DISCUSSION BY CHRIS SVENDSGAARD

Abstract

When the chi-square test is used in the manner suggested by Mr. Dropkin, the hypothesis tested is that all insureds have the same Poisson frequency distribution (with identical means). Rejection of the hypothesis does not necessarily imply a non-Poisson frequency distribution, if individual insureds have different mean frequencies.

The "Binomial Variance" test suggested by Mr. Dropkin is invalid.

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1. INTRODUCTION

Mr. Dropkin's 1959 paper is important in two major respects. It shows conclusively that classifying risks solely on their driving records is not correct. (Class and territory cannot be ignored.) And it introduced the negative binomial distribution to the casualty actuarial community.

Since the paper is still on the syllabus of examinations, it is clearly relevant today. However, it has been discovered that one of the hypothesis-testing techniques used by Mr. Dropkin is invalid. Additionally, even when using the correct technique (as Mr. Dropkin does for his main results), pitfalls in interpreting the result may trap the unwary.

A common practical application of Mr. Dropkin's paper has been to apply either the chi-square or "Binomial Variance" test to a frequency distribution, reject the Poisson assumption, and conclude that frequency is negative binomial. In this review, I will show that the "Binomial Variance" test is invalid. In addition, I will show that the negative binomial is not necessarily a correct conclusion, even if the particular Poisson hypothesis tested is rejected using a valid test. Finally, I will make a quibble.

2. THE "BINOMIAL VARIANCE" TEST IS INVALID

Mr. Dropkin states, "The Binomial Variance is equal to the product of the mean and the complement of the mean[1]." His test is to compare the "Binomial Variance" to the "variance." If the latter were greater than the former, Mr. Dropkin would reject the Poisson hypothesis.

The semantics are a little fuzzy here—are we talking about parameters or estimates of parameters? Is the "mean" μ or $\Sigma X_i/n$? Parameters are unknowable. Hence Mr. Dropkin's "mean" must be the sample mean. Similarly, the "variance" must be one of the two popular sample estimates of the variance.

In the Appendix, I demonstrate that the "Binomial Variance" is always less than or equal to either sample estimate of the variance. The inequality is strict if any insured has more than one claim. This means that (if any insured has as many as 2 claims) the Poisson hypothesis would *always* be rejected using Mr. Dropkin's "Binomial Variance" test.

An alternative might be to compare the sample mean to the sample variance. Since the mean and the variance of the Poisson distribution are equal, the sample mean and the sample variance ought to be close to one another. This comparison would be a valid (although perhaps not the most powerful) test if care were taken to calculate the significance levels associated with various differences between the sample mean and sample variance. (Simply rejecting the Poisson hypothesis if the sample variance were greater than the sample mean would probably lead to a 50% rejection rate—even if the data were generated by a Poisson process, no matter how extensive the data were.) However, the chi-square test is already available—why go to the trouble of calculating significance levels for another test?

3. NEGATIVE BINOMIAL NOT NECESSARILY CORRECT

What hypothesis is being tested? Even assuming the chi-square test is performed, the answer is not "Poissoness." The null hypothesis is really: "Each driver's number of accidents is Poisson AND each driver has the same mean frequency (Poisson parameter)." When we reject the null hypothesis using the chi-square test, we know that either the distribution is not Poisson, OR the drivers do not have the same mean frequency (Poisson parameter), OR both.

The data must all come from a single territory, a single class, and a single time period. Otherwise, drivers will have different mean frequencies, and the chi-square test will be useless, whether or not it rejects the null hypothesis.

For example, suppose that our portfolio consists of 200 drivers. Suppose 100 Class A drivers have mean claim frequency 1 per year and 100 Class B drivers have mean claim frequency 100 per year. Finally, suppose each driver's claim frequency distribution is Poisson, and each driver's claim frequency is independent of every other driver's claim frequency.

In this case, it is extremely unlikely that the chi-square test as applied by Mr. Dropkin would accept the Poisson hypothesis.[1] Yet each driver's distribution is Poisson, and the total number of claims, being the sum of independent Poisson random variables, is Poisson. The negative binomial distribution is clearly inappropriate, and the prior distribution is clearly not gamma (or any other continuous distribution).

4. A QUIBBLE

Mr. Dropkin makes the assumption (repeated by other authors) that the claim propensity must be constant over time in order to have a Poisson distribution. This assumption is false. Consider a risk that has constant Poisson frequency with mean 1 for the first half of the year, and constant Poisson frequency with mean 2 in the second half of the year. The sum of independent Poisson random variables is Poisson, so the frequency for the whole year is Poisson with mean 3. (Note that the claim propensity is not even differentiable with respect to time, let alone constant.) Nor is it necessary that claim propensity be piecewise constant. Bühlmann [2], for instance, shows this conclusion.

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[1] On the average, the estimated mean would be 50.5. If every insured had this mean, the expected number of insureds having between 40 and 60 claims would be over 170. In fact, in this situation the expected number of insureds in the range is (much) less than one.

REFERENCES

[1] L. Dropkin, "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records," *PCAS* XLVI, 1959, p. 168.

[2] H. Bühlmann, *Mathematical Methods in Risk Theory*, Springer-Verlag, New York, 1970, pp. 49–51.

APPENDIX

Proof that "Binomial Variance" is *always* less than or equal to variance (equality) only if no insured has more than 1 claim).

Situation:

m = Number of insureds

n_i = Number of claims for ith insured

$$\text{Variance estimate} = \frac{\Sigma n_i^2}{m} - \left(\frac{\Sigma n_i}{m}\right)^2$$

$$\text{"Binomial variance"} = \left(\frac{\Sigma n_i}{m}\right)\left(1 - \frac{\Sigma n_i}{m}\right) = \frac{\Sigma n_i}{m} - \left(\frac{\Sigma n_i}{m}\right)^2$$

$$\text{Variance Estimate} - \text{"Binomial Variance"} = \frac{\Sigma n_i^2}{m} - \frac{\Sigma n_i}{m}$$

Since the number claims n_i is always an integer, $n_i^2 \geq n_i$ always.

The variance estimate used in the above has m as a denominator. This variance estimate is less than the variance estimate which has $m - 1$ in the denominator. Hence the above proof shows that the Variance Estimate minus the "Binomial Variance" is always non-negative. (Note also that when the mean number of claims is greater than 1, the "Binomial Variance" is negative.)