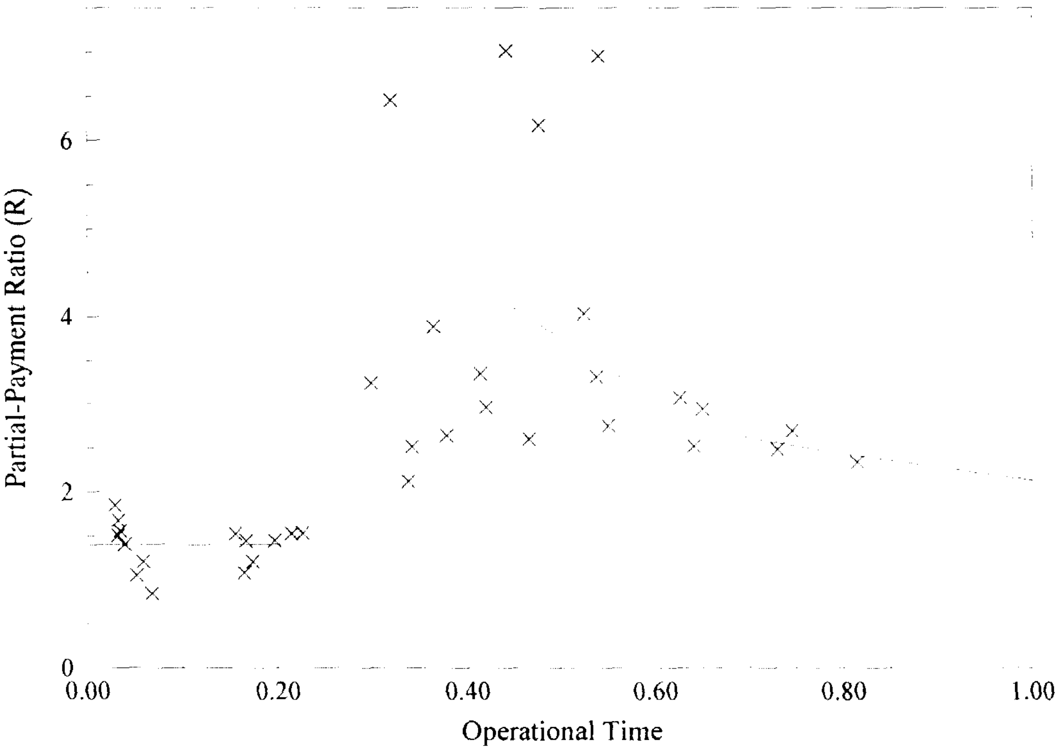
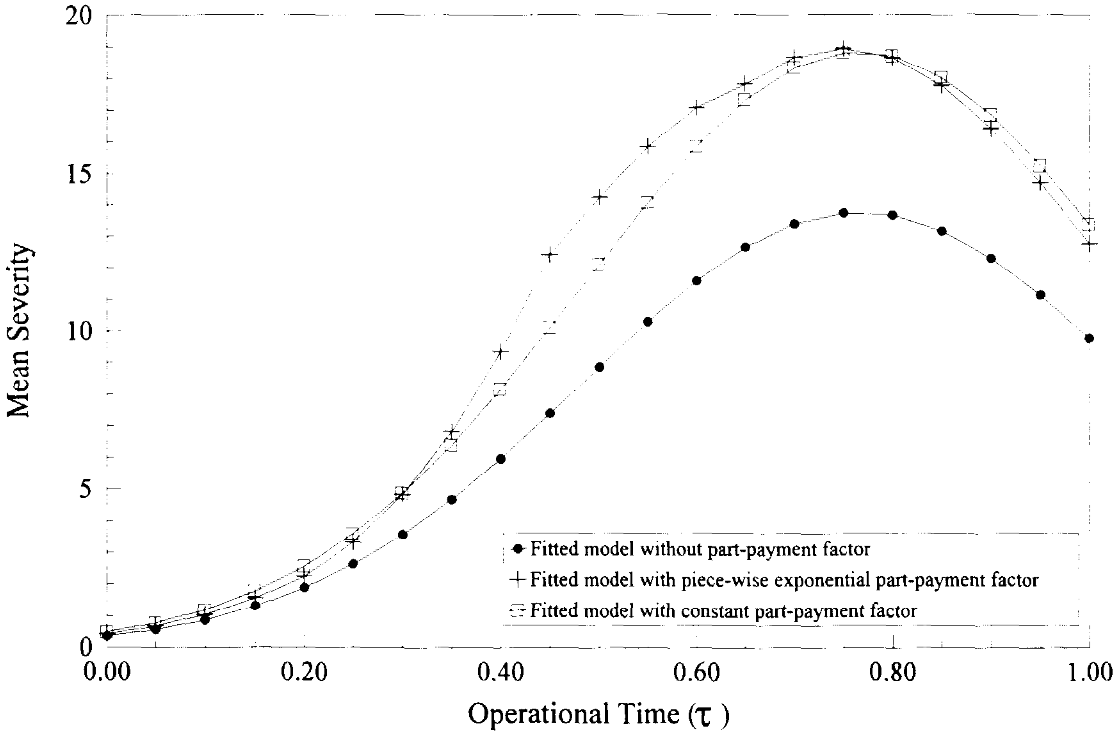


FIGURE 11

FITTED MODELS FOR MEAN SEVERITY



The slope of the exponent $\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2$ is $\beta_1 + 2 \cdot \beta_2 \cdot \tau$, so the function does not decrease for τ in the range (0, 1) if, and only if, β_1 is not negative and β_2 is not less than $-\beta_1/2$. If β_3 is defined by $\beta_3 = \beta_2 + \beta_1/2$, the exponent can be expressed as $\beta_0 + \beta_1 \cdot (\tau - \tau^2/2) + \beta_3 \cdot \tau^2$, and the condition for the function to be non-decreasing is that β_1 and β_3 should both be non-negative. This condition can be tested by refitting the model using the new explanatory variable $(\tau - \tau^2/2)$ instead of τ. This is essentially the same model as before so the parameter estimates are unchanged. β_3 is estimated to be -1.46, which follows from the estimates of β_1 and β_2 given above. However, fitting the model in this new form gives a standard error for β_3 which cannot be calculated from the previous parameter estimates and standard errors. The value is 0.72—less than half the absolute value of the estimate itself, indicating that β_3 is significantly negative. This is confirmed by refitting with β_3 set to zero. The minimized deviance becomes 3,154, an increase of 381, giving an F-statistic of 4.3 on one and 31 degrees of freedom.

The analysis described in the earlier paragraph shows that, for this data set, the decrease in the mean claim amount for large values of τ is not fully explained by the partial payment factor $\exp(c \cdot R)$. However, this does not imply that the mean settlement payment decreases with τ. A more plausible explanation is that the factor $\exp(c \cdot R)$ only partly accounts for the effects of partial payments. Full details of how this might occur are given Appendix G. Briefly, the explanation is that the rate at which partial payments are made on an open claim tends to decrease the longer the claim remains open.

As the coefficient c is significant, it is necessary to estimate a value of R for each future operational time in order to project the fitted model. Experience with other data sets suggests that R shows little variation as τ approaches one. With this in mind, a continuous piecewise linear approximation for R has been estimated by eye from the observed values:

$$
\begin{aligned}
R_\tau &= 1.40 && \text{for} && \tau < 0.21 \\
&= -0.96 + 11.25 \times \tau && \text{for} && 0.21 < \tau < 0.45 \\
&= 7.25 - 7.00 \times \tau && \text{for} && 0.45 < \tau < 0.65 \\
&= 3.80 - 1.70 \times \tau && \text{for} && 0.65 < \tau .
\end{aligned}
$$

Figure 10 shows both the observed values of R and the piecewise linear approximation R_τ.

In 1976 money terms, the formula for R_τ gives the results in Table 12. Column 3 has been calculated using Equation 7.3.

TABLE 12

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Additional Uncertainty	(5) Root-Mean-Square Error
1969	6,187	1,711	1,323	1,224	2,485
1970	10,119	2,434	1,710	1,869	3,514
1971	20,757	4,138	2,467	2,705	5,525
1972	31,795	5,287	3,049	3,741	7,159
1973	42,811	6,029	3,506	5,074	8,625
1974	46,865	5,554	3,598	6,277	9,121
1975	54,903	5,913	3,828	7,930	10,607
1976	68,591	7,197	4,240	12,079	14,686
All	282,027	37,839	8,828	17,327	42,544

As no allowance has been made for uncertainty in the projected values of R, it is interesting to examine the sensitivity of the results to these projections. The piecewise linear function defined earlier implies an average value of R over the entire range (0, 1) of τ of 2.47. Table 13 was obtained using this constant value for R.

TABLE 13

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Additional Uncertainty	(5) Root-Mean-Square Error
1969	6,372	1,778	1,337	1,244	2,549
1970	10,345	2,512	1,723	1,878	3,578
1971	21,017	4,234	2,477	2,679	5,589
1972	31,867	5,373	3,051	3,611	7,156
1973	42,180	6,080	3,489	4,710	8,446
1974	45,069	5,541	3,555	5,864	8,816
1975	52,742	5,873	3,781	7,687	10,386
1976	66,257	7,154	4,190	11,677	14,321
All	275,850	38,159	8,767	16,653	42,547

Between the two sets of results given in Tables 12 and 13 the estimate for the entire triangle differs by just over $7 million, which is quite small compared to the RMS error of about $43 million. Thus, the uncertainty in future values of R appears to be relatively unimportant. Experience with other data sets suggests this is true quite generally. Figure 11 shows the fitted model for m_τ obtained using both the piecewise linear model for R (Curve 2) and the constant model (Curve 3). The difference between these two curves is slight, which explains the similarity in the two sets of results. Curve 3 is simply a scaled up version of curve 1 which shows the fitted model with the partial payment factor exp $(c \cdot R)$ excluded.

Table 14 gives results based on the piecewise linear model for R in current money terms. Future inflation has been included using the methods described in Section 6. The run-off of settlements over real development time was taken to be exponential with the same parameters as in Section 6, and the 95% confidence interval for the future force of inflation was taken as 0.10 to 0.18.

TABLE 14

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Additional Uncertainty (Inflation)	(4) Standard Deviation	(5) Additional Uncertainty (Number of Claims)	(6) Root-Mean-Square Error
1969	15,193	4,796	3,317	3,176	1,461	6,799
1970	24,570	7,202	5,370	4,097	2,378	10,156
1971	50,298	13,404	11,132	5,994	3,780	18,810
1972	77,679	18,976	17,392	7,550	5,809	27,447
1973	106,385	24,112	24,200	8,932	8,855	36,404
1974	121,292	25,510	28,390	9,628	13,371	41,572
1975	155,292	31,440	38,697	11,172	21,330	55,369
1976	220,251	44,179	59,611	13,823	38,668	84,803
All	770,959	169,504	188,109	24,661	47,574	258,821

Berquist and Sherman produced several sets of projections, with totals ranging from \$430 million to \$750 million. The best estimate shown in Table 14 is comparatively high, but the RMS error is large. It would be interesting to see how these estimates compare to the actual experience.

Another Numerical Example

As a final example, the data set from Taylor [9] is analyzed using the methods developed in the present paper. This relates to a compulsory third party motor portfolio for the 12 accident years 1969 to 1980, broken down by development year. There were no claims closed in 1980 for origin year 1969, so the number of data points is 77. The triangles Y_{wd} and N_{wd} are given in Tables A.6 and A.7. The estimates of ultimate numbers M given by Taylor have been used. For the purposes of this example, the standard errors of these estimates have been taken as 5% of the number of claims estimated to be remaining (not yet settled). These figures are given in Table A.8. As the reported counts triangle is not given in [9], values for $R = L/N$ have been calculated using the alternative definition of L given earlier in Section 7. This triangle is given in Table A.10.

Figures 12 and 13 show the data S and R plotted against operational time τ. Figure 14 shows the quantity $S/\exp(c \cdot R)$ plotted against τ using the prior estimate 0.1 for c. This quantity is the mean payment per claim closed adjusted to remove the partial payment effect (see Appendix G). One data point has been excluded from Figures 12 and 14 for the sake of clarity: the value of (τ, S) for $d = 11$ of origin year 1970 is (0.995, 108.2). This value for S is more than double the largest value shown in Figure 12. However, it is based on only two settlements, and standardized residual plots show that it is not an outlier, therefore, it has not been excluded from the analysis.

Table 15 gives the results of fitting models with both a partial payment parameter and an inflation parameter to the unadjusted data shown in Figures 12 and 13. The variance index was taken to be $\alpha = 2$, and the prior estimate of the partial payment parameter for use in the weights was taken as $c = 0.1$. The plot of standardized residuals from Model 0 against operational time is shown in Figure 15. There is no evidence of heteroscedasticity, so the F-statistics are valid. The standardized residual for the point excluded from Figures 12 and 14 is included in Figure 15.

TABLE 15

Model	Deviance	df	F
0	618.6	63	—
1	860.4	72	2.74
2	848.2	72	2.60

The F-statistics indicate that neither Model 1 nor Model 2 fit the data very well. However, the more general model with both τ^2 and $\ln(\tau)$ in the linear predictor (as in the numerical examples of Sections 3 and 4) fits well. The minimized deviance is 704.8, giving an F-statistic of 1.10 on eight and 63 degrees of freedom.

FIGURE 12

OBSERVED MEAN SEVERITY VS. OPERATIONAL TIME

(DATA FROM TAYLOR [9])

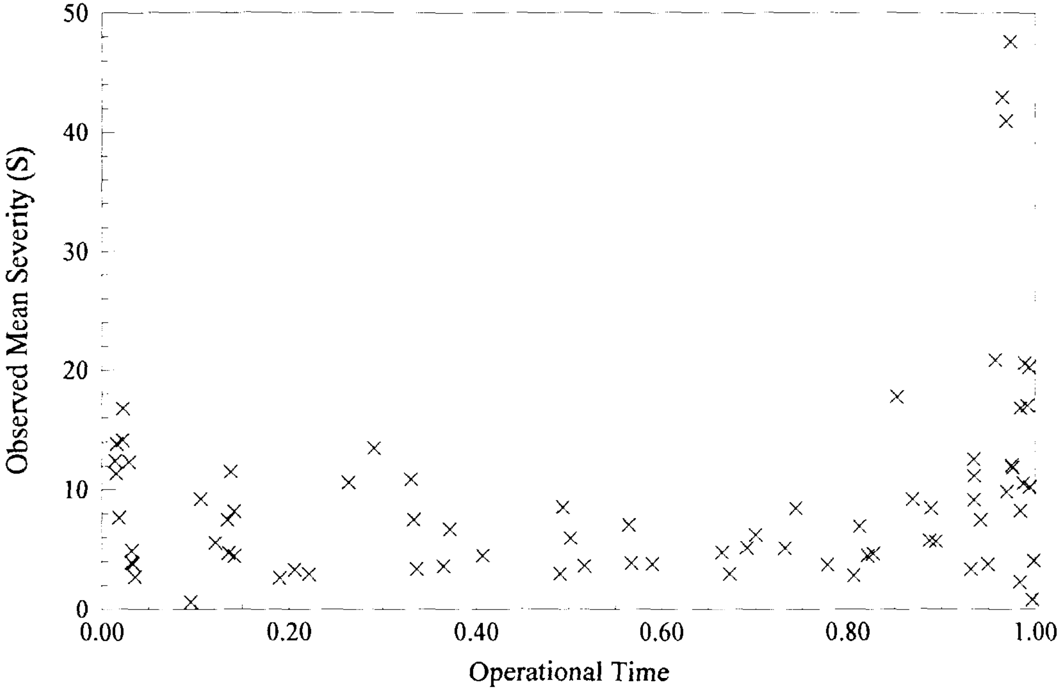


FIGURE 13

PARTIAL PAYMENT RATIO (R) VS. OPERATIONAL TIME

(DATA FROM TAYLOR [9])

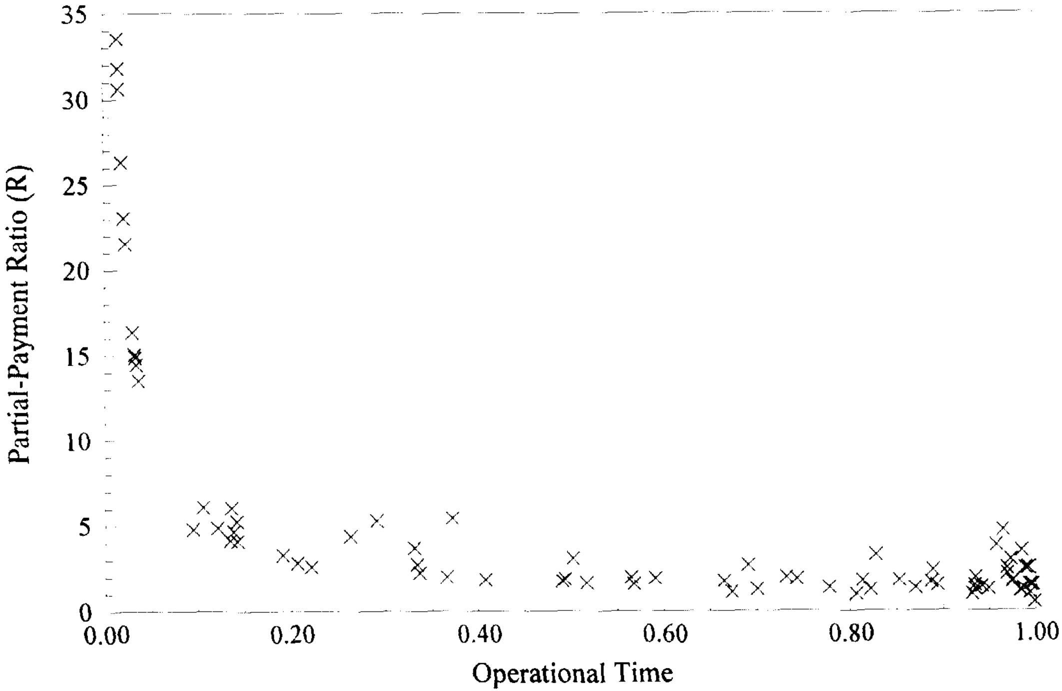


FIGURE 14

ADJUSTED MEAN SEVERITY VS. OPERATIONAL TIME

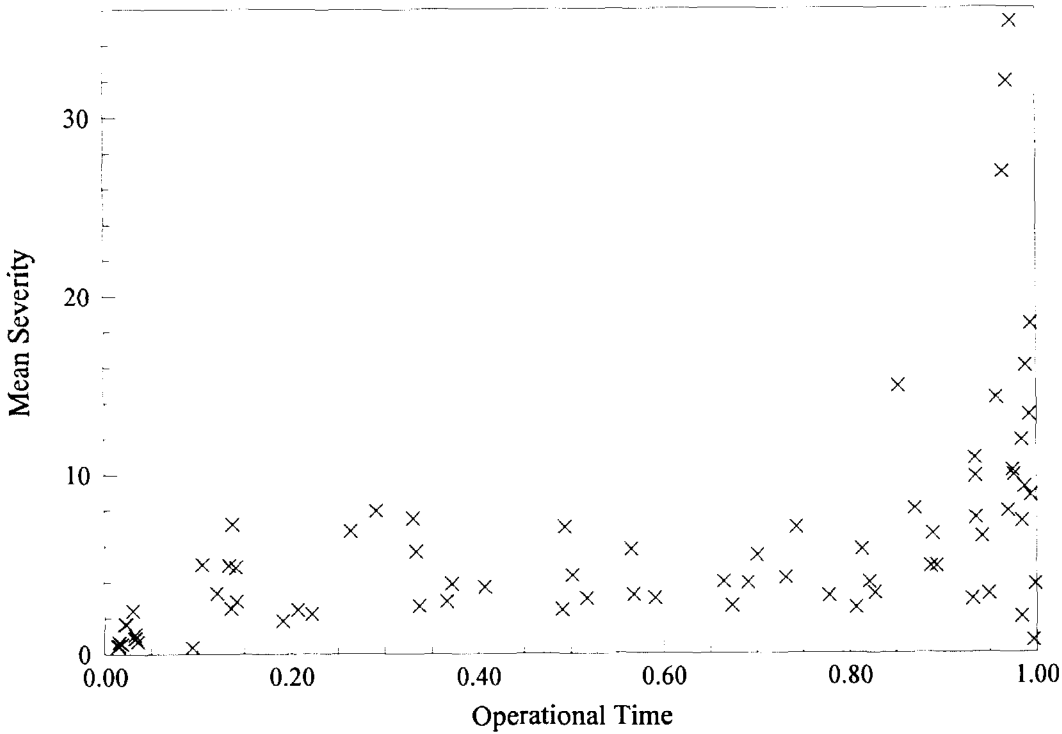
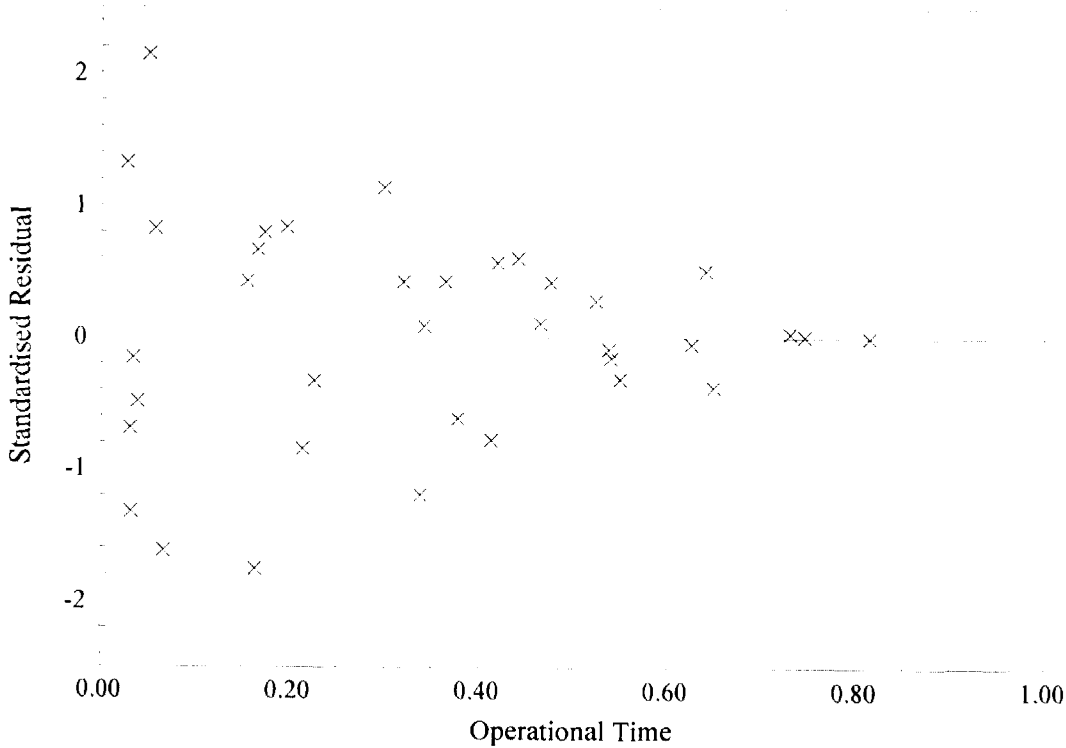


FIGURE 15

RESIDUAL PLOT FOR MODEL ZERO WITH $\alpha = 2$



The parameter estimates are:

c	0.111	(0.024)	
i	0.131	(0.013)	
β_0	5.17	(0.81)	
β_1	-7.11	(1.88)	
β_2	4.54	(1.18)	(coefficient of τ^2)
β_3	1.25	(0.34)	(coefficient of $\ln(\tau)$) .

Note that the estimate of c is not significantly different from the value of 0.1 used in the weights. Also, the average force of inflation estimated from the data does not differ significantly from Taylor's prior estimate of 0.117 (derived from the Australian Capital Territory Average Weekly Earnings Index).

The function $\exp(\beta_0 + \beta_1 \cdot \tau + \beta_2 \cdot \tau^2 + \beta_3 \cdot \ln(\tau))$ represents the payment per claim closed in constant 1980 terms, with the partial payment factor excluded. This is shown in Figure 16. This should be compared to the adjusted data shown in Figure 14. The slight decrease from 7.0 to 6.6 over the range $\tau = 0.27$ to $\tau = 0.52$ is attributed to a declining partial payment rate on open claims, as explained in Appendix G.

Figure 13 shows the observed values of R used in fitting the model. The form $R = \alpha_0 + \alpha_1/\tau$ fits reasonably well to these data. Least squares estimation gives: $\alpha_0 = 1.32$ and $\alpha_1 = 0.463$. The fitted curve is shown in Figure 17. Figure 18 shows the fitted mean payment per claim closed with the factor $\exp(c \cdot R)$ included, using the estimates $c = 0.111$ and $R = 1.32 + 0.463/\tau$. This should be compared to Figure 12. The fitted curve tends to infinity as τ tends to zero, but this is unimportant because projection is unnecessary for τ less than 0.043. Table 16 gives the forecasts obtained from this model, in constant terms; the units are thousands of 1980 Australian dollars.

FIGURE 16

FITTED MODEL WITH PARTIAL PAYMENT FACTOR EXCLUDED

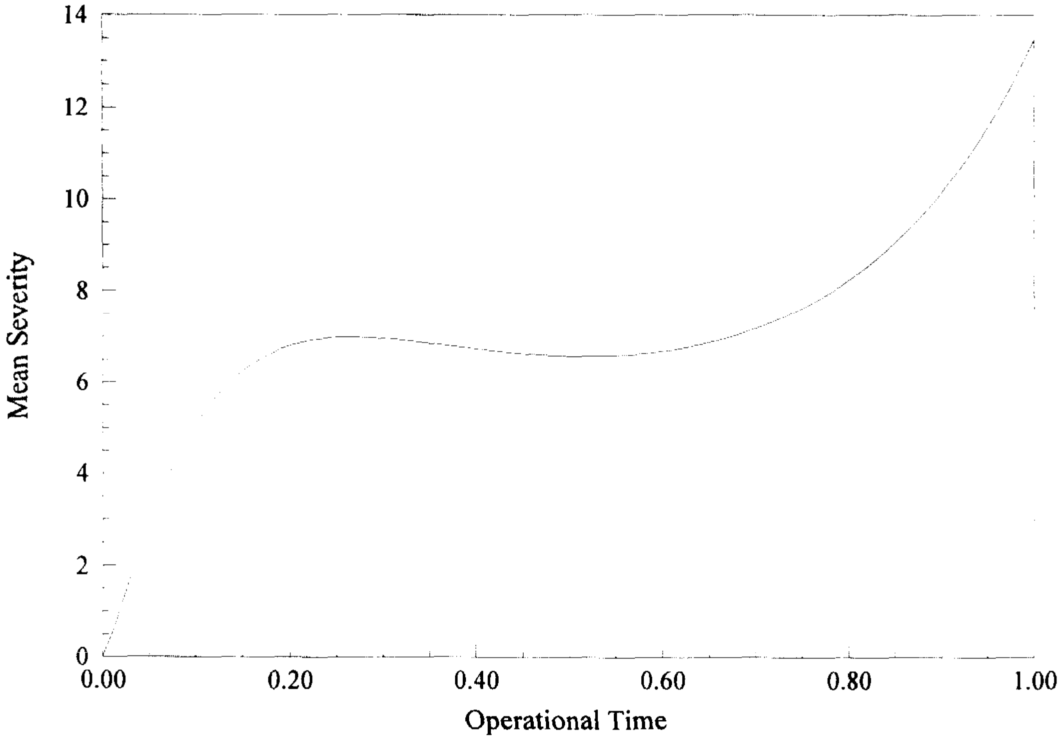


FIGURE 17

FITTED MODEL FOR PARTIAL PAYMENT RATIO

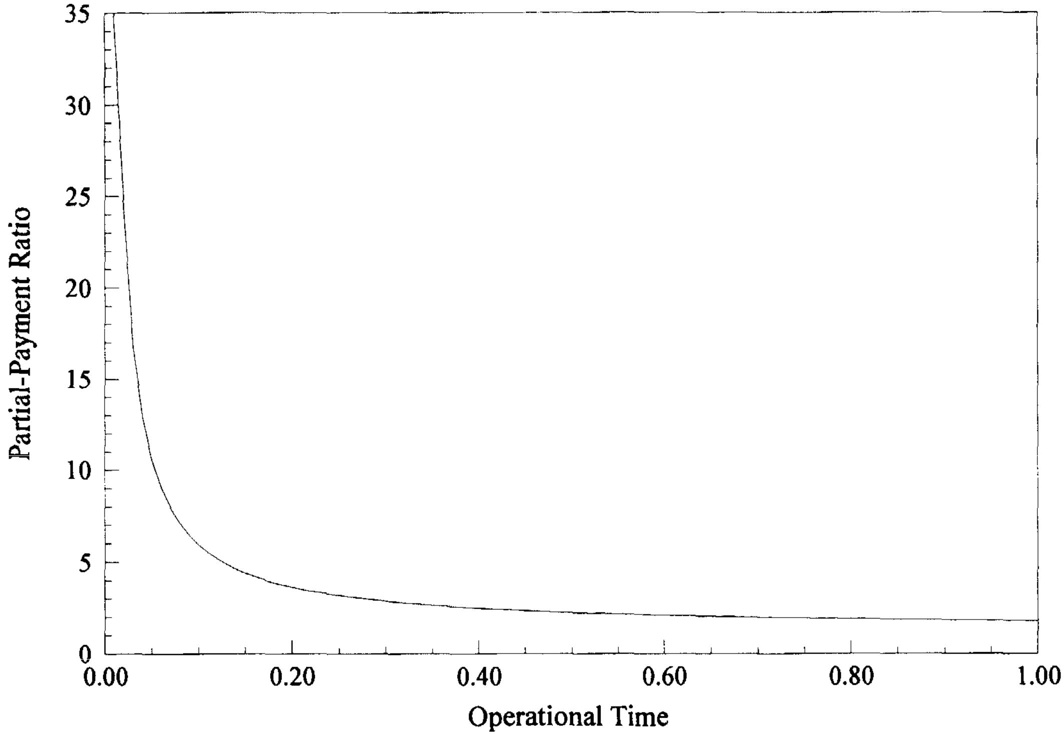


FIGURE 18

FINAL FITTED MODEL FOR DATA FROM TAYLOR [9]

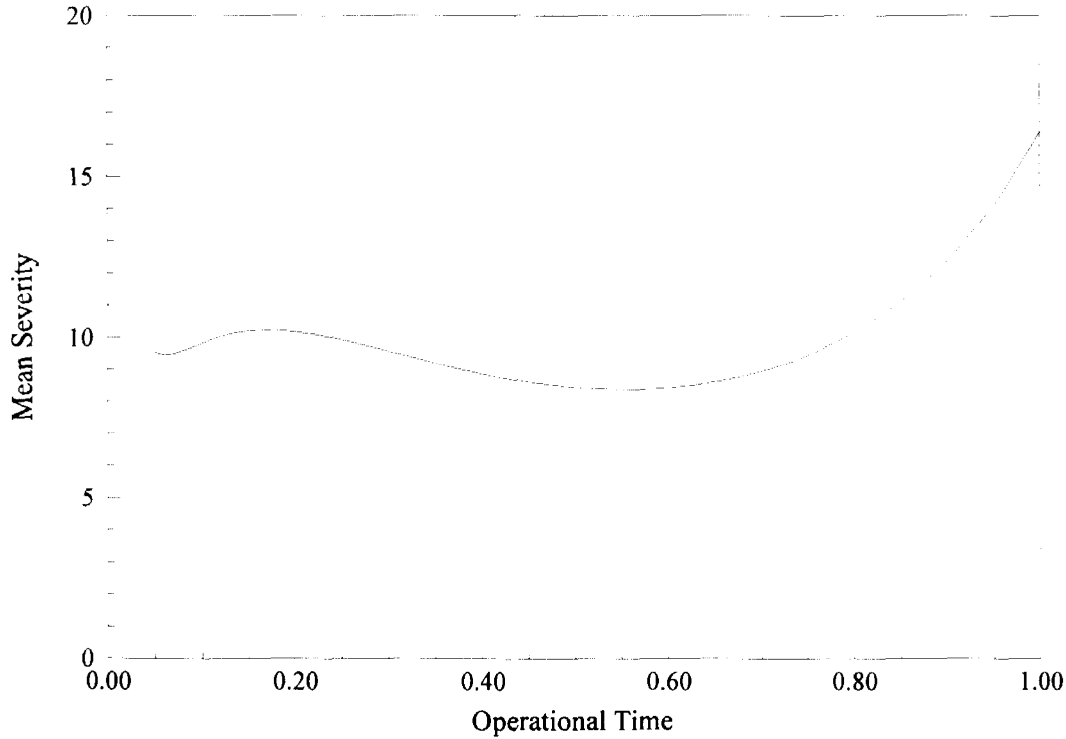


TABLE 16

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Standard Deviation	(4) Additional Uncertainty	(5) Root-Mean-Square Error
1969	0	0	0	0	0
1970	33	5	66	0	67
1971	33	5	66	0	67
1972	97	16	114	0	115
1973	315	49	203	15	209
1974	444	66	238	14	247
1975	774	102	301	37	320
1976	1,278	147	372	54	404
1977	2,550	227	491	113	553
1978	4,650	338	627	223	747
1979	5,041	349	645	249	775
1980	8,494	571	814	426	1,082
All	23,708	1,741	1,436	558	2,325

The RMS error (Column 5) for all years combined indicates that about 10% must be added to the best estimate (Column 1) for a reasonably safe reserve. This gives \$26 million in 1980 terms.

To introduce future claims inflation, the method given in [10] has been applied to the triangle of the number of settlements. This indicates that the run-off pattern does not differ significantly across origin years, the rate of settlement being proportional to $\exp(1.422 \times \ln(t) - 0.897 \times t)$, where t is real development time. Operational times have been converted to expected real development times by numerically inverting the corresponding Gamma distribution function, and future claims inflation introduced using the method outlined in Appendix F. The best estimate of the force of inflation in the future was taken as 0.12, and the combined uncertainty of this estimate and the real time scale was taken to be 0.02^2. The results are given in Table 17:

TABLE 17

Year	(1) Expected Total Future Payments	(2) Standard Error	(3) Additional Uncertainty (Inflation)	(4) Standard Deviation	(5) Additional Uncertainty (Number of Claims)	(6) Root-Mean-Square Error
1969	0	0	0	0	0	0
1970	38	6	8	77	0	78
1971	38	6	8	77	0	78
1972	115	18	23	137	0	140
1973	380	59	65	250	15	265
1974	540	81	85	296	15	319
1975	959	129	129	388	38	430
1976	1,612	193	195	494	58	568
1977	3,329	317	343	692	128	844
1978	6,275	486	542	934	275	1,216
1979	6,884	505	552	976	318	1,271
1980	12,113	851	866	1,305	600	1,881
All	32,283	2,539	2,817	2,141	748	4,419

The additional uncertainty of future inflation means that the best estimate of 32,283 must be augmented by 13.7% for a reasonably safe reserve. This gives \$36.7 million.

8. CONCLUDING REMARKS

Origin Year Effects

All the methods described in this paper are based on the hypothesis that, in real terms, the mean claim amount as a function of operational time is the same for all origin years. The plausibility of this hypothesis should be considered for each data set before proceeding to apply the methods. If there is a trend change in the mean claim amount over the origin years, for fixed operational time, the test of Appendix B should give a warning. However, the basic hypothesis could be violated in other ways. A change in the mix of claim types over origin years might cause a significant violation. For example, in property insurance, the proportion of land subsidence claims may be increased for a certain origin year because of hot dry weather. Since subsidence claims tend to be large, the mean claim amount in real terms would be higher for such an origin year. The approach of Appendix B can be modified to test for the presence of such phenomena. For example, Model 0 could be generalized by having a separate level parameter (β_0 in Appendix B) for certain origin years. Plots of residuals against origin year should help in deciding which origin years are affected.

If the basic hypothesis is violated because of a changing mix of claim types across origin years, there are two possible remedies:

1. if there are sufficient data points in each group of similar origin years, a separate "level parameter" for each group can be retained throughout the analysis, or
2. if the data are available, each claim type can be analyzed separately.

The Purpose of Modelling Partial Payments

Frequently in practice, the only data available is of type (c) (Section 1), so the methods of Section 7 are appropriate. These methods are particularly valuable when the triangle does not contain data over the full range of operational times, so that projection is necessary. For the Berquist and Sherman data, the highest observed operational time is about 0.85. Figure

4 shows the curve for the mean payment per claim closed fitted with no allowance for the presence of partial payments. This should be compared to Curve 2 of Figure 11 which was obtained by the methods of Section 7. The two fitted curves are in close agreement over the range of the data, because both fit the data well. However, their projections into the range (0.85, 1) of operational time are very different, causing a substantial effect on the forecasts (compare the results in Table 12 to those in Table 8). The improvement in projections possible by considering the effect of partial payments does not, of course, negate the need for caution when projecting a fitted curve beyond the range of the data. An informal Bayesian approach is appropriate. The indications of the particular data set under analysis (via F-tests, etc.) should be tempered by experience of more fully developed triangles for similar lines of business.

Even when the full range of operational time is covered by the data, so that no projection of m_τ is necessary, the methods of Section 7 are recommended for data of type (c). The models of Section 7 usually explain more of the variation in such data than the models of earlier sections. This is indicated by a significant estimate of the partial payment parameter c. Consequently, the other parameters of m_τ will be more reliably estimated if allowance is made for the partial payment effect. As the observed values for $R = L/N$ differ between origin years, the models of Section 7 effectively allow a different function m_τ to be fitted for each origin year.

Distribution of Settlements Over Real Development Time

The similarity between projecting the partial payment ratio R and projecting the run-off of settlements over real development time for the purpose of introducing future inflation, was mentioned in Section 7. For future inflation, the distribution function F_t of settlement delays must be projected. The relationship between τ and t can then be approximated using $\tau = F_t$. The partial payment ratio $R = L/N$ could similarly be approximated using $R = (1 - F_t)/f_t$ where f_t is the probability density function of the delays ($f_t = dF_t/dt$). This is the reciprocal of the hazard function of the delay distribution. Thus, projection of R could be based on the same model for the run-off of settlements over real development time as used to introduce future inflation. This possibility has not been fully explored but would probably require unusually accurate estimation of F_t for reasonably reliable estimates of R.

For triangles which are not well developed (such as the Berquist and Sherman triangle used in the examples) experience with more fully developed triangles for similar lines of business would be very valuable in estimating the real time scale of the remaining run-off. The technique used in the example in Section 6 of estimating the "half life" by examining the triangle of operational times is not recommended for general use. Although not illustrated in the examples, the projected total of payments remaining for each origin year could, of course, be broken down by development year, given a projection for the remaining run-off of settlements.

Integration into a Comprehensive Approach

The question of how the methods proposed in this paper can be combined with results obtained by other methods and additional items of information can obviously not be answered definitively because every reserving problem is different. A few suggestions are given below.

If reliable case estimates are available for some of the outstanding claims, the estimate M can be reduced by the number of claims concerned so that the fitted model m_τ is summed over those claims for which reliable case estimates are not available. This involves an assumption that the operational times of those claims with reliable case estimates are uniformly distributed over the remaining interval of operational time, and that the presence of a claim in this class does not depend on its size.

Because large claims are often assessed individually, and as accurately as possible, the entire method could be restricted to smaller claims only.

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APPENDIX A

DATA USED FOR THE EXAMPLES

The first data set consists of medical malpractice triangles taken from Berquist and Sherman [1]. The triangle below is Y_{wd} , the total amount of all payments made in development year d for each origin year w. This has been calculated from Exhibit E of Berquist and Sherman. The units are thousands of dollars, not adjusted for inflation.

TABLE A.1

	d							
Year	0	1	2	3	4	5	6	7
1969	125	281	1,037	1,543	1,481	3,712	4,459	3,177
1970	43	486	1,487	1,625	3,882	6,772	4,688	
1971	295	852	1,332	2,592	6,328	6,308		
1972	50	736	3,024	5,961	8,747			
1973	213	620	2,766	7,693				
1974	172	1,415	4,680					
1975	210	1,355						
1976	209							

The triangle below is N_{wd} , the number of claims closed in development year d for each origin year w. This has been derived from Exhibits C and E of Berquist and Sherman [1]. The final column is N_0, the total of the N_{wd} for each origin year.

TABLE A.2

	d								
Year	0	1	2	3	4	5	6	7	N_0
1969	311	521	349	179	161	293	261	191	2,266
1970	391	529	271	178	303	367	240		2,279
1971	418	764	236	526	487	422			2,853
1972	311	854	523	629	621				2,938
1973	294	1,146	691	657					2,788
1974	332	1,015	613						1,960
1975	406	907							1,313
1976	398								398

Table A.3 is the number of non-zero claims reported in each development year, from Exhibits C, D, and E of Berquist and Sherman [1]. The final two columns are the best estimate of the ultimate number of claims, and its standard error. These were obtained by applying the stochastic method detailed in [10] to the reported numbers triangle.

TABLE A.3

	d									
Year	0	1	2	3	4	5	6	7	M	v
1969	1,060	612	511	383	-11	24	29	17	2,664	70
1970	1,051	826	463	379	48	27	24		2,896	102
1971	1,296	1,215	627	605	116	50			4,065	148
1972	1,354	1,372	790	695	249				4,771	215
1973	1,382	1,446	843	994					5,280	314
1974	1,365	1,400	858						4,837	461
1975	1,544	1,241							5,169	690
1976	1,594								6,257	1,097

The triangle below gives the mean operational times τ_{wd} calculated from the triangle N_{wd} given in Table A.2 and the estimates M given in Table A.3 using Equation 2.1.

TABLE A.4

	d							
Year	0	1	2	3	4	5	6	7
1969	0.058	0.215	0.378	0.477	0.541	0.626	0.730	0.815
1970	0.068	0.226	0.364	0.442	0.525	0.641	0.746	
1971	0.051	0.197	0.320	0.414	0.538	0.650		
1972	0.033	0.155	0.299	0.420	0.551			
1973	0.027	0.164	0.338	0.466				
1974	0.034	0.174	0.342					
1975	0.039	0.166						
1976	0.032							

Below is the triangle of average numbers of claims outstanding calculated from the triangles given in Tables A.2 and A.3. These figures are the L_{wd} of Section 7.

TABLE A.5

	d							
Year	0	1	2	3	4	5	6	7
1969	375	795	921	1,104	1,120	900	649	446
1970	330	809	1,053	1,250	1,223	925	647	
1971	439	1,104	1,525	1,760	1,614	1,242		
1972	522	1,302	1,695	1,861	1,708			
1973	544	1,238	1,464	1,709				
1974	517	1,226	1,541					
1975	569	1,305						
1976	598							

The second data set is taken from Taylor [9]. The triangle below is Y_{wd}. The units are thousands of Australian dollars.

TABLE A.6

Year	*d* 0	1	2	3	4	5	6	7	8	9	10	11
1969	57.4	514.1	418.0	305.7	208.8	77.8	82.8	52.6	20.4	0.8	4.0	0.0
1970	122.2	559.6	674.9	453.8	156.9	246.3	200.5	88.2	20.4	34.1	216.3	
1971	213.0	619.8	561.5	398.5	438.1	161.3	120.3	50.5	61.7	81.1		
1972	168.3	426.5	518.5	705.6	378.7	400.2	214.7	285.6	73.9			
1973	201.4	536.5	912.8	707.7	717.0	246.3	187.8	409.4				
1974	191.0	612.9	959.9	608.6	483.6	261.3	376.3					
1975	181.8	511.0	942.9	798.3	599.9	747.5						
1976	198.7	850.1	1,074.8	502.5	796.1							
1977	455.9	830.6	1,129.4	1,426.5								
1978	318.2	963.6	1,279.2									
1979	424.3	1,402.7										
1980	638.0											

The triangle below is N_{wd}. The final column, N_0, is the total of each row.

TABLE A.7

						d							
Year	0	1	2	3	4	5	6	7	8	9	10	11	N_0
1969	99	154	112	84	37	21	7	5	2	1	1	0	523
1970	46	193	187	89	34	43	27	9	9	2	2		641
1971	44	191	193	78	99	49	10	3	3	4			674
1972	45	166	78	185	136	36	5	6	9				666
1973	52	115	256	240	78	27	9	10					787
1974	25	140	216	129	70	31	30						641
1975	16	93	126	113	71	42							461
1976	16	114	99	88	128								445
1977	37	102	84	168									391
1978	23	105	121										249
1979	30	122											152
1980	38												38

Table A.8 gives the estimated ultimate number of claims, M, for each origin year, and the standard error, v, of this estimate. M has been taken directly from Taylor [9], and v has been calculated (for the purposes of the second example in Section 7) as 5% of the number not yet settled ($M - N_0$).

TABLE A.8

Year	M	v
1969	523	0
1970	643	0
1971	676	0
1972	672	0
1973	807	1
1974	670	1
1975	516	3
1976	544	5
1977	622	12
1978	715	23
1979	660	25
1980	894	43

The triangle below gives the mean operational times τ_{wd} calculated using Equation 2.1 from the data in Table A.8.

TABLE A.9

	d											
Year	0	1	2	3	4	5	6	7	8	9	10	11
1969	0.095	0.337	0.591	0.778	0.894	0.949	0.976	0.988	0.994	0.997	0.999	1.000
1970	0.036	0.222	0.517	0.732	0.827	0.887	0.942	0.970	0.984	0.992	0.995	
1971	0.033	0.206	0.490	0.691	0.822	0.931	0.975	0.984	0.989	0.994		
1972	0.033	0.190	0.372	0.568	0.807	0.935	0.965	0.973	0.984			
1973	0.032	0.136	0.366	0.673	0.870	0.935	0.957	0.969				
1974	0.019	0.142	0.407	0.665	0.813	0.889	0.934					
1975	0.016	0.121	0.333	0.565	0.743	0.853						
1976	0.015	0.134	0.330	0.502	0.700							
1977	0.030	0.141	0.291	0.494								
1978	0.016	0.106	0.264									
1979	0.023	0.138										
1980	0.021											

The triangle in Table A.10 gives the mean number of claims not yet settled, calculated as the difference between M (from Table A.8) and the cumulative values of N_{wd} (from Table A.7). These figures were used for L_{wd} in the second example of Section 7.

TABLE A.10

	d										
Year	0	1	2	3	4	5	6	7	8	9	10
1969	473.5	347.0	214.0	116.0	55.5	26.5	12.5	6.5	3.0	1.5	0.5
1970	620.0	500.5	310.5	172.5	111.0	72.5	37.5	19.5	10.5	5.0	3.0
1971	654.0	536.5	344.5	209.0	120.5	46.5	17.0	10.5	7.5	4.0	
1972	649.5	544.0	422.0	290.5	130.0	44.0	23.5	18.0	10.5		
1973	781.0	697.5	512.0	264.0	105.0	52.5	34.5	25.0			
1974	657.5	575.0	397.0	224.5	125.0	74.5	44.0				
1975	508.0	453.5	344.0	224.5	132.5	76.0					
1976	536.0	471.0	364.5	271.0	163.0						
1977	603.5	534.0	441.0	315.0							
1978	703.5	639.5	526.5								
1979	645.0	569.0									
1980	875.0										

APPENDIX B

TESTING MODEL ZERO

This appendix is concerned with testing the hypothesis that the mean claim amount in real terms is a function of operational time only. This hypothesis underlies all the methods proposed in this paper, and should be checked for each data set before applying these methods. Intuitively, the most likely violation is that there may be a trend across the origin years in the mean claim amount at a certain operational time. Such a trend would give a different mean claim size for the earlier origin years than for the later origin years, for a certain operational time. This can be tested as follows.

The run-off triangle is bisected into Regions A and B as shown in Figure 19. If P is the number of development periods per annum (for example, $P = 4$ for quarterly development) then calendar time t' is given by:

$$t' = d + P \cdot (w - 1). \qquad \text{(B.1)}$$

If d runs from 0 to $T - 1$, then t' also varies from 0 to $T - 1$, so data for the last calendar period (represented by the hypotenuse of the run-off triangle) is given by $T - 1 = d + P \cdot (w - 1)$, which is equivalent to:

$$w = 1 + (T - 1 - d)/P\ . \qquad \text{(B.2)}$$

Note that data exists for the latest calendar period only for those d-values that give an integer value for w.

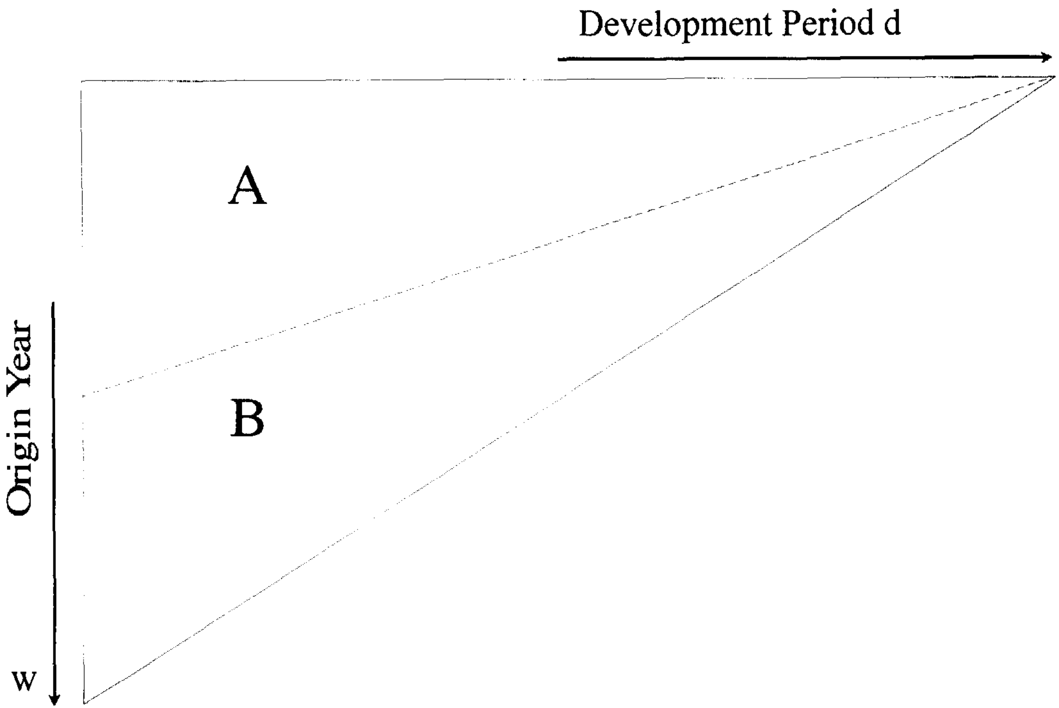
The boundary between regions A and B is therefore given by:

$$w = 0.5 \times \{ 1 + (T - 1 - d)/P \},$$

so region B is defined as those data points satisfying:

$$w > 0.5 \times \{ 1 + (T - 1 - d)/P \}.$$

FIGURE 19



Model 0 of Section 2 is generalized to allow the mean claim amount as a function of operational time to differ between the Regions A and B. This less restrictive model can be expressed as:

$$m_\tau^A = \exp\,(\beta_0^A + \sum_j \beta_j^A \cdot \tau_j) \tag{B.3}$$

$$m_\tau^B = \exp\,(\beta_0^B + \sum_j \beta_j^B\, \tau_j)\,, \tag{B.4}$$

where the explanatory variables τ_j are the same simple functions of τ used in Model 0. Model 0 is the special case $\beta_j^A = \beta_j^B$ for all j, and can be tested against the more general model using an F-test in the usual way. However, Model 0 could appear to be unacceptable when tested in this way if an incorrect inflation rate has been used to preadjust the data. For this reason it is better to use unadjusted data and include an inflation parameter. Model 0 of Section 4 is generalized to:

$$m_{w\tau}^A = \exp\,(i \cdot (w + d/P) + \beta_0^A + \sum_j \beta_j^A \cdot \tau_j) \tag{B.5}$$

$$m_{w\tau}^B = \exp\,(i \cdot (w + d/P) + \beta_0^B + \sum_j \beta_j^B \cdot \tau_j)\,. \tag{B.6}$$

In the example of Section 4, Model 0 has 10 parameters (including i) and gives a minimized deviance of 1,961 using an index $\alpha = 1.5$ in the variance function. The less restrictive model specified in Equations B.3 and B.4 has 19 parameters, but only 18 can be estimated due to an absence of data in Region A for the last operational time band. The minimized deviance (using the variance function with $\alpha = 1.5$) is 1,676. Residual plots confirm the assumption $\alpha = 1.5$. As there are 36 data points, the model has 18 degrees of freedom, giving an estimate of 93.1 for the scale parameter. The mean increase in the deviance per degree of freedom under Model 0 is (1,961-1,676)/8 = 35.6, so the F-statistic is 35.6/93.1 = 0.38 on eight and 18 degrees of freedom. The lack of significance indicates that Model 0 fits the data as well as the 18-parameter model, so the hypothesis that the mean m_τ does not vary across origin years for any τ is verified.

APPENDIX C

INTERPRETATION OF OPERATIONAL TIME MODELS FOR MEAN CLAIM AMOUNT

When formulating models for the mean claim amount as a function of operational time, it is helpful to discover what such models imply about the mean claim amount as a function of real development time. This Appendix describes how this can be done, and illustrates the techniques by giving the real-time interpretation of certain special cases and generalizations of the models proposed in Section 2. The following notation is used:

m_τ = mean claim amount as function of operational time τ,

μ_t = mean claim amount as function of real development time t,

M = ultimate number of claims closed,

F_t = distribution function (over individual settlements) of delay t,

N_t = cumulative number of claims closed by real development time t,

C_t = expected cumulative amount paid by real development time t.

By definition of τ, τ and t are related by:

$$\tau = N_t/M\,. \tag{C.1}$$

Each of the M claims has probability F_t of being closed by time t (by definition of F_t), so N_t is binomially distributed with parameters M and F_t. Therefore we have:

$$E(N_t) = M \cdot F_t \text{ and } \operatorname{Var}(N_t) = M \cdot F_t \cdot (1 - F_t)\,. \tag{C.2}$$

Hence, using Equation C.1:

$$E(\tau \mid t) = F_t \text{ and } \operatorname{Var}(\tau \mid t) = F_t \cdot (1 - F_t)/M. \tag{C.3}$$

So, if M is reasonably large, to a good approximation we have:

$$\tau = F_t \ . \tag{C.4}$$

That is, operational time is simply the distribution function of the real delay.

By definition of m and μ, we have:

$$\mu_t = m_\tau \ .$$

Using Equation C.4, this gives:

$$\mu_t = m(F_t) \ . \tag{C.5}$$

Given a functional form for m_τ, equation C.5 immediately gives a relationship between μ_t and F_t. For example, Model 1 of Section 2 in the case $\beta_1 = 0$ is:

$$\begin{aligned} m_\tau &= \exp\left(\beta_0 + \beta_2 \cdot \ln(\tau)\right) \\ &= k \cdot \tau^\beta . \end{aligned}$$

Using Equation C.5, this is equivalent to:

$$\mu_t = k \cdot F_t^\beta .$$

That is, in real time, the mean claim size is a power function of the delay distribution function.

By definition,

$$C_t = M \cdot \int_0^t \mu_s \, dF_s \ .$$

If M is sufficiently large, then from Equation C.4 we have, to a good approximation,

$$C_t = M \cdot \int_0^t m_\sigma \, d\sigma. \tag{C.6}$$

For certain functional forms m_τ, the integral on the right of Equation C.6 is analytically tractable and can be expressed simply in terms of m_τ. In such cases the equation can be rearranged to express the mean claim amount in terms of C_t.

For example, consider Model 1 or 2 of Section 2 in the case $\beta_2 = 0$:

$$m_\tau = \exp\,(\beta_0 + \beta_1 \cdot \tau) \Rightarrow \int_0^\tau m_\sigma \, d\sigma = (m_\tau - m_0)/\beta_1\ ,$$

and, using Equation C.6, we have:

$$C_t = M \cdot (m_\tau - m_0)/\beta_1\ .$$

Rearranging gives:

$$\mu_t = \mu_0 + \beta_1 \cdot C_t\,/M. \tag{C.7}$$

As a second example, consider the generalization of Model 3 of Section 2:

$$m_\tau = (\beta_0 + \beta_1 \cdot \tau)^\delta\ .$$

It is straightforward to show that this implies:

$$\int_0^\tau m_\sigma \, d\sigma = [m_\tau^{(\delta+1)/\delta} - m_0^{(\delta+1)/\delta}]/[\beta_1 \cdot (\delta + 1)\ ].$$

Hence, using Equation C.6 and rearranging:

$$\mu_t = [\beta_0^{(\delta+1)} + (\delta + 1) \cdot \beta_1 \cdot C_t/M]^{\delta/(\delta+1)}.$$

Model 3 is the case $\delta = 2$, so is equivalent to the real-time relationship:

$$\mu_t = [\ \beta_0^3 + 3 \cdot \beta_1\ \ C_t/M\ ]^{2/3}\ .$$

APPENDIX D

PREDICTION WHEN ULTIMATE NUMBERS ARE KNOWN

We have $E(X_\tau) = m_\tau$ and Var $(X_\tau) = \varphi^2 \cdot m_\tau^\alpha$ where:

X_τ is the size of the individual claim payment (in constant money terms) made at operational time τ, and

m_τ is a function of several parameters which can be expressed as a vector β.

Sections 2 through 4 of the paper describe how the data triangles can be used to decide on the functional form of m_τ, and to estimate the parameters β, α, and φ. The estimation algorithm (Fisher's scoring method) also gives the variance-covariance matrix V for the estimates of β. This Appendix describes how the fitted model can be used to predict totals of future claims, under the assumption that the ultimate number of claims M is fully known for each origin year.

Consider a single origin year. If M is the ultimate number of claims, and N_0 is the number to date, then there are $M - N_0$ claims in the future. The operational times of these future claims are:

$$\tau = (N_0 + 0.5)/M, (N_0 + 1.5)/M, \ldots, (M - 0.5)/M. \qquad \text{(D.1)}$$

If R represents the total of future claims, and μ and σ^2 denote the mean and variance of R, respectively, then since separate claim amounts are mutually independent we have:

$$\mu = \sum_\tau m_\tau \text{ and } \sigma^2 = \varphi^2 \cdot \sum_\tau m_\tau^\alpha, \qquad \text{(D.2)}$$

the summation being over the values of τ given at Equation D.1.

Hence, an estimate $\hat{\mu}$ can be obtained using the fitted model:

$$\hat{\mu} = \sum_\tau \hat{m}_\tau, \qquad \text{(D.3)}$$

where $\hat{m}_\tau$ means m_τ evaluated using the estimated values of β.

This is the "best estimate" of R given in Columns 1 of the example result tables in Sections 3 through 7.

Now consider the mean-square-error of the estimate for a single origin year given by Equation D.3:

$$\begin{aligned} E(R - \hat{\mu})^2 &= E[(R - \mu) + (\mu - \hat{\mu})]^2 \\ &= E(R - \mu)^2 + E(\hat{\mu} - \mu)^2 - 2 \cdot E(R - \mu)(\hat{\mu} - \mu) \; . \end{aligned} \tag{D.4}$$

The last term of Equation D.4 is zero because R and $\hat{\mu}$ are stochastically independent. The randomness in R comes from future claims, and the randomness in $\hat{\mu}$ comes from past claims.

The first term of Equation D.4 is simply σ^2 and can be estimated by using the estimated parameters in the expression at Equation D.2. This gives the quantity in the standard deviation columns of the examples.

For the middle term of equation D.4, note that μ is a known function of the parameters β. See Equation D.2. Using a first order Taylor series, we have:

$$\hat{\mu} \approx \mu + (\hat{\beta} - \beta)^T \cdot \delta,$$

where δ is the vector of first derivatives of μ with respect to the βs.

Hence:

$$(\hat{\mu} - \mu)^2 \approx \delta^T \cdot (\hat{\beta} - \beta) \cdot (\hat{\beta} - \beta)^T \cdot \delta.$$

Taking expected values,

$$E(\hat{\mu} - \mu)^2 \approx \delta^T \cdot V \cdot \delta, \tag{D.5}$$

where V is the variance-covariance matrix of the estimates $\hat{\beta}$.

The vector of derivatives δ can be calculated from Equation D.2:

$$\delta_i = d\mu / d\beta_i = \sum_{\tau} dm_\tau / d\beta_i \,. \tag{D.6}$$

Each term $dm_\tau / d\beta_i$ can be evaluated using the estimated β-values.

The standard error for a single origin year is given by the square root of the estimated right-hand side of Equation D.4:

$$SE = \sqrt{(\hat{\sigma}^2 + \hat{\delta}^T \cdot \boldsymbol{V} \cdot \hat{\delta})} \,.$$

This is the quantity given in the RMS columns of Tables 4 and 6. The standard error columns of these tables are from:

$$\sqrt{(\hat{\delta}^T \cdot \boldsymbol{V} \cdot \hat{\delta})} \,.$$

Now consider the prediction of total future payments for all origin years combined. The best estimate is obtained simply by summing the estimates given by Equation D.3 for each origin year. The mean-square-error of this estimate cannot be obtained so simply. The estimate of σ^2 (given by using the estimated parameters in Equation D.2) can be summed over all origin years, because all future claims are mutually independent. However, the "estimation error" component (given by Equation D.5 for a single origin year) cannot be summed over all origin years, because the same estimates $\hat{\beta}$ are used in equation D.3 for each origin year, so the estimates are not mutually independent.

Corresponding to the middle term of Equation D.4, we need to evaluate $E[\,(\Sigma\hat{\mu}) - (\Sigma\mu)\,]^2$, where the summation is over all origin years.

An analysis similar to that in Equation D.5 shows that:

$$E[\,(\Sigma\hat{\mu}) - (\Sigma\mu)\,]^2 \approx \boldsymbol{D}^T \cdot \boldsymbol{V} \cdot \boldsymbol{D}, \tag{D.7}$$

where $\boldsymbol{V}$ is the variance-covariance matrix of the estimates β, and $\boldsymbol{D}$ is the vector of first derivatives of $\Sigma\mu$ with respect to the βs. That is,

$$\boldsymbol{D} = \Sigma\, \delta,$$

where, again, summation is over all origin years.

The figure for the total over all origin years, in the second columns of the examples, is the square root of the right-hand side of Equation D.7.

If the number of outstanding claims $M - N_0$ is large for some origin years, the amount of computation involved in evaluating Equations D.2, D.3, and D.6 may be substantial. However, when M is large the variation in the summands from one point in operational time to the next is usually so small that the sums can be well approximated by integrals. That is, Equation D.2 can be replaced by:

$$\mu = M \cdot \int_{N_0/M}^{1} m_\tau \, d\tau \ \text{ and } \ \sigma^2 = \varphi^2 \cdot M \cdot \int_{N_0/M}^{1} m_\tau^\alpha \, d\tau. \qquad \text{(D.8)}$$

If the form of m_τ is such that either or both of these definite integrals are analytically tractable, the burden of computation can be substantially reduced. If the first of these is tractable, the δ_i can easily be obtained from the first part of Equation D.6:

$$\delta_i = d\mu / d\,\beta_i,$$

after finding μ as a function of β by analytic integration.

Otherwise, the integration corresponding to the second part of Equation D.6 may be tractable for some i :

$$\delta_i = M \cdot \int dm_\tau / d\,\beta_i \, d\tau.$$

Note that for models m_τ with a log link function, the partial derivatives required for calculating the δ_i have a particularly simple form:

$$m_\tau = \exp\,(\boldsymbol{x}_\tau \cdot \beta) \Rightarrow dm_\tau / d\,\beta_i = x_{\tau i} \cdot m_\tau\,.$$

APPENDIX E

PREDICTION WHEN ULTIMATE NUMBERS ARE UNCERTAIN

Appendix D deals with the distributions of future claim totals conditional on the ultimate number of claims M. These conditional distributions are relevant for forecasting when M is known. In practice, M is rarely fully known. This appendix shows how uncertainty in M can be allowed for in forecasting. Briefly, the problem is that uncertainty in M for a particular origin year implies uncertainty in the operational times of future claims, given at Equation D.1. The uncertainty affects both the number of future values of τ and the values themselves.

Replacing the summation in Equation D.2 by integration (as suggested in Appendix D), and making the conditioning on M explicit, we have:

$$\mathrm{E}(R \mid M) = M \cdot \int_{\tau/0}^{1} m_\tau \, d\tau, \tag{E.1}$$

and

$$\mathrm{Var}\,(R \mid M) = \varphi^2 \cdot M \cdot \int_{\tau_0}^{1} m_\tau^\alpha \, d\tau, \tag{E.2}$$

where the integration is from the present operational time τ_0 $(= N_0/M)$ to 1.

It is shown below that both these functions are sufficiently near linear in M (compared to the magnitude of the uncertainty in M) to ensure that good approximations of their expected values are given by replacing M by its expected value. That is,

$$\mathrm{E}(\mathrm{E}(R \mid M)) \approx \mathrm{E}(R \mid M = \mathrm{E}(M)), \tag{E.3}$$

and

$$\mathrm{E}(\mathrm{Var}(R \mid M)) \approx \mathrm{Var}(R \mid M = \mathrm{E}(M)).$$

In this appendix, μ and σ^2 are used to denote the unconditional mean and variance of the total R of future claims for an origin year. Thus,

$$\mu = \mathrm{E}(R) = \mathrm{E}(\mathrm{E}(R \mid M)) \approx \mathrm{E}(\mathrm{R} \mid M = \mathrm{E}(M)) \qquad \text{(E.4)}$$

using Equation E.3, and

$$\sigma^2 = \mathrm{Var}(R) = \mathrm{E}(\mathrm{Var}(R \mid M)) + \mathrm{Var}(\mathrm{E}(R \mid M))$$

$$\approx \mathrm{Var}(R \mid M = \mathrm{E}(M)\ ) + \mathrm{Var}(\mathrm{E}(R \mid M))\,. \qquad \text{(E.5)}$$

Equation E.4 implies that the best estimate of R is given by evaluating the right-hand side of Equation E.1 using the estimate $\hat{M} = \mathrm{E}(M)$ in place of M. Similarly, the first term on the right of Equation E.5 is simply the right-hand side of Equation E.2 evaluated using the estimate $\hat{M} = \mathrm{E}(M)$. Thus, the estimates of the unconditional μ and σ^2 are exactly the same as the estimates of the conditional μ and σ^2 given in Appendix D, except for the addition of the second term on the right of Equation E.5. Equation D.4 remains valid, and as the estimate $\hat{\mu}$ is unchanged, the middle term of Equation D.4 (the estimation error) can be evaluated exactly as described in Appendix D. Therefore, the only change necessary to the mean-square-error given in Appendix D is the addition of the second term on the right of Equation E.5.

Using the usual approximation derived from a first order Taylor series:

$$\mathrm{Var}(\mathrm{E}(R \mid M)) \approx [d\mathrm{E}(R \mid M)/dM]^2 \cdot \mathrm{Var}(M).$$

From Equation E.1:

$$d\mathrm{E}(R \mid M)/dM = \int_{\tau_0}^{1} m_\tau \, d\tau + M \cdot d/dM \int_{\tau_0}^{1} m_\tau \, d\tau\,.$$

But,

$$d/dM \int_{\tau_0}^{1} m_\tau \, d\tau = -m_0 \cdot d\tau_0/dM = m_0 \cdot \tau_0/M \, ,$$

where

$$m_0 = m_\tau \text{ evaluated at } \tau = \tau_0 \, ,$$

and

$$\int_{\tau_0}^{1} m_\tau \, d\tau = \mathrm{E}(R \mid M)/M.$$

Hence,

$$\frac{d\,\mathrm{E}(R \mid M)}{dM} = \frac{\mathrm{E}(R \mid M)}{M} + m_0 \cdot \tau_0 \, .$$

Using the estimate $\hat{M} = \mathrm{E}(M)$ to evaluate the first term gives:

$$\frac{d\,\mathrm{E}(R \mid M)}{dM} \approx \hat{\mu}/\hat{M} + m_0 \cdot \tau_0 \; . \tag{E.6}$$

The accuracy of this approximation is demonstrated in the example of Section 5.

The approximations quoted at Equation E.3 are derived in this section. For any analytic function $h(M)$, a second order Taylor series about $\mathrm{E}(M)$ gives:

$$\mathrm{E}(h(M)) \approx h(\mathrm{E(M)}) + \tfrac{1}{2} \cdot h''(\mathrm{E(M)}) \cdot \mathrm{Var}\,(M),$$

so

$$\mathrm{E}(h(M)) \approx h(\mathrm{E}(M)) \tag{E.7}$$

$$\text{if } \tfrac{1}{2} \cdot \mid h''(\mathrm{E}(M))\mid \cdot \mathrm{Var}\,(\mathrm{E}(M)) << h(\mathrm{E}(M)).$$

In the remainder of this section, E(M) is shortened to $\hat{M}$, because the best estimate of M is its prior expected value.

It is straight-forward to show that if:

$$h(M) = M \cdot \int_{N/M}^{1} g(\tau)\, d\tau, \qquad \text{(E.8)}$$

where g is any analytic function, then:

$$h''M = -(N^2/M^3) \cdot g'(N/M).$$

Also, if g is increasing,

$$h(M) > (M - N) > g(N/M)\,, \qquad \text{(E.9)}$$

so for functions $h(M)$ of the form in Equation E.8 with g increasing, a sufficient condition for Equation E.7 is:

$$½ \cdot (N^2/\hat{M}^3) \cdot g'(N/\hat{M}) \cdot \mathrm{Var}(M) << (\hat{M} - N) \cdot g(N/\hat{M}).$$

Writing $f(\tau)$ for $\ln(g(\tau))$ so that $f'(\tau) = g'(\tau)/g(\tau)$, and assuming $g(\tau) > 0$, this becomes:

$$f'(N/\hat{M}) << 2 \cdot \hat{M}^3 \cdot (\hat{M} - N)/(N^2 \cdot \mathrm{Var}(M)).$$

Writing τ for N/M and Ω for the coefficient of variation of M ($\Omega^2 = \mathrm{Var}(M)/\hat{M}^2$) this becomes:

$$f'(\tau) << 2 \cdot (1 - \tau)/(\Omega \cdot \tau)^2\,. \qquad \text{(E.10)}$$

The functions in Equations E.1 and E.2 are of the form given in Equation E.8 with $g(\tau)$ positive and increasing. In terms of $f(\tau) = \ln(g(\tau))$, the mean (Equation E.1) is the case: $f(t) = \ln(m_\tau)$, and the variance (Equation E.2) is the case: $f(\tau) = 2 \cdot \ln(\varphi) + \alpha \cdot \ln(m_\tau)$.

Therefore, using Equation E.10, a sufficient condition for Equation E.3 is:

$$\alpha \cdot d\ln(m_\tau)/d\tau << 2 \cdot (1 - \tau)/(\Omega \cdot \tau)^2 . \qquad \text{(E.11)}$$

This is invariably satisfied by fitted models m_τ. This appendix demonstrates this for the example of Section 5.

Although the fitted model m_τ of Section 5 is not strictly increasing (because of partial payments), it is mostly increasing and the inequality in Equation E.9 remains true for most τ. The fitted model has $\alpha = 1.5$, and

$$\ln(m_\tau) = -3.71 + 17.8 \cdot \tau - 12.5 \cdot \tau^2 - 0.80 \cdot \ln(\tau) .$$

Therefore,

$$d\ln(m_\tau)/d\tau = 17.8 - 25 \cdot \tau - 0.8/\tau .$$

Table E.1 gives both sides of Equation E.11 evaluated for each origin year using figures $\tau = \tau_0$ and $\Omega = \sqrt{\text{Var }(M)}/M$ from Table 7.

TABLE E.1

Year	τ_0	Ω	LHS	RHS
1969	0.85	0.026	-6.6	614
1970	0.79	0.035	-4.4	549
1971	0.70	0.036	-1.3	945
1972	0.62	0.045	1.5	976
1973	0.53	0.060	4.6	930
1974	0.41	0.095	8.4	778
1975	0.25	0.134	12.5	1,337
1976	0.06	0.175	4.5	17,052

These figures indicate that, for this example, the exclusion of non-linear terms at Equations E.4 and E.5 leads to an error of no more than about 1% in the mean and variance of total future payments for each origin year. The near-linearity of E(R | M) is demonstrated more directly in Section 5.

APPENDIX F

PREDICTION WITH UNCERTAIN FUTURE INFLATION

From Appendix D, the expected constant-price total of future claims for a single origin year is given by:

$$\hat{\mu} = \sum_{\tau} \hat{m}_{\tau}, \tag{F.1}$$

where $\hat{m}_{\tau}$ is the fitted mean claim amount in constant terms, and summation is over the operational times τ of all expected future claims (given in Equation D.1).

If t represents the real calendar time corresponding to operational time τ (with the convention $t = 0$ when $\tau = \tau_0$), and if the force of future claims inflation is i, the current price total is obviously given by:

$$\hat{\mu}' = \sum_{\tau} \exp(i \cdot t) \cdot \hat{m}_{\tau} . \tag{F.2}$$

To evaluate the current price prediction $\hat{\mu}'$ from equation F.2, the relationship between t and τ is needed. This can be approximated using a continuous curve. A typical shape is shown in Figure 20. The shape can usually be well approximated using an exponential distribution function, although a Gamma distribution function is sometimes necessary for later origin years. There is usually uncertainty about the real time scale.

Thus, we have $\tau = F(\varphi' \cdot t)$ for some known function F. The uncertainty in the real time scale is represented by the random variable φ'.

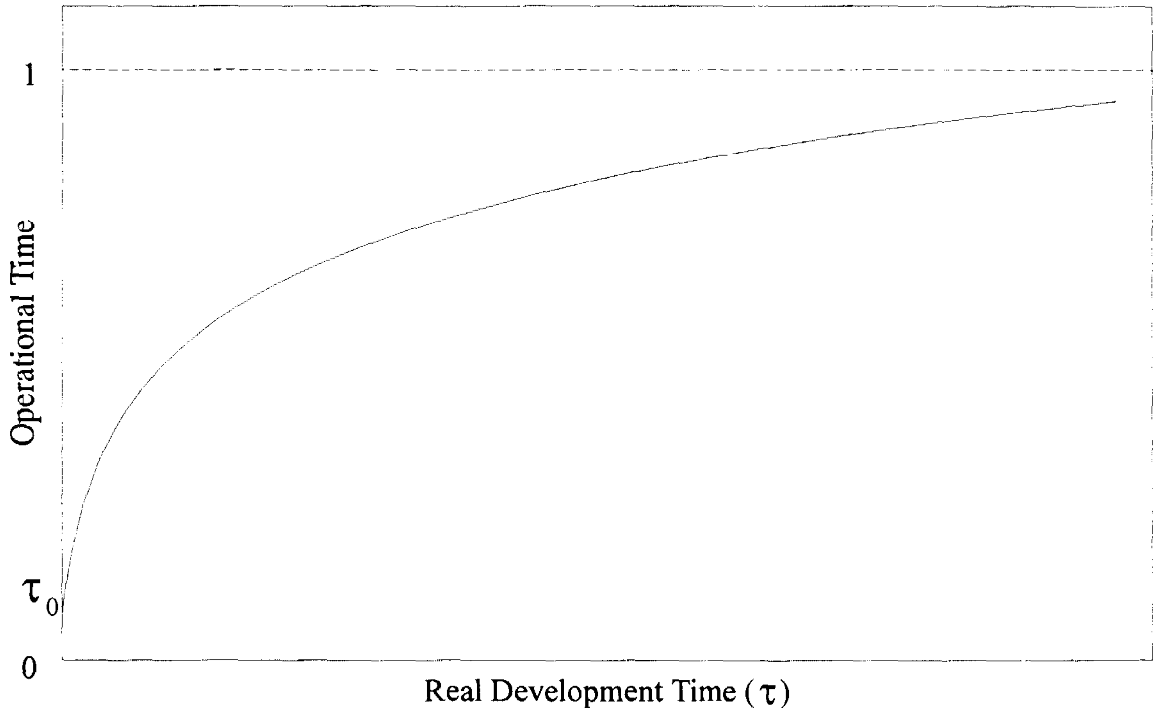
Rearranging:

$$t = \varphi \cdot \mathrm{H}(\tau) , \tag{F.3}$$

where H is the inverse of F, and $\varphi = 1/\varphi'$. A following section will describe how the function H can be found in cases where the run-off of settlements is approximately exponential.

FIGURE 20

TYPICAL TAIL RELATIONSHIP BETWEEN OPERATIONAL TIME AND REAL DEVELOPMENT TIME



Substituting from Equation F.3 into Equation F.2, and writing θ for $i \cdot \varphi$ gives:

$$\hat{\mu}' = \sum_{\tau} \exp(\theta \cdot H(\tau)) \cdot \hat{m}_{\tau} . \tag{F.4}$$

In general, there is uncertainty in both the rate of future claims inflation and the real time scale, so both i and φ are random variables. The expected values are denoted $\hat{i}$ and $\hat{\varphi}$, and the variances (representing the uncertainty) are denoted U_i and U_φ , respectively. Since θ is the product of these two random variables, its mean is given by $\hat{\theta} = \hat{i} \cdot \hat{\varphi}$ and its variance by:

$$U_\theta = U_i \cdot U_\varphi + \hat{i}^2 \cdot U_\varphi + \hat{\varphi}^2 \cdot U_i . \tag{F.5}$$

It is always possible to have $\hat{\varphi} = 1$ by scaling the function H (Equation F.3). When this is done, we have $\hat{\theta} = \hat{i}$. Equations F.4 and F.5 become:

$$\hat{\mu}' = \sum_{\tau} \exp(\hat{i} \cdot H(\tau)) \cdot \hat{m}_{\tau} , \tag{F.6}$$

and

$$U_\theta = U_i \cdot U_\varphi + \hat{i}^2 \cdot U_\varphi + U_i . \tag{F.7}$$

Writing A_τ for the inflation factor $\exp(\hat{i} \cdot H(\tau))$, we have:

$$\hat{\mu}' = \sum_{\tau} A_\tau \cdot \hat{m}_{\tau} . \tag{F.8}$$

Current price predictions can be calculated using the methods described in Appendices D and E with the following changes:

1. The estimate of the total of future payments is given by Equation F.8 instead of Equation D.3.
2. The "future process variance" (given in the standard deviation columns in the examples) is not given by Equation D.2, but by:

$$\hat{\sigma}^2 = \varphi^2 \cdot \sum_{\tau} A_\tau^2 \cdot \hat{m}_\tau^\alpha$$

if the model was fitted using inflation adjusted data (as in Sections 2 and 3), or:

$$\hat{\sigma}^2 = \varphi^2 \cdot \sum_{\tau} (A_\tau \cdot \hat{m}_\tau)^\alpha \tag{F.9}$$

if the model was fitted using unadjusted data (as in Section 4).

3. In calculating the estimation error as described in Appendix D, the variance-covariance matrix $\boldsymbol{V}$ of the estimated β-parameters is extended to:

$$\begin{pmatrix} U_\theta \; 0 \ldots 0 \\ 0 \\ . \quad \boldsymbol{V} \\ . \\ 0 \end{pmatrix}$$

and the vector δ is extended to have a first component δ_0 given by:

$$\delta_0 = d\mu'/di = \sum_{\tau} H(\tau) \cdot A_\tau \cdot m_\tau. \tag{F.10}$$

The effect of this is simply to augment the mean-square-error by an amount:

$$(d\mu'/di)^2 \cdot U_\theta\,. \tag{F.11}$$

Note that $\mathrm{E}(R' \mid i)$ and $\mathrm{Var}\,(R' \mid i)$ are not nearly linear in i, (where R' is the total of future claims for an origin year in current prices). Therefore, using the best estimate i in these functions, as done in Equations F.8 and F.9, will not give such good approximations to the unconditional mean and variance as in the case of M (Equations E.4 and E.5). Also, it may be noted that the additional element of variance was regarded as part of the "future process variance" σ^2 in Equation E.5, whereas it is regarded here as an element of the "estimation error." This is unimportant, but it seems more natural to regard Var $(E(R' \mid i))$ as estimation error because in cases where m_τ has a log link-function (such as Models 1 and 2 of Sections 3 and 4), i is exactly like one of the β-parameters of the model m_τ.

The exponential model for the run-off of the remaining $(1 - \tau_0) \cdot M$ claim settlements over real development time is:

$$F(t) = 1 - (1 - \tau_0) \cdot \exp(-t/\beta)\,, \tag{F.12}$$

where

$$\beta \cdot \ln(2) = \text{expected time for half the remaining claims to be closed.} \tag{F.13}$$

Inverting this gives:

$$t = -\beta \cdot \ln[(1 - \tau)/(1 - \tau_0)]\,.$$

Comparing with Equation F.3, in order to have $\hat{\varphi} = 1$, we must have:

$$H(\tau) = -\hat{\beta} \cdot \ln[1 - \tau)/(1 - \tau_0)]\,, \tag{F.14}$$

where $\hat{\beta}$ is the best estimate given by Equation F.13. The uncertainty in the estimate $\hat{\beta}$ obtained from Equation F.13 can be used to give a value for the variance U_φ of φ. A numerical example is given in Section 6.

APPENDIX G

MODELS FOR PARTIAL PAYMENTS

Introduction

Each figure from the claim amounts triangle has two components:

$$Y_d = Y_{d\,1} + Y_{d\,2}\,,$$

where

$Y_{d\,1}$ = total of settlement payments, and

$Y_{d\,2}$ = total of partial payments on claims not yet closed.

In this appendix, the second subscript is used to distinguish between these two components. The origin year subscript w is dropped to simplify the presentation. Thus:

$N_{d\,1}$ = number of claims closed, and

$N_{d\,2}$ = number of partial payments, in development period d.

This appendix is concerned with the distribution of $Y_{d\,1}$ and $Y_{d\,2}$ conditional on $N_{d\,1}$. This is relevant for the modelling of Y_d when $N_{d\,1}$ is known.

Conditional Distribution of $Y_{d\,1}$

If:

m_τ = expected size of settlement payment at operational time τ, and

φ = coefficient of variation of settlement payments (assumed the same for all τ),

then:

$$\mathrm{E}(Y_{d\,1}) = N_{d\,1} \cdot m_\tau \text{ and } \mathrm{Var}(Y_{d\,1}) = N_{d\,1} \cdot \varphi^2 \cdot m_\tau^2\,, \qquad \text{(G.1)}$$

where τ is the "mean" operational time of development period d.

Conditional Distribution of Y_{d2}

It is assumed that:

- the expected size of a non-zero partial payment $= \Gamma \cdot m_\tau$ (for some constant Γ, the same for all τ), and
- the coefficient of variation of non-zero partial payments $= \varphi$ (the same as for settlement payments).

It is assumed that each open claim generates partial payments according to a Poisson process with parameter p (p is the expected number of partial payments per year of delay until the claim is closed). Therefore, if L_d represents the mean number of claims outstanding at the end of development period d, the number of prepayments N_{d2} is Poisson distributed with parameters $L_d \cdot p$. Initially it is assumed that p is constant for all w and d, that is, the partial payment Poisson process is homogeneous over real development time and the rate is the same for all claims. Alternatives are considered shortly.

Under the above assumptions, Y_{d2} has a compound Poisson distribution. Using standard results from risk theory,

$$E(Y_{d2}) = p \cdot \Gamma \cdot L_d \cdot m_\tau \ ,$$

and

$$\mathrm{Var}(Y_{d2}) = (1 + \varphi^2) \cdot p \cdot \Gamma^2 \cdot L_d \cdot m_\tau^2 \, . \tag{G.2}$$

Conditional Distribution of Payment Per Claim Closed

Conditional on N_{d1} and L_d, Y_{d1} and Y_{d2} are mutually independent, so the variance (as well as the mean) is additive. Adding Equations G.1 and G.2 gives:

$$E(Y_d) = N_{d1} \cdot [1 + c \cdot R_d] \cdot m_\tau \, ,$$

and

$$\mathrm{Var}(Y_d) = N_{d1} \cdot [\varphi^2 + (1 + \varphi^2) \cdot \Gamma \cdot c \cdot R_d] \cdot m_\tau^2 \, ,$$

where:

$$R_d = L_d/N_{d\,1} \text{ and } c = p \cdot \Gamma\ .$$

The payment per claim closed, $S_d = Y_d/N_{d\,1}$ therefore has:

$$E(S_d) = [1 + c \cdot R_d] \cdot m_\tau\ , \tag{G.3}$$

and

$$\text{Var}\,(S_d) = \varphi^2 \cdot [1 + (1 + 1/\varphi^2) \cdot \Gamma \cdot c \cdot R_d] \cdot m_\tau^2/N_{d\,1}\ .$$

If $c \cdot R_d < 1$ (as is usually the case) then we have:

$$E(S_d) \approx \exp\,(c \cdot R_d) \cdot m_\tau\ . \tag{G.4a}$$

If $(1 + 1/\varphi^2) \cdot \Gamma \approx 1$ (this seems plausible; e.g., $\varphi = 1, \Gamma = 0.5$), then we have:

$$\text{Var}\,(S_d) \approx \varphi^2 \cdot \exp\,(c \cdot R_d) \cdot m_\tau^2/N_{d\,1}$$

$$= \varphi^2 \cdot E(S_d)^2/[\exp\,(c \cdot R_d) \cdot N_{d\,1}]. \tag{G.4b}$$

Equations G.4 can be used to estimate the parameter c by Fisher's scoring method in the case of a log-link model for m_τ.

Equation G.4b for the variance of S is more approximate than the expression for the mean. This is acceptable because the variance is only of secondary importance. Its role is to determine the weights in fitting the model for the mean. The expression for the variance can be checked empirically via residual plots. The approximation $(1 + 1/\varphi^2) \cdot \Gamma \approx 1$ can be monitored by plotting standardized residuals against R_d. Fanning out indicates that $(1 + 1/\varphi^2) \cdot \Gamma > 1$.

If c is known, the data S can be preadjusted to remove the partial payment effect:

$$S_d' = S_d/\exp\,(c \cdot R_d)\ .$$

We then have:

$$\mathrm{E}(S_d') = m_\tau = \text{ mean size of settlement payments,}$$

and

$$\mathrm{Var}\,(S_d') = \varphi^2 \cdot m_\tau^2/[\exp\,(c \cdot R_d) \cdot N_{d\,1}]. \tag{G.5}$$

If $c = 0$ (i.e., $p = 0$ or $\Gamma = 0$; no non-zero partial payments), Equations G.3 and G.4 reduce to Equations 2.2 and 2.3.

Alternatives and Generalizations

In previous paragraphs it was assumed that until a claim is closed, it yields partial payments at a constant rate p per development year. This implies that the total number of partial payments (and hence the total claim amount) depends on the real time scale. An obvious generalization is to allow the partial payment rate to be a function of d, say p_d. Intuitively, one might expect the rate to decrease with d; for example, $p_d = k/d^{\alpha}$ or $p_d = k \cdot \exp(-\alpha \cdot d)$. The form of Equation G.4 is unchanged if p_d is not constant. For example, $p_d = k/d^{\alpha}$ gives the same equations but with $c = k \cdot \Gamma$ and $R_d = L_d/(d^{\alpha} \cdot N_{d\,1})$. However, for any function p_d, the expected total number of partial payments, and hence the expected amount of each claim, will depend on the real time scale.

Earlier portions of this appendix have all been concerned with the partial payment rate over real development time. An alternative is to consider the partial payment rate over operational time (defined in terms of the number of claims closed). The simplest model of this type (analogous to the constant p_d model presented earlier) is that the partial payment rate over operational time is constant. That is, each claim outstanding at operational time τ has a fixed probability of yielding a partial payment in the next increment of operational time, the probability not depending on τ. In terms of the partial payment rate over real development time this can be expressed as $p_d = k \cdot N_{d\,1}/M$, for some constant k (because $N_{d\,1}/M$ is the increase in operational time). This implies that the expected number of partial payments on a claim is proportional to the operational time of settlement of the claim, and does not depend on the real time scale. (This is invariance in Taylor's sense; see [9]).

Using this expression for p_d in the development leads to equations of the same form as Equation G.4 but with $c = k \cdot \Gamma$ and $R_d = L_d/M$ instead of $R_d = L_d/N_{d\,1}$.

If the ratio of the number of claims outstanding (i.e., reported but not closed) to the total number not yet closed is approximately constant, then we have $L_d/M = \pi \cdot (1 - \tau)$, where π is a constant. In such a case we have

$$E(S_d) = \exp(c \cdot \pi \cdot (1 - \tau)) \cdot m_\tau , \qquad \text{(G.6a)}$$

and

$$\operatorname{Var}(S_d) = \varphi^2 \cdot E(S_d)^2 / [\exp(c \cdot \pi \cdot (1 - \tau)) \cdot N_{d\,1}] , \qquad \text{(G.6b)}$$

where $c = k \cdot \Gamma$. Note that the factor $\exp(c \cdot \pi \cdot (1 - \tau))$ decreases monotonically as τ increases from zero to one. Thus, although the mean settlement payment m_τ will usually increase, the mean payment per claim closed may decrease for some values of τ. As operational time progresses, the number of claims which may contribute partial payments to the payment per claim closed decreases, so $E(S_d)$ may decrease.

Throughout this appendix, $E(S_d)$ denotes the expected value of S_d conditional on $N_{d\,1}$. The absence of any dependence on $N_{d\,1}$ in Equation G.6a implies that the $\operatorname{Cov}(S_d, N_{d\,1}) = 0$, so the negative association between S and N_1 described in Section 7 does not exist under the assumptions presented earlier. However, this is not usually plausible for the reasons given in Section 7. Empirical studies have confirmed the existence of a negative association between S and N_1 (for example, Taylor [8], Section 6).

In previous paragraphs, the notation p_d has been used for the rate (in real time) at which partial payments are made on each claim outstanding in development period d. Similarly, p_τ will be used to denote the rate (in operational time) at which partial payments are made on each claim outstanding in the development period corresponding to operational time τ.

Thus,

p_τ/M = expected number of partial payments per $1/M$ increase in operational time (per outstanding claim).

Therefore:

$$p_d = N_{d\,1} \cdot p_\tau/M.$$

Previous sections have considered the cases:

p_d = constant (above, in this appendix),

p_τ = constant, that is, $p_d = k \cdot N_{d\,1}/M$.

In the real world, the truth probably lies somewhere between these two extremes. The arguments of Section 7 suggest that p_d increases with the number of settlements $N_{d\,1}$ but not proportionately. In other words, p_d increases but p_τ decreases as $N_{d\,1}$ increases.

This can be modelled using $p_d = p_0 + k \cdot N_{d\,1}/M$, for some constants p_0 and k. This implies that p_d decreases from some value of d onwards, as suggested earlier. By varying the ratio of p_0 to k, any situation between the extremes of p_d constant and p_τ constant can be attained. Repeating the development and using this expression for p_d leads to:

$$\mathrm{E}(S_d) = \exp\,(c_0 \cdot L_d/N_{d\,1} + c_1 \cdot L_d/M) \cdot m_\tau\,,$$

and

$$\mathrm{Var}\,(S_d) = \varphi^2 \cdot \mathrm{E}(S_d)^2/[\exp\,(c_0 \cdot L_d/N_{d\,1} + c_1 \cdot L_d/M) \cdot N_{d\,1}]\,, \quad \text{(G.7)}$$

where $c_0 = p_0 \cdot \Gamma$ and $c_1 = k \cdot \Gamma$.

If the ratio of the number of claims outstanding to the total number not yet closed is approximately constant at π, then we have $L_d/M \approx \pi \cdot (1 - \tau)$. If $\ln(m_\tau)$ includes a constant term and a term linear in τ (for example Models 1 and 2 of Section 2), the factor $\exp\,(c_1 \cdot L_d/M)$ can therefore be subsumed into m_τ and we have:

$$E(S_d) = \exp(c_0 \cdot L_d/N_{d1}) \cdot m_\tau, \tag{G.8}$$

which is the form obtained in Equations G.4a and G.4b under the assumption, p_d, constant. The only difference is that in Equation G.4a, m_τ is the mean settlement payment, so would normally be a monotonic increasing function of τ, whereas, here, with p_d not necessarily constant, m_τ includes the factor $\exp(c_1 \cdot \pi \cdot (1 - \tau))$ of the partial payment effect, and is not necessarily monotonic increasing.

Since c_1 is not estimated (it is confounded with other parameters of m_τ), the factor $\exp(c_1 \cdot L_d/M)$ cannot easily be included in the denominator of Var (S_d). Since c_1 is probably small, it seems reasonable to omit this factor from the prior weights in fitting the model. Residual plots will indicate if this is unreasonable.