DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXIV

AN ANALYSIS OF EXCESS LOSS DEVELOPMENT

EMANUEL PINTO AND DANIEL F. GOGOL

DISCUSSION BY ROBERT A. BEAR

Abstract

Messrs. Pinto and Gogol have made a valuable contribution to actuarial literature through their analyses of industry excess loss development patterns. Based upon application of a theoretical model to industry data, the authors have convincingly demonstrated that paid and incurred loss and ALAE development patterns increase significantly as the retention increases. This is due to the phenomenon that the severity distribution becomes thicker-tailed as claims mature. This review presents a generalization of the Pinto-Gogol formula that shows how the authors' methodology can be applied to estimate account-specific development patterns for relatively high excess layers.

This reviewer would like to thank Kurt A. Reichle for encouraging him to write this discussion, and Daniel F. Gogol for his helpful suggestions.

1. INTRODUCTION

Messrs. Pinto and Gogol [1] have made a valuable contribution to actuarial literature through their analyses of industry excess incurred loss development patterns. They have convincingly demonstrated that incurred loss and allocated loss adjustment expense (ALAE) development increases significantly as the retention increases. The same is likely to be true of paid loss and ALAE development, as the following argument shows.

The authors note that estimates of paid excess loss development factors can be computed by multiplying each incurred excess loss development factor by the quotient of the paid-to-reported ratios for the later and earlier valuations. The paid-to-reported ratios are simply ratios of excess paid losses and ALAE to excess incurred losses and ALAE. These ratios are computed at all valuations for a representative retention, since the authors found that they do not vary substantially as a function of the retention. Thus, paid excess loss development factors may be expressed as the product of incurred excess loss development factors (which increase with the retention) and a quantity which does not vary substantially with the retention. This implies that paid loss and ALAE development factors can be expected to increase significantly as the retention increases. Thus, the retention should be appropriately reflected in the estimation of discounted excess losses using paid development factors.

2. COMMENTS ON THE UNDERLYING MODEL

The function $y = ax^b$ was used by the authors to fit excess development factors as a function of the retention, based on Insurance Services Office (ISO) data. Basic properties of the underlying Single Parameter Pareto (SPP) severity distribution [2] are summarized in Appendix A. As the retention, x, was normalized through division by \$10,000, the parameter a represents the factor for development excess of \$10,000. The incremental factors, $a - 1$, are then fitted to the inverse power function $y = cx^d$ as recommended by Sherman [3]. The inverse power function is used for interpolation and to yield tail factors for development beyond 99 months. The use of this same functional form to extrapolate b-parameters beyond 99 months appears to have been based on goodness-of-fit tests rather than on theoretical considerations, because the parameter b represents the decline in the SPP q-parameter between the valuations underlying the age-to-age factor.

Philbrick [2] and Reichle and Yonkunas [4] noted that the tails of fitted SPP severity distributions are thicker than the tails of empirical casualty loss distributions at very large loss sizes. This implies that empirical average claim sizes in excess of high retentions will be less than those implied by the SPP distribution. Fits to more recent ISO data led Bear and Nemlick [5] to conclude that the SPP q-parameter varies with the truncation point used in the fitting procedure. This increase in the estimated q-parameter as a function of the truncation point supports the earlier findings of Philbrick and Reichle-Yonkunas. Pinto and Gogol note that the impact of this error will be reduced by using a ratio to estimate a development factor if the error is of comparable magnitude in the numerator and denominator.

Bear and Nemlick found that if the truncation point used in the fitting procedure is less than 50% of the attachment point for a particular analysis, the errors due to the redundant estimates of excess severities from the SPP distribution become unacceptably large. They used development triangles of SPP parameter estimates to derive the shape parameter q at various stages of development and to project ultimate estimates of this parameter by class of business and truncation point. For the casualty classes of business analyzed, their fits of more recent ISO data confirmed the result noted by Philbrick and Reichle-Yonkunas; i.e., the q-parameter tends to decline as a function of the stage of development. This implies that the severity distribution becomes thicker-tailed as claims mature. (See Appendix A.) This is also confirmed by the fact that the b-parameters estimated by Pinto and Gogol were positive.

3. ESTIMATING ACCOUNT-SPECIFIC DEVELOPMENT

The authors concentrated on the estimation of industry loss development patterns for unbounded layers (with losses capped by policy limits), and suggested a reasonable approach for estimating industry incurred loss development patterns for reinsurance layers. This same approach can be applied to industry paid loss development patterns for unbounded layers to estimate paid loss development patterns for reinsurance layers.

This reviewer observes that the basic Pinto-Gogol formula for computation of industry loss development factors for unbounded layers as a function of the retention can be applied in large account primary pricing and in account-specific reinsurance pricing. This formula can also be applied to estimate account-specific development patterns for reinsurance layers and for large account primary excess layers.

A generalization of the Pinto-Gogol formula is presented below and proven in Appendix B. This generalization permits one to estimate the account-specific development pattern for a relatively high layer as a function of the development pattern for a lower layer, assuming the ratios of the gross limit to the retention for both layers are equal.

Proposition

Let d represent the incurred loss development factor from valuation i to valuation j for losses in the layer from k_1 to k_2. Assume the SPP distribution is an appropriate severity model for claims in excess of k_1. Let q_i and q_j represent the estimated values of the SPP parameter based on claims at valuations i and j, respectively, and let $e = q_i - q_j$. Then the incurred loss development factor from valuation i to valuation j for losses in the layer from x_1 to x_2 is given by

$$dc^e ,$$

$$\text{where } c = \frac{x_1}{k_1} \geq 1$$

$$\text{and } \frac{x_2}{x_1} = \frac{k_2}{k_1} = b .$$

This result also holds for unbounded layers (i.e., k_2 and x_2 are infinite) if the SPP parameters exceed one. If q_j represents the projected value of q, the SPP parameter for fully developed claims, this result may be used to estimate age-to-ultimate development factors.

Applications

The SPP parameters can be estimated from account-specific data in large account primary pricing and in reinsurance pricing. The parameters estimated from account-specific data can be credibility weighted with parameters estimated from industry data ([4],[6]).

A key assumption in the above proposition is that the ratio of the gross limit to the retention (in reinsurance pricing), or self-insured retention (in primary pricing), are equal for both layers:

$$b = \frac{x_2}{x_1} = \frac{k_2}{k_1} .$$

Thus, one would want to select k_1 to be sufficiently high so that the SPP distribution is an appropriate severity model for claims in excess of k_1. On the other hand, one would want to select k_1 to be sufficiently low so that credible development patterns can be estimated for a layer in excess of k_1 . One would select k_2 so that

$$k_2 = bk_1 ,$$

$$\text{where } b = x_2/x_1 .$$

The proposition could then be applied to estimate the development pattern for a relatively high layer (where the account-specific data are not sufficiently credible) from the development pattern of a relatively low layer (where the account-specific data are more credible).

For example, suppose that the SPP parameter for a particular account and line of business after 24 months has been estimated to be 1.25, and the projected value of this parameter for fully mature claims is 1.10. These parameters have been estimated based on the account's claims in excess of $100,000. The incurred loss development factor from 24 months to ultimate, for the layer from $100,000 to $300,000, has been estimated to be 3.5 based upon the account's historical development pattern. The development factor from 24 months to ultimate for the layer from $200,000 to $600,000 is given by

$$3.5\,(2.0)^{.15} = 3.88.$$

Note that $d = 3.5$, $c = 200{,}000/100{,}000 = 2.0$, $e = 1.25 - 1.10 = .15$, and $b = 600{,}000/200{,}000 = 300{,}000/100{,}000 = 3$. In fact, the gross limit for the lower layer was selected to be three times the retention of $100,000 because this is the ratio of the gross limit to the retention for the layer for which we wished to estimate the development pattern.

Development patterns for layers with retentions in excess of \$200,000 (more than twice the \$100,000 truncation point used in estimating the SPP parameters) could be estimated with reasonable confidence using this procedure, if one had reason to believe the SPP parameters remained relatively stable as higher truncation points were used in the fitting procedure. (Recall that errors arising from this source may be reduced by using a ratio to estimate a development factor.)

Finally, it should be noted that paid loss development factors for bounded layers may be estimated by applying the Pinto-Gogol approach of multiplying each incurred loss development factor by the quotient of the paid-to-reported ratios for the later and earlier valuations. This reviewer suggests that the paid-to-reported ratios be estimated for the particular layer of interest (or at least for a similar layer), but possibly from a broader data source than was used to estimate the incurred loss development factors.

4. SUMMARY

Based upon application of a theoretical model to industry data, the authors have convincingly demonstrated that paid and incurred loss and ALAE development patterns increase significantly as the retention increases. This is due to the phenomenon (confirmed by recent ISO casualty data) that the severity distribution becomes thicker-tailed as claims mature. The proposition presented above shows how the Pinto-Gogol methodology can be applied to estimate account-specific development patterns for relatively high excess layers.

REFERENCES

[1] Pinto, Emanuel, and Gogol, Daniel F., "An Analysis of Excess Loss Development," *PCAS* LXXIV, 1987, p.227.

[2] Philbrick, Stephen W., "A Practical Guide to the Single Parameter Pareto Distribution," *PCAS* LXXII, 1985, p.44.

[3] Sherman, Richard E., "Extrapolating, Smoothing, and Interpolating Development Factors," *PCAS* LXXI, 1984, p.122.

[4] Reichle, Kurt A., and Yonkunas, John P., Discussion of Philbrick: "A Practical Guide to the Single Parameter Pareto Distribution," *PCAS* LXXII, 1985, p.85.

[5] Bear, Robert A., and Nemlick, Kenneth J., "Pricing the Impact of Adjustable Features and Loss Sharing Provisions of Reinsurance Treaties," *PCAS* LXXVII, 1990, p. 60.

[6] Rytgaard, Mette "Estimation in the Pareto Distribution," *ASTIN Colloquium* XXI, 1989, p.389.

APPENDIX A

SINGLE PARAMETER PARETO SEVERITY DISTRIBUTION

1. MODEL DEFINITION*

Assume ground-up loss occurrences, W, above the truncation point, k, are distributed according to the following cumulative distribution function:

$$F(w) = 1 - \left(\frac{k}{w} \right)^q , \text{ where } k > 0,\ q > 0,\ w \geq k .$$

Note that

$$F(w) = 1 - \left(\frac{k}{k + (w - k)} \right)^q .$$

Let $Y = W - k$ represent the occurrence size excess of k.

Then

$$F(y) = 1 - \left(\frac{k}{k + y} \right)^q , \text{ where } y \geq 0 .$$

Thus, occurrence losses excess of the truncation point k are distributed according to the two-parameter shifted Pareto distribution, with scale parameter equal to k and shape parameter equal to q [5].

If we "normalize" the losses W (which are all greater than or equal to k) by dividing each loss by the truncation point k, we have the well-known Single Parameter Pareto (SPP) severity distribution [2]:

$$F(z) = 1 - \frac{1}{z^q} = 1 - z^{-q}$$

$$\text{where } Z = W/k \geq 1 \text{ and } q > 0 .$$

---

* Using standard statistical notation, capital letters in this appendix are used to represent random variables. Lower case represents actual values.

2. CLAIM FREQUENCIES

As $F(z)$ represents the proportion of normalized occurrence losses which are less than or equal to z, $G(z) = 1 - F(z) = z^{-q}$ represents the proportion of normalized losses which exceed z. Let n_i represent the expected number of claims in excess of k at valuation i, and let m_i represent the expected number of claims in excess of x at valuation i. Let c represent the normalized value of x, $c = x/k$. Because c^{-q} represents the proportion of normalized losses which exceed c, c^{-q} is also the proportion of claims in excess of k which are also larger than x. Then $m_i = n_i c^{-q}$ is the expected number of claims in excess of x at valuation i, given that n_i claims are expected to be in excess of k at valuation i.

For example, the proportion of claims excess of \$500,000 that also exceed \$1,000,000 is $c^{-q} = 2^{-2} = 0.25$ if

$$q = 2\ (c = 1{,}000{,}000/500{,}000 = 2).$$

Thus, if $n_3 = 400$ claims are expected to exceed \$500,000 at the third valuation, then $m_3 = n_3 c^{-q} = 400(0.25) = 100$ claims are expected to exceed \$1,000,000 at the third valuation. Note that if $q = 1.5$, then $c^{-q} = 0.354$. The proportion c^{-q} becomes 0.5 if $q = 1$, and 0.707 if $q = 0.5$. Thus, the proportion of claims in excess of \$500,000 that also exceed \$1,000,000 increases as the q-parameter declines. Thus, the tail of the SPP distribution is thicker for lower values of the q-parameter.

3. MEAN SEVERITIES

The formula for the average ground-up unlimited claim which exceeds x_1 is given in [2]:

$$x_1\left(\frac{q}{q-1}\right), \quad \text{if } q > 1\ .$$

The average unlimited claim in excess of x_1 is given by

$$s = \mathrm{E}\,(Y) = x_1\left(\frac{q}{q-1} - 1\right) = \frac{x_1}{q-1}\ , \quad \text{if } q > 1\ ,$$

where s represents the expected value of $Y = W - x_1$ and $W \geq x_1$.

The formula for the average ground-up claim which exceeds x_1 but is limited to x_2 is given in [2]:

$$x_1 \left(\frac{q - b^{1-q}}{q - 1} \right), \text{ if } q \neq 1,$$

$$\text{and } x_1 (1 + \ln(b)), \text{ if } q = 1,$$

where $b = x_2/x_1$ and $\ln(b)$ represents the natural logarithm of b. The formula for the average claim in the layer from x_1 to x_2 (total losses in the layer divided by the number of claims in excess of x_1) is given by

$$s = \mathrm{E}(\min(Y, x_2 - x_1)) = x_1 \left(\frac{q - b^{1-q}}{q - 1} - 1 \right) = x_1 \left(\frac{b^{1-q} - 1}{1 - q} \right),$$

$$\text{if } q \neq 1, \text{ and}$$

$$s = x_1 ((1 + \ln(b)) - 1) = x_1 \ln(b), \text{ if } q = 1, \text{ where } b = x_2/x_1 .$$

Note that s represents the expected value of $Y = W - x_1$, where Y is capped by the layer limit $x_2 - x_1$ and $Y \geq 0$.

For example, the average claim in the layer from \$500,000 to \$1,000,000 is calculated as follows, assuming $q = 2$:

$$s = (500{,}000) \left(\frac{2^{1-2} - 1}{1 - 2} \right) = \$250{,}000,$$

$$\text{since } b = 1{,}000{,}000/500{,}000 = 2.$$

If $q = 1.5$, then s is similarly calculated to be \$292,893.

If $q = 1$, then $s = (500{,}000)(\ln(2)) = \$346{,}574$.

If $q = 0.5$, then

$$s = (500{,}000) \left(\frac{2^{1-0.5} - 1}{1 - .5} \right) = \$414{,}214.$$

This example illustrates the property of the SPP distribution that lower values of the q-parameter are associated with higher mean severities. This is because the distribution becomes thicker-tailed (more probability in excess of any large value) as the q-parameter declines.

For casualty classes of business, the q-parameter tends to decline as a function of the stage of development ([2], [4], [5]). This implies that casualty severity distributions tend to become thicker-tailed as claims mature, and so the average claim in any layer (where the SPP distribution is an appropriate model) will increase as claims mature.

APPENDIX B

PROOF OF PROPOSITION

The proof of the proposition is based upon estimating the incurred losses in a layer as the product of the expected number of claims above the retention and the average claim in the layer. The incurred loss development factors for both the relatively low and high layers are computed as ratios of layer incurred losses at the appropriate valuations. Simple algebra leads to the formula in the proposition when one assumes that the ratios of the gross limit to the retention for both layers are equal.

Recall that the SPP distribution is assumed to be an appropriate severity model for claims in excess of k_1. If n_i represents the expected number of claims in excess of k_1 at valuation i, then $m_i = n_i c^{-q_i}$ represents the expected number of claims in excess of x_1 at valuation i, where $c = x_1/k_1$. The average claim in the layer from x_1 to x_2 (total losses in the layer divided by the number of claims in excess of x_1) at valuation i is given by

$$s_i = x_1\left(\frac{b^{1-q_i}-1}{1-q_i}\right), \text{ if } q_i \neq 1,$$

where $b = x_2/x_1$.

The formulas for m_i and s_i follow from the properties of the SPP distribution and are proven in Appendix A.

Incurred losses in the layer from x_1 to x_2 at valuation i are given by $m_i\, s_i$. Similarly, incurred losses in the layer from the retention x_1 to the gross limit x_2 at valuation j are given by $m_j s_j$, where $m_j = n_j c^{-q_j}$ and

$$s_j = x_1\left(\frac{b^{1-q_j}-1}{1-q_j}\right), \text{ if } q_j \neq 1.$$

(Note that n_j represents the expected number of claims in excess of k_1 at valuation j, and m_j represents the expected number of claims in excess of x_1 at valuation j.)

The incurred loss development factor from valuation i to valuation j for losses in the layer from x_1 to x_2 is given by

$$f = \frac{m_j s_j}{m_i s_i} = \left(\frac{n_j (1 - q_i)(b^{1-q_j} - 1)}{n_i (1 - q_j)(b^{1-q_i} - 1)} \right) c^{q_i - q_j}.$$

Recall that $x_2/x_1 = k_2/k_1 = b$ and $c = x_1/k_1$. If $c = 1$ then $x_1 = k_1$ and $x_2 = k_2 = bk_1$. The formula for f then yields the incurred loss development factor from valuation i to valuation j for losses in the layer from k_1 to k_2, which is denoted d:

$$f = d = \frac{n_j (1 - q_i)(b^{1-q_j} - 1)}{n_i (1 - q_j)(b^{1-q_i} - 1)}.$$

Hence, the formula for the incurred loss development factor from valuation i to valuation j for losses in the layer from x_1 to x_2 simplifies to

$$f = dc^{q_i - q_j} = dc^e,$$

$$\text{if } q_i \neq 1 \text{ and } q_j \neq 1.$$

If j represents the valuation at which claims are fully developed, then $q_j = q$ and f represents the development factor from valuation i to ultimate.

In the case of unbounded layers (i.e., k_2 and x_2 are infinite), m_i and m_j do not change but s_i and s_j are as given below (see Appendix A):

$$s_i = \frac{x_1}{q_i - 1}, \text{ if } q_i > 1,$$

$$\text{and } s_j = \frac{x_1}{q_j - 1}, \text{ if } q_j > 1.$$

Then f is given by

$$f = \frac{m_j s_j}{m_i s_i} = \left(\frac{n_j (q_i - 1)}{n_i (q_j - 1)} \right) c^{q_i - q_j}.$$

If $c = 1$, then $x_1 = k_1$ and so

$$f = d = \frac{n_j (q_i - 1)}{n_i (q_j - 1)} .$$

This implies that the incurred loss development factor from valuation i to valuation j for losses in the unbounded layer above x_1 is given by

$$f = dc^{q_i - q_j} = dc^e,$$

where d is the incurred loss development factor from valuation i to valuation j for losses in the unbounded layer above k_1, $c = x_1/k_1$, and the SPP parameters are assumed to exceed one.

For bounded layers with $q_i = q_j = 1$, the averages of the claims in the layer from x_1 to x_2 at valuations i and j (s_i and s_j) are as follows (see Appendix A):

$$s_i = s_j = x_1 \ln(b), \text{ where } b = x_2/x_1.$$

The expected numbers of claims in excess of x_1 at valuations i and j are given respectively by

$$m_i = n_i c^{-1} \text{ and } m_j = n_j c^{-1}.$$

This implies that

$$f = \frac{m_j s_j}{m_i s_i} = \frac{n_j}{n_i} .$$

However, the averages of the claims in the layer from k_1 to k_2 at valuations i and j (t_i and t_j) are given by

$$t_i = t_j = k_1 \ln(b), \text{ where } b = k_2/k_1.$$

This implies that the development factor d for the layer from k_1 to k_2 is given by

$$d = \frac{n_j\, t_j}{n_i\, t_i} = \frac{n_j}{n_i}.$$

Hence, $f = d$, which is in agreement with the formula in the proposition due to the unchanging q-parameter.

If $q_i \neq 1$ and $q_j = 1$, then the averages of the claims in the layer from the retention x_1 to the gross limit x_2 at valuations i and j are proven in Appendix A to be

$$s_i = x_1 \left(\frac{b^{1-q_i} - 1}{1 - q_i} \right),$$

$$\text{and } s_j = x_1 \ln(b).$$

The incurred loss development factor from valuation i to valuation j for losses in the layer from x_1 to x_2 is given by

$$f = \frac{m_j\, s_j}{m_i\, s_i} = \frac{n_j}{n_i} \left(\frac{\ln(b)}{(b^{1-q_i} - 1)/(1 - q_i)} \right) c^{q_i - q_i}.$$

If $c = 1$, then $x_1 = k_1$ and so $x_2 = k_2 = bk_1$. The formula for f then reduces to the formula for d,

$$d = \frac{n_j}{n_i} \left(\frac{\ln(b)}{(b^{1-q_i} - 1)/(1 - q_i)} \right).$$

Substituting into the formula for f,

$$f = dc^{q_i - q_i} = dc^e.$$

An analogous proof would hold if $q_i = 1$ and $q_j \neq 1$. Q.E.D.