DISCUSSION BY ROBERT A. BAILEY
REPRINTED FROM VOLUME XLVII

As Mr. R. E. Beard, secretary and editor of ASTIN, said,[1]

> "The literature in the English language relating to analytical expressions of the risks involved in general insurance is scanty and largely limited to papers presented to International Congresses of Actuaries and the *Proceedings of the Casualty Actuarial Society.* There are, however, a number of contributions to the subject in various other languages, scattered over various journals, mainly, insurance publications of European countries, e.g. *Skandinavisk Aktuarietidskrift* and a few books."

The C.A.S. can rightfully be proud of its contributions in this field which have been ably enhanced by Mr. Dropkin's treatment of the negative binomial distribution.

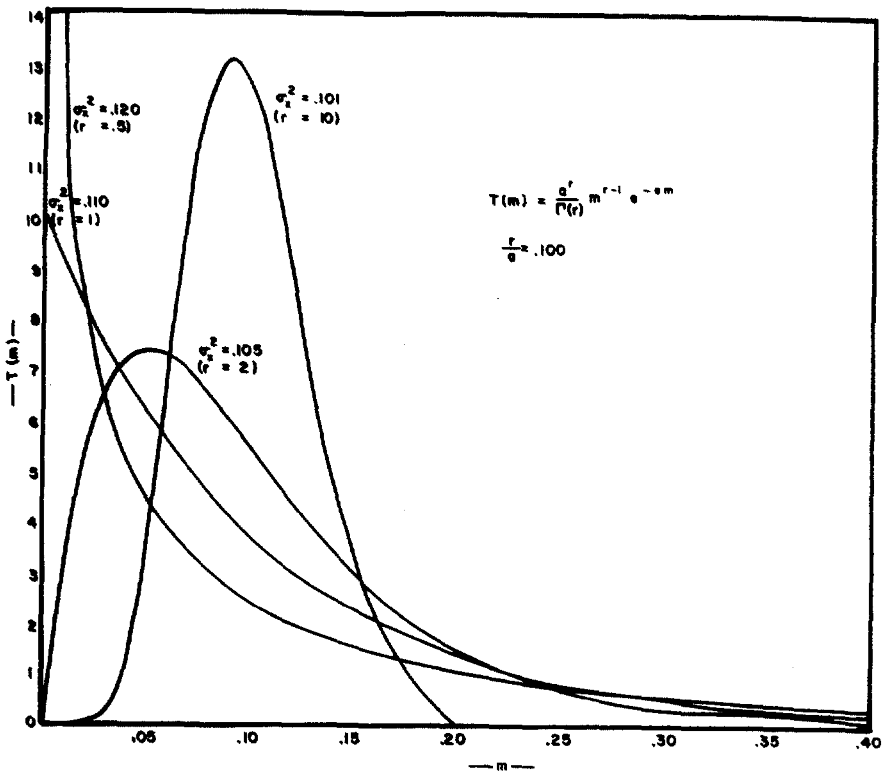
The analytical expression of risk distributions provides a valuable insight into many practical problems. One of the important results of Mr. Dropkin's paper is a realization of the large amount of variation among individual risks. Automobile risks even within a single class or merit rating group are far from being all alike. In order to help visualize this variation, there are shown in Figure 1 the graphs of the distribution of risks which Mr. Dropkin shows to be inherent in the negative binomial distribution. Four graphs are shown, all for an average accident frequency $\frac{r}{a} = .100$, and with variances of the accident frequency (not the variances of m, the inherent hazard) of $.120(r = \frac{1}{2})$, $.110(r = 1)$, $.105(r = 2)$ and $.101(r = 10)$.

One of the many practical applications to which Mr. Dropkin's development can be applied is the calculation of the discount for n accident-free years. This application was suggested to the writer by Mr. Dropkin's paper because it provided a means of deriving mathematically what had been derived empirically in the paper presented at the same time as Mr. Dropkin's, "An Actuarial Note on the Credibility of Experience of a Single Private Passenger Car," since the discount from the overall average rate for n accident-free years is equal to the "credibility" as defined in the paper just cited.

The chance that any individual risk with inherent hazard (m) will be accident-free for 1 year is e^{-m} where e^{-m} is the value of the Poisson distribution

[1] *Transactions of the XVth International Congress of Actuaries,* Volume II, 1957, p. 230.

FIGURE 1



$P(x) = \dfrac{m^x e^{-m}}{x!}$ when $x = 0$. Mr. Dropkin shows that the total distribution of individual risks can be described by the distribution

$$T(m) = \frac{a^r}{\Gamma(r)} m^{r-1} e^{-am} .$$

Therefore, the distribution of risks with 1 or more accident-free years is

$$T_1(m) = \frac{T(m)e^{-m}}{\int_0^\infty T(m)e^{-m}dm} = \left(\frac{a+1}{a}\right)^r T(m)e^{-m} .$$

Likewise the distribution of risks with 2 or more accident-free years is

$$T_2(m) = \left(\frac{a+2}{a}\right)^r T(m)e^{-2m} .$$

This provides us a means of immediately calculating the expected claim frequency of claim-free risks. Mr. Dropkin shows that the claim frequency for all risks $= E(x)$

$$= \sum_{x=0}^{\infty} x \int_0^{\infty} \frac{m^x e^{-m}}{x!} \frac{a^r m^{r-1} e^{-ma} dm}{\Gamma(r)}$$

$$= \frac{r}{a} .$$

Therefore, the claim frequency for risks with 1 or more accident-free years

$$= \sum_{x=0}^{\infty} x \int_0^{\infty} \frac{m^x e^{-m}}{x!} \frac{(a+1)^r m^{r-1} e^{-m(a+1)} dm}{\Gamma(r)}$$

$$= \frac{r}{a+1} .$$

Similarly, the expected claim frequency for risks with 2 or more accident-free years is $\frac{r}{a+2}$, and for 3 or more accident-free years is $\frac{r}{a+3}$, and so on.

Therefore, the expected claim frequency for risks accident-free for n or more years relative to the expected claim frequency for all risks, assuming that the inherent hazard (m) for each individual risk remains unchanged from one year to the next, is $\frac{a}{a+n}$ and the corresponding discount from the average rate is $\frac{n}{a+n}$. This is the same as saying that these risks are $\frac{n}{a+n}$ better than average.

The expression $\frac{n}{a+n}$ is equal to the "credibility" of risks accident-free for n or more years, as defined in the paper cited above, and it is the same result obtained independently by Dr. F. Bichsel, in a paper entitled "Une méthode pour calculer une ristorne adéquate pour années sans sinistres" (A method of calculating an adequate no-claim bonus for years without accidents) presented at the ASTIN Colloquy in La Baule, France, in June, 1959. Furthermore, if this expression for the credibility of the experience of an individual risk for n years,

$$Z = \frac{n}{a + n},$$

is multiplied in the numerator and denominator by the premium for one car year, it becomes

$$Z = \frac{P}{P + K},$$

where P is the premium during the experience period and where K is a constant which equals the parameter a multiplied by the premium for one car year. This is the credibility formula derived by Mr. A. W. Whitney in "The Theory of Experience Rating," *PCAS*, Vol. IV, and used ever since in almost all experience rating plans.

Another application which Mr. Dropkin's development suggested is a comparison of the variation of hazard among licensed drivers and among licensed automobiles. In Appendix B, Mr. Dropkin fits the negative binomial to the total distribution of California drivers and obtains $r = .8927$. From the graphs shown in Figure 1 and also from an analysis of the formula for $T(m)$ it can be seen that when $0 < r \leq 1$, $T(m)$ is a "J" shaped curve with a maximum height at $m = 0$. ($T(m)$, it should be remembered, is the distribution of the inherent hazard of the individual drivers and is to be distinguished from $N(x)$, the distribution of the resulting accidents.) It is reasonable that the California data should be described by a "J" shaped curve since some drivers licensed in California do not drive in California for a number of reasons, such as they do not have a car or they live outside the state. Since such licensed drivers will have an inherent hazard $m = 0$, a "J" shaped curve is a reasonable distribution of hazard for licensed drivers. On the other hand, however, the distribution of hazard for licensed automobiles should not be a "J" shaped curve, since practically no automobiles have a hazard $m = 0$ and therefore for the distribution of hazard for licensed automobiles, r should be greater than 1.

This proposition can be tested by using the Canadian merit rating experience for insured automobiles. By setting the one-year credibility for Class 1 cars of

$.055^2$ equal to the expression derived above for the one-year credibility, $\frac{1}{a+1}$, we obtain $a = 17.2$. Since the average frequency for Class 1 $= .087 = \frac{r}{a}$, we obtain $r = 1.50$ which is greater than 1 as we would expect. From this we can draw the conclusion that there is more variation of hazard among drivers than among cars.

There are undoubtedly many other applications which can be made of Mr. Dropkin's work and we are fortunate to have a development of the negative binomial distribution in the *Proceedings*, especially at this time when merit rating is of such great concern. We are entering a time of great competitive effort in the search for more accurate classification systems, not only in private passenger automobile insurance but in other lines as well, as Mr. Pruitt pointed out so forcefully last November in his presidential address, "St. Vitus's Dance." The negative binomial distribution, which has also been called the "accident proneness" distribution, provides a valuable tool for that search.

[2] R. A. Bailey and L. J. Simon, "An Actuarial Note on the Credibility of Experience of a Single Private Passenger Car," *PCAS* XLVI, Table 4, p. 163.