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AN ANALYSIS OF EXCESS LOSS DEVELOPMENT

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Abstract

There is very little information available regarding excess loss development, despite its importance in excess of loss pricing and reserving. In this study, paid and reported excess loss development patterns are estimated at various retentions for certain casualty lines of business. The effects of allocated loss adjustment expense and policy limits on excess development are discussed. The pattern of change, as development progresses, of Pareto distributions fitted to casualty loss distributions was considered in developing curve fitting methods. A method is described for determining development factors by layer. Applications to excess loss pricing, loss reserving, and increased limits factors are mentioned.

Special thanks to ISO, which provided us with a great deal of data, and to Susan Greiff, Thomas Highet, Madelyn Esposito and Francine Leong who assisted in the data processing and compilation.

1. INTRODUCTION

Loss development patterns for both reported and paid excess losses are of fundamental importance in excess of loss pricing as well as in estimating loss reserves for excess of loss insurance and reinsurance. Excess of loss reinsurance constitutes a major portion of the business written by reinsurers and is the area involving the greatest degree of independent pricing and reserving activity.

There is a paucity of published information regarding both reported and paid excess loss development. The Reinsurance Association of America (RAA) publishes a study biennially of reported excess casualty loss development patterns for certain lines of business, based on data supplied by member companies. Incurred[1] loss development patterns for automobile liability, general liability, workers' compensation and medical malpractice have been described in these studies. Certain of these lines of business have well over twenty years of significant reported excess loss development, indicating that excess reporting patterns vary significantly from first dollar reporting patterns. In that study, however, excess losses in various layers are all grouped together, so the data does not indicate the development patterns by line for various individual layers. Since the data indicates that excess business generally exhibits much slower reporting than that normally associated with primary business, there appears to be a relationship between the layer for which business is written and the resulting development pattern. It is this relationship that we intend to analyze in this paper for both paid and reported losses. Applications to increased limits and excess of loss pricing are also noted.

The protracted development of excess losses reflected in the RAA study suggests that the development is not only caused by late reported claims and increases in the average reported loss per claim but also by changes at successive maturities in the proportion of claims with losses which are large multiples of the average. Thus, the shape of the size of loss distribution changes at successive valuations. Accordingly, we requested and received from the Insurance Services Office various data comprising size of loss distributions at successive maturities. Specifically, included in the data were size of loss distributions of incurred losses, for policy year evaluations up to 99 months, or the latest evaluation, for policy years 1972 through 1982. This countrywide monoline data was provided separately for OL&T; M&C and Products with each size of loss distribution containing 118 intervals.

These size of loss distributions combine data from business written at different policy limits. Thus, the data includes losses censored at each of the policy limits. While no adjustments were made to this data, the implications of using combined limits data are discussed in Appendix B.

[1] "Incurred" is used in this study to mean the same as reported, i.e., it excludes IBNR.

Finally, the treatment of allocated loss adjustment expense in these distributions should be mentioned. Losses were assigned to a given size of loss interval based on reported loss size (paid plus outstanding) excluding allocated loss adjustment expenses. The total allocated loss adjustment expense associated with the losses in each interval was given separately. As loss adjustment expense is treated in different ways in excess reinsurance, the treatment of these expenses will be discussed further in the context of deriving excess development factors.

Size of loss distributions listing paid losses and outstanding losses separately, as well as paid and outstanding allocated loss adjustment expense separately, were also provided by ISO for OL&T and M&C. The latest valuation available with this policy year data was 63 months. The RAA study provides reported loss development data for over twenty years of development for general liability and other lines on an accident year basis.

2. INCURRED EXCESS LOSS DEVELOPMENT FACTORS

In this section, we will display and discuss the incurred excess loss development factors derived from the size of loss distributions.

In developing these factors, we adjusted the retentions for policy years prior to 1982 to recognize changing levels of average cost per occurrence. For policy year 1982, the retentions used were $10,000, $25,000, $50,000, $100,000, $250,000, $500,000 and $1,000,000. For prior policy years, these retentions were multiplied by relativities reflecting the average cost per occurrence for the given policy year relative to the average cost per occurrence for the 1982 year. (Although ISO has used higher trend for higher layers in determining increased limits factors, we did not find support for this procedure in the data provided. Higher trend for higher layers would produce a trend towards smaller maximum likelihood estimates of the Pareto parameter, but this is not the case, as shown in Reichle and Yonkunas [2].) Thus, the relativity for 1982 was 1.00, while for each prior policy year N it was computed by multiplying the relativity for the policy year $N + 1$ by the ratio of the average cost per occurrence for year N to the average cost per occurrence for year $N + 1$. The ratio was based on the latest available pair of reports at the same stage of development, excluding claims closed without payment. As the resulting deflated retentions did not correspond with endpoints of the 118 size of loss intervals, the closest possible endpoints were selected.

Allocated loss adjustment expense (ALAE) is handled in different ways in excess reinsurance contracts. The three most common treatments are as follows:

1) ALAE is added to the pure loss amount and the total is treated as one in determining coverage.
2) ALAE is assigned to an excess layer on a pro rata basis. That is, the ratio that the excess portion of the pure loss bears to the total loss is applied to the total ALAE to determine the excess ALAE.
3) ALAE is not included in the coverage.

Separate sets of excess loss development factors were calculated to reflect each of the above treatments of ALAE. This was done, respectively, as follows:

1) All ALAE on occurrences with pure loss greater than a given retention was included with the pure losses excess of that retention.
2) The total ALAE on occurrences for which the pure loss exceeded a given retention was multiplied by the ratio of the pure excess losses to the ground up losses on these same occurrences to determine the excess ALAE. This excess ALAE was then included with the pure excess losses.
3) No ALAE was added to the pure excess losses.

A discussion of the degree of accuracy of these methods of assigning ALAE can be found in Appendix A.

The factors shown in Exhibits 1 through 3 are dollar weighted averages of the factors by policy year. The retentions shown are retentions on policy year 1982 level, although they actually correspond to different retentions for different policy years. By estimating the factor for the increase in average cost per occurrence from policy year 1982 to accident year 1987, for example, one could bring the retentions to accident year 1987 level.

A review of the factors will show that the development is not materially affected after 39 months by the treatment of allocated loss adjustment expense. Therefore, future discussion will only deal with the case in which ALAE is included in the limit. This is probably the most common treatment in reinsurance, and it corresponds to the factors for excess losses plus ALAE. It is also clear from these factors that the development increases as the retention increases. Some exceptions to this trend occur at retentions of $500,000 and $1,000,000 for individual stages of development. This may be due to the fact that there is a lesser amount of data at these retentions which increases the variability of the factors. Despite the exceptions, these higher retentions tend to have the largest development factors.

EXHIBIT 1

OL&T BI Development Factors

Excess Losses Plus ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.2113	1.1178	1.0682	1.0437	1.0504	1.0094
10,000	1.3356	1.1799	1.1056	1.0664	1.0710	1.0118
25,000	1.3849	1.2200	1.1402	1.0877	1.0909	1.0146
50,000	1.4055	1.2549	1.1764	1.1128	1.1134	1.0167
100,000	1.4021	1.2942	1.2168	1.1506	1.1424	1.0235
250,000	1.3512	1.3517	1.2963	1.2120	1.2015	1.0383
500,000	1.2742	1.3940	1.4080	1.2787	1.2626	1.0613
$1,000,000	1.0688	1.3061	1.6135	1.3662	1.3534	1.1111

Excess Losses Plus Pro Rata ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.2113	1.1178	1.0682	1.0437	1.0504	1.0094
10,000	1.3437	1.1870	1.1111	1.0695	1.0729	1.0127
25,000	1.3909	1.2291	1.1483	1.0926	1.0938	1.0160
50,000	1.4098	1.2655	1.1860	1.1189	1.1172	1.0191
100,000	1.4023	1.3070	1.2287	1.1573	1.1468	1.0264
250,000	1.3563	1.3611	1.3150	1.2180	1.2077	1.0446
500,000	1.2648	1.3957	1.4292	1.2838	1.2701	1.0684
$1,000,000	1.0503	1.3501	1.6417	1.3731	1.3576	1.1182

Excess Losses Only

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.2064	1.1185	1.0702	1.0458	1.0504	1.0115
10,000	1.3451	1.1940	1.1181	1.0735	1.0737	1.0155
25,000	1.3955	1.2389	1.1578	1.0981	1.0943	1.0193
50,000	1.4148	1.2777	1.1963	1.1249	1.1176	1.0239
100,000	1.4107	1.3191	1.2404	1.1626	1.1474	1.0319
250,000	1.3689	1.3690	1.3277	1.2199	1.2067	1.0517
500,000	1.2753	1.3981	1.4340	1.2832	1.2663	1.0740
$1,000,000	1.0316	1.3888	1.6258	1.3629	1.3504	1.1197

EXHIBIT 2

M&C BI DEVELOPMENT FACTORS

Excess Losses Plus ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.4959	1.2077	1.0865	1.0297	1.0285	1.0210
10,000	1.6246	1.2630	1.1100	1.0401	1.0360	1.0267
25,000	1.6816	1.2974	1.1316	1.0513	1.0449	1.0319
50,000	1.7201	1.3280	1.1509	1.0642	1.0554	1.0382
100,000	1.7528	1.3583	1.1771	1.0788	1.0724	1.0491
250,000	1.7481	1.3775	1.2214	1.1008	1.1194	1.0782
500,000	1.6110	1.3845	1.2520	1.1340	1.1898	1.1192
$1,000,000	1.4056	1.5619	1.2130	1.1942	1.4206	1.2383

Excess Losses Plus Pro Rata ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.4959	1.2077	1.0865	1.0297	1.0285	1.0210
10,000	1.6326	1.2682	1.1128	1.0414	1.0375	1.0274
25,000	1.6909	1.3044	1.1354	1.0531	1.0475	1.0332
50,000	1.7297	1.3353	1.1556	1.0660	1.0594	1.0401
100,000	1.7689	1.3654	1.1828	1.0811	1.0789	1.0525
250,000	1.7652	1.3862	1.2306	1.1049	1.1267	1.0826
500,000	1.6093	1.4190	1.2534	1.1372	1.1993	1.1264
$1,000,000	1.4064	1.5551	1.1934	1.1901	1.4891	1.2350

Excess Losses Only

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.4865	1.2039	1.0838	1.0273	1.0300	1.0216
10,000	1.6294	1.2690	1.1136	1.0410	1.0410	1.0285
25,000	1.6933	1.3090	1.1367	1.0533	1.0519	1.0349
50,000	1.7368	1.3418	1.1587	1.0659	1.0649	1.0423
100,000	1.7835	1.3723	1.1871	1.0814	1.0858	1.0551
250,000	1.7878	1.3927	1.2346	1.1070	1.1300	1.0839
500,000	1.6334	1.4367	1.2555	1.1372	1.2014	1.1250
$1,000,000	1.4010	1.5516	1.1970	1.1846	1.5060	1.2276

EXHIBIT 3

PRODUCTS BI DEVELOPMENT FACTORS

Excess Losses Plus ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.6284	1.1974	1.1032	1.0545	1.0707	1.0332
10,000	1.7891	1.2906	1.1276	1.0632	1.0800	1.0293
25,000	1.9089	1.3561	1.1501	1.0776	1.0932	1.0369
50,000	1.9563	1.3844	1.1736	1.0928	1.1058	1.0405
100,000	2.0207	1.4221	1.1993	1.1165	1.1165	1.0421
250,000	2.1053	1.4790	1.2301	1.1453	1.0944	1.0440
500,000	2.3936	1.5098	1.4073	1.1660	1.1180	0.9605
$1,000,000	1.8026	1.5847	1.9141	1.2074	1.2271	0.7657

Excess Losses Plus Pro Rata ALAE

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.6284	1.1974	1.1032	1.0545	1.0707	1.0332
10,000	1.7995	1.3065	1.1302	1.0653	1.0812	1.0311
25,000	1.8940	1.3571	1.1538	1.0805	1.0939	1.0398
50,000	1.9255	1.3847	1.1777	1.0961	1.1053	1.0443
100,000	1.9550	1.4214	1.2041	1.1203	1.1135	1.0465
250,000	1.9284	1.4790	1.2514	1.1494	1.0924	1.0302
500,000	2.1034	1.5104	1.4556	1.1520	1.1271	0.9303
$1,000,000	1.7797	1.5970	1.9188	1.2199	1.2676	0.7245

Excess Losses Only

Retention	27–39	39–51	51–63	63–75	75–87	87–99
$ -0-	1.5635	1.1844	1.0958	1.0511	1.0636	1.0347
10,000	1.7291	1.2966	1.1266	1.0663	1.0758	1.0403
25,000	1.8118	1.3416	1.1505	1.0810	1.0885	1.0483
50,000	1.8340	1.3699	1.1752	1.0969	1.0993	1.0536
100,000	1.8344	1.4096	1.2034	1.1199	1.1081	1.0546
250,000	1.7100	1.4690	1.2601	1.1528	1.0942	1.0252
500,000	1.5748	1.5052	1.4556	1.1485	1.1267	0.9242
$1,000,000	1.4736	1.5162	1.9311	1.2105	1.2719	0.7226

The excess development factors shown were all derived directly from the underlying size of loss distributions. We now use these factors to estimate curves which, in addition to smoothing the underlying factors, will generate excess development factors beyond 99 months as well as for retentions other than those previously treated. This would be necessary for computing development factors at policy year 1982 retentions that are equivalent to various retentions at accident year 1987 level, for example.

For each development interval, a curve is estimated to fit the excess loss development factors as a function of retention. These curves are then fitted to a smoothly progressing series of curves. The procedure is done separately for each line of business.

The curve selected to fit the excess development factors as a function of retention was $y = ax^b$ where x is the retention divided by $10,000. Thus, a is the value given by the curve for development excess of $10,000.

The use of this function was motivated by the qualities of the single parameter Pareto distribution used to model size of loss distributions. This is discussed further in Section 4.

Separate curves of the form $y = a_n x^{b_n}$ were fit to the excess loss development factors by retention for each interval of development of the form 27 to 27 + $12n$ months, for $n = 1, 2, 3, 4, 5$ or 6. These intervals were used rather than individual successive intervals of development in order to stabilize the curve fitting process. Only retentions up to $250,000 were used, since the data for larger retentions had much less credibility.

The a_n and b_n values were determined from the corresponding data points x,y by fitting the values of log y and log x to a least squares line which gives:

$$\log y = \log a_n + b_n \log x.$$

Thus, values for a_n and b_n were determined for each of the development intervals. These values were then separately fit to curves as a function of the stage of development. The method is illustrated in Exhibit 4 for the a_n values for M&C BI.

Thus, it is actually the values of $a_n' - 1$ that are fitted to the curve $y = cx^d$ to obtain the fitted values. Sherman [3] recommends this type of approach for fitting loss development factors. An exactly analogous procedure is used to obtain fitted b_n'' values. The formulas chosen to determine the fitted values a_n'' and b_n'' through 99 months are used to produce the tail beyond 99 months. In

EXHIBIT 4

M&C BI ILLUSTRATION

	27–39	27–51	27–63	27–75	27–87	27–99
a_n values-actual	1.6401	2.0770	2.2928	2.3764	2.4395	2.4879
	27–39	39–51	51–63	63–75	75–87	87–99
a_n' values (ratios of a_n values)	1.6401	1.2664	1.1039	1.0365	1.0266	1.0198
$a_n' - 1$ values	.6401	.2664	.1039	.0365	.0266	.0198
$a_n'' - 1$ values (fitted to $a_n' - 1$ values)		.2566	.0948	.0468	.0270	.0173
a_n'' values-fitted	1.6401	1.2566	1.0948	1.0468	1.0270	1.0173

a few cases, an adjustment was made to an a_n or b_n value to produce what seemed to be a more reasonable curve. The resulting fitted excess development factors by retention through 363 months of development are shown by line on Exhibits 5 through 7. The "a values" and "b values" displayed are the a_n'' and b_n'' values. The corresponding actual factors derived from the data are shown at the bottom of each exhibit.

A simple method for converting the policy year development factors shown to approximately equivalent accident year development factors is based on the fact that, for a policy year as of 27 months, the time elapsed since the average accident date is 15 months; and, for an accident year as of 21 months, the average time elapsed is 15 months. A policy year development factor from 27 + $12n$ to 27 + $12(n + 1)$ months, for $n \geq 0$, is assumed to be equivalent to an accident year development factor from 21 + $12n$ to 21 + $12(n + 1)$ months. Accident year development factors from 24 + $12n$ to 24 + $12(n + 1)$ months could then be estimated by linear interpolation or by fitting an exponential curve to the excess over one of the two adjacent factors.

Although application of calculus would yield more refined results, the accuracy of this approach improves rapidly after the estimated 24–36 month accident year factor. This is because the expected rate of development as a function of maturity becomes more nearly linear as maturity increases.

EXHIBIT 5
OL&T BI Excess Loss & ALAE Development Factors
Fitted Factors

Development Interval	Fitted b Values	Retention 10,000*	25,000	50,000	100,000	250,000	500,000	1,000,000
27– 39	.01000	1.36556	1.37813	1.38771	1.39736	1.41023	1.42004	1.42991
39– 51	.03986	1.15206	1.19492	1.22839	1.26281	1.30978	1.34647	1.38420
51– 63	.05066	1.08024	1.13157	1.17202	1.21390	1.27158	1.31703	1.36410
63– 75	.03873	1.05099	1.08895	1.11858	1.14901	1.19051	1.22290	1.25617
75– 87	.02528	1.03587	1.06014	1.07889	1.09796	1.12370	1.14356	1.16378
87– 99	.01616	1.02691	1.04222	1.05396	1.06583	1.01873	1.09391	1.10623
99–111	.01055	1.02110	1.03102	1.03859	1.04622	1.05638	1.06414	1.07195
111–123	.00712	1.01710	1.02375	1.02882	1.03391	1.04068	1.04583	1.05100
123–135	.00497	1.01420	1.01882	1.02234	1.02586	1.03054	1.03409	1.03766
135–147	.00357	1.01203	1.01534	1.01785	1.02037	1.02372	1.02625	1.02879
147–159	.00263	1.01035	1.01279	1.01464	1.01649	1.01895	1.02081	1.02267
159–171	.00199	1.00902	1.01086	1.01226	1.01365	1.01550	1.01690	1.01830
171–183	.00153	1.00795	1.00937	1.01044	1.01152	1.01294	1.01401	1.01509
183–195	.00120	1.00708	1.00819	1.00903	1.00987	1.01098	1.01182	1.01267
195–207	.00096	1.00635	1.00723	1.00790	1.00857	1.00946	1.01013	1.01080
207–219	.00077	1.00573	1.00645	1.00699	1.00753	1.00824	1.00878	1.00933
219–231	.00063	1.00521	1.00579	1.00624	1.00668	1.00726	1.00770	1.00815
231–243	.00052	1.00476	1.00524	1.00561	1.00597	1.00646	1.00682	1.00719
243–255	.00044	1.00437	1.00477	1.00508	1.00538	1.00579	1.00609	1.00640
255–267	.00037	1.00403	1.00437	1.00463	1.00489	1.00523	1.00548	1.00574
267–279	.00031	1.00373	1.00402	1.00424	1.00446	1.00475	1.00497	1.00518
279–291	.00027	1.00347	1.00372	1.00390	1.00409	1.00434	1.00452	1.00471
291–303	.00023	1.00323	1.00345	1.00361	1.00377	1.00398	1.00414	1.00430
303–315	.00020	1.00302	1.00321	1.00335	1.00349	1.00367	1.00381	1.00395
315–327	.00018	1.00284	1.00300	1.00312	1.00324	1.00340	1.00352	1.00365
327–339	.00015	1.00267	1.00281	1.00291	1.00302	1.00316	1.00327	1.00338
339–351	.00014	1.00251	1.00264	1.00273	1.00282	1.00295	1.00304	1.00314
351–363	.00012	1.00237	1.00248	1.00257	1.00265	1.00276	1.00284	1.00293
		Actual Factors						
27– 39		1.33560	1.38490	1.40550	1.40210	1.35120	1.27420	1.06880
39– 51		1.17990	1.22000	1.25490	1.29420	1.35170	1.39400	1.30610
51– 63		1.10560	1.14020	1.17640	1.21680	1.29630	1.40800	1.61350
63– 75		1.06640	1.08770	1.11280	1.15060	1.21200	1.27870	1.36620
75– 87		1.07100	1.09090	1.11340	1.14240	1.20150	1.26260	1.35340
87– 99		1.01180	1.01460	1.01670	1.02350	1.03830	1.06130	1.11110
		Cumulative Comparison						
27– 99 Actual		2.01300	2.31900	2.61400	2.97100	3.58000	4.28500	4.62700
27– 99 Fitted		1.90000	2.24200	2.54100	2.88000	3.39900	3.85200	4.36600

* These equal the fitted *a* values.

EXHIBIT 6

M&C BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS

Fitted Factors

DEVELOPMENT INTERVAL	FITTED b VALUES	RETENTION 10,000*	25,000	50,000	100,000	250,000	500,000	1,000,000
27– 39	.02402	1.64008	1.67658	1.70472	1.73334	1.77190	1.80165	1.83189
39– 51	.02784	1.25665	1.28913	1.31425	1.33986	1.37449	1.40127	1.42848
51– 63	.02666	1.09481	1.12188	1.14280	1.16412	1.19290	1.21515	1.23781
63– 75	.02266	1.04677	1.06874	1.08566	1.10285	1.12599	1.14382	1.16193
75– 87	.01867	1.02704	1.04476	1.05836	1.07214	1.09064	1.10484	1.11923
87– 99	.01534	1.01728	1.03168	1.04270	1.05385	1.06876	1.08018	1.09173
99–111	.01270	1.01183	1.02367	1.03272	1.04185	1.05405	1.06337	1.07277
111–123	.01063	1.00852	1.01839	1.02592	1.03351	1.04362	1.05133	1.05911
123–135	.00899	1.00638	1.01471	1.02106	1.02744	1.03594	1.04242	1.04894
135–147	.00769	1.00493	1.01204	1.01745	1.02289	1.03012	1.03563	1.04117
147–159	.00665	1.00390	1.01003	1.01470	1.01938	1.02561	1.03034	1.03510
159–171	.00579	1.00315	1.00849	1.01255	1.01662	1.02203	1.02614	1.03027
171–183	.00509	1.00259	1.00728	1.01084	1.01441	1.01915	1.02276	1.02637
183–195	.00451	1.00216	1.00630	1.00945	1.01261	1.01680	1.01998	1.02317
195–207	.00402	1.00182	1.00551	1.00832	1.01113	1.01486	1.01769	1.02052
207–219	.00360	1.00155	1.00486	1.00737	1.00989	1.01323	1.01576	1.01830
219–231	.00325	1.00134	1.00432	1.00658	1.00885	1.01185	1.01413	1.01642
231–243	.00294	1.00116	1.00386	1.00591	1.00796	1.01068	1.01274	1.01481
243–255	.00267	1.00102	1.00347	1.00534	1.00720	1.00967	1.01155	1.01342
255–267	.00244	1.00090	1.00314	1.00484	1.00655	1.00880	1.01051	1.01223
267–279	.00224	1.00080	1.00285	1.00441	1.00597	1.00804	1.00961	1.01118
279–291	.00206	1.00071	1.00260	1.00404	1.00547	1.00738	1.00882	1.01026
291–303	.00190	1.00064	1.00238	1.00371	1.00503	1.00679	1.00812	1.00945
303–315	.00176	1.00057	1.00219	1.00342	1.00465	1.00627	1.00750	1.00873
315–327	.00164	1.00052	1.00202	1.00316	1.00430	1.00581	1.00695	1.00809
327–339	.00153	1.00047	1.00187	1.00293	1.00399	1.00539	1.00646	1.00752
339–351	.00142	1.00043	1.00174	1.00272	1.00371	1.00502	1.00602	1.00701
351–363	.00133	1.00039	1.00161	1.00254	1.00347	1.00469	1.00562	1.00655
		ACTUAL FACTORS						
27– 39		1.62460	1.68160	1.72010	1.75280	1.74610	1.61100	1.40560
39– 51		1.26300	1.29740	1.32800	1.35830	1.37750	1.38450	1.56190
51– 63		1.11000	1.13160	1.15090	1.17710	1.22140	1.25200	1.21300
63– 75		1.04010	1.05130	1.06420	1.07880	1.10080	1.13400	1.19420
75– 87		1.03600	1.04490	1.05540	1.07240	1.11940	1.18980	1.42060
87– 99		1.02670	1.03190	1.03820	1.04910	1.07820	1.11920	1.23830
		CUMULATIVE COMPARISON						
27– 99 Actual		2.52000	2.79900	3.06600	3.40100	3.90800	4.21700	5.59400
27– 99 Fitted		2.46800	2.79300	3.06800	3.36900	3.81300	4.18800	4.59900

* These equal the fitted a values.

EXHIBIT 7

PRODUCTS BI EXCESS LOSS & ALAE DEVELOPMENT FACTORS
Fitted Factors

DEVELOPMENT INTERVAL	FITTED *b* VALUES	RETENTION 10,000*	25,000	50,000	100,000	250,000	500,000	1,000,000
27– 39	.04877	1.80564	1.88815	1.95307	2.02022	2.11254	2.18517	2.26030
39– 51	.04373	1.27527	1.32740	1.36825	1.41036	1.46802	1.51320	1.55977
51– 63	.02738	1.13277	1.16155	1.18381	1.20649	1.23715	1.26086	1.28502
63– 75	.01617	1.07914	1.09525	1.10759	1.12007	1.36791	1.14960	1.16256
75– 87	.00997	1.05298	1.06265	1.07002	1.07744	1.08733	1.09487	1.10246
87– 99	.00650	1.03817	1.04438	1.04909	1.05383	1.06013	1.06492	1.06973
99–111	.00446	1.02893	1.03314	1.03634	1.03954	1.04380	1.04703	1.05027
111–123	.00318	1.02275	1.02574	1.02801	1.03028	1.03329	1.03557	1.03786
123–135	.00235	1.01841	1.02061	1.02228	1.02395	1.02616	1.02784	1.02951
135–147	.00179	1.01523	1.01690	1.01816	1.01943	1.02110	1.02237	1.02364
147–159	.00140	1.01283	1.01413	1.01511	1.01609	1.01739	1.01838	1.01937
159–171	.00111	1.01097	1.01200	1.01278	1.01356	1.01459	1.01537	1.01616
171–183	.00090	1.00950	1.01033	1.01096	1.01159	1.01242	1.01306	1.01369
183–195	.00074	1.00832	1.00900	1.00951	1.01003	1.01071	1.01123	1.01175
195–207	.00061	1.00735	1.00791	1.00834	1.00877	1.00934	1.00977	1.01020
207–219	.00052	1.00654	1.00702	1.00738	1.00774	1.00821	1.00858	1.00894
219–231	.00044	1.00587	1.00627	1.00658	1.00688	1.00729	1.00759	1.00790
231–243	.00038	1.00529	1.00564	1.00590	1.00616	1.00651	1.00677	1.00704
243–255	.00033	1.00480	1.00510	1.00533	1.00556	1.00585	1.00608	1.00631
255–267	.00028	1.00438	1.00464	1.00484	1.00503	1.00530	1.00549	1.00569
267–279	.00025	1.00401	1.00424	1.00441	1.00459	1.00481	1.00499	1.00516
279–291	.00022	1.00369	1.00389	1.00404	1.00420	1.00440	1.00455	1.00470
291–303	.00019	1.00341	1.00358	1.00372	1.00385	1.00403	1.00417	1.00430
303–315	.00017	1.00316	1.00331	1.00343	1.00355	1.00371	1.00383	1.00395
315–327	.00015	1.00293	1.00307	1.00318	1.00329	1.00343	1.00354	1.00365
327–339	.00014	1.00273	1.00286	1.00296	1.00305	1.00318	1.00328	1.00338
339–351	.00013	1.00255	1.00267	1.00276	1.00284	1.00296	1.00305	1.00313
351–363	.00011	1.00239	1.00250	1.00258	1.00265	1.00276	1.00284	1.00292
		ACTUAL FACTORS						
27– 39		1.78910	1.90890	1.95630	2.02070	2.10530	2.39360	1.80260
39– 51		1.29060	1.35610	1.38440	1.42210	1.47900	1.50980	1.58470
51– 63		1.12670	1.15010	1.17360	1.19930	1.23010	1.40730	1.91410
63– 75		1.06320	1.07760	1.09280	1.11650	1.14530	1.16600	1.20740
75– 87		1.08000	1.09320	1.10580	1.11650	1.09440	1.11800	1.22710
87– 99		1.02930	1.03690	1.04050	1.04210	1.04400	.96050	.76570
		CUMULATIVE COMPARISON						
27– 99 Actual		3.07700	3.63700	3.99600	4.47700	5.01200	6.36800	6.20300
27– 99 Fitted		3.07700	3.53900	3.93300	4.37200	5.02800	5.58800	6.21100

* These equal the fitted *a* values.

As has been mentioned, the RAA Loss Development Study combines business written at various retentions. The subline mix underlying the "General Liability Excluding Asbestos" experience is also difficult to estimate. For these reasons, as well as the fact that the RAA experience is accident year, it is difficult to make a precise comparison of our results with those of the RAA. Nevertheless, Exhibit 8 shows a rough comparison based on the following assumptions:

1) A retention of $250,000 is used to reflect the development characteristics of the various retentions and limits underlying the RAA experience.
2) An equal weighting of the excess loss development factors for OL&T, M&C and Products is used to approximate the subline mix of the RAA data.
3) A weighting of 25% of the accident year factor from $12 + 12k$ months to $12 + 12(k + 1)$ months and 75% of the accident year factor from $12 + 12(k + 1)$ months to $12 + 12(k + 2)$ months is used to estimate the policy year factor from $27 + 12k$ months to $27 + 12(k + 1)$ months.
4) Dollar weighted factors are derived using the most recent five years of RAA experience.

EXHIBIT 8

DEVELOPMENT FACTOR COMPARISON

Development Interval	Fitted ISO Data Excess $250,000	RAA (as of 12/84)
27–39	1.765	1.801
39–51	1.384	1.392
51–63	1.234	1.242
63–75	1.151	1.153
75–87	1.101	1.097
87–99	1.070	1.072
99–111	1.051	1.067
111–123	1.039	1.049
123–135	1.031	1.038
135–147	1.025	1.038
147–159	1.021	1.030
159–171	1.017	1.029
171–183	1.015	1.036
183–Ult.	1.105	1.228

The RAA data begins to show higher developments than the curves fitted to ISO data after 99 months. This could be partially due to the effects of reinsurance coverage on an aggregate basis showing up later in the development. Also, the RAA study points out that unidentified longer tailed medical malpractice losses are present in the RAA data, particularly in the older years. This could have a great effect on development at later valuations. It is also possible that the distribution of RAA retentions and limits results in larger development at later stages relative to earlier stages than the development associated with the fixed ISO retention. Higher layer losses have more relative weight at later stages since they develop more slowly. The RAA data, unlike the ISO data, includes excess and surplus lines and umbrella business written with large policy limits. Finally, as mentioned, the curves chosen to fit the ISO data through 99 months are used to produce the tail beyond 99 months. The RAA development, despite its limitations, is based on actual data at all maturities.

It is possible, if so desired, to calculate development factors by retention beyond 99 months that are more consistent with the RAA factors. One simple method is as follows. Suppose the OL&T, M&C, and Products factors for a retention of $250,000 are $1 + a$, $1 + b$ and $1 + c$, respectively, and the RAA factor is $1 + d$. Solve for x such that $(a + b + c)x \div 3 = d$ and let $1 + ax$, $1 + bx$ and $1 + cx$ be the OL&T, M&C, and Products factors for a $250,000 retention. (This is based on the approximation that the 3 sublines comprise equal portions of the RAA data.) Then use the fitted factors by subline for a retention of $10,000 to solve for the b value using $y = ax^b$. Factors at other retentions can then be calculated.

In calculating adjusted development factors at other retentions, this method assumes the fitted factors at the $10,000 retention are accurate. The lower development of the $10,000 retention, as well as the substantial amount of data available for determining factors at the $10,000 retention, support this as a reasonable method. This method operates identically for producing factors to ultimate as for age-to-age factors.

Commercial Auto Liability

The commercial auto liability study was based on a total of almost $4 billion in losses from accident years 1980, 1981 and 1982. These were the only years available to us and our study is of the only available development factors: 21 to 33, 33 to 45, and 45 to 57 months.

The development factors for losses plus ALAE excess of various retentions (on an accident year 1982 level) are:

Retention	21–33	33–45	45–57	21–57	33–57
-0-	1.084	1.031	1.011	1.130	1.042
10,000	1.137	1.044	1.012	1.201	1.057
25,000	1.152	1.050	1.014	1.227	1.065
50,000	1.159	1.053	1.016	1.240	1.070
100,000	1.172	1.058	1.013	1.256	1.072
250,000	1.177	1.030	1.043	1.264	1.074
500,000	1.444	.949	1.168	1.601	1.108

A pattern of increasing development with increasing retentions can be observed, especially in the 21–57 month factors. The factors for the $500,000 retention have limited credibility. Due to the small change in development factors from one retention to another, no curve fitting was performed.

The breakdown of premium by policy limits for accident year 1982 can be approximated as 5% at $100,000, 15% at $300,000, 60% at $500,000, and, 20% at $750,000 or $1,000,000.

Accident year development factors for excess losses based on a weighted average of RAA development data for the last five years as of 12/31/84 for auto liability are:

12–24	24–36	36–48	48–60	60–72	72–84	84–ultimate
1.804	1.204	1.093	1.062	1.052	1.026	1.076

3. EXCESS PAID LOSS & ALAE DEVELOPMENT

In this section, ratios of excess paid losses and ALAE to excess incurred losses and ALAE were determined at policy year valuations from 27 months to ultimate for OL&T BI and M&C BI. (Sufficient data was not available for Products BI.) These ratios of paid to reported, in conjunction with excess incurred loss and ALAE development, will produce excess paid loss and ALAE development factors.

The procedure previously discussed which was used in developing excess incurred losses and ALAE by retention at various valuations was used for both paid and reported losses and ALAE from 27 months to 63 months of development. The resulting ratios of paid to reported are shown in Exhibit 9 for policy year 1982 cost levels.

EXHIBIT 9

RATIO OF PAID TO REPORTED EXCESS LOSS AND ALAE

OL&T BI

Retention	27 mo.	39 mo.	51 mo.	63 mo.
$ 10,000	.1937	.3587	.5041	.6356
25,000	.1616	.3217	.4634	.5964
50,000	.1518	.3080	.4469	.5754
100,000	.1585	.3210	.4519	.5838
250,000	.1852	.3616	.4919	.5640
$500,000	.2269	.3103	.5106	.4205

M&C BI

Retention	27 mo.	39 mo.	51 mo.	63 mo.
$ 10,000	.1417	.2427	.4098	.5350
25,000	.1425	.2358	.4069	.5294
50,000	.1526	.2364	.4054	.5233
100,000	.1751	.2473	.4142	.5279
250,000	.2312	.2924	.4464	.5094
$500,000	.2209	.3586	.4285	.4794

It appears that the paid-to-reported ratios shown for excess loss and ALAE do not vary meaningfully as a function of the retention. Accordingly, we selected the paid-to-reported ratios for loss and ALAE excess of $25,000 as characteristic of the various retentions shown in producing a development pattern of paid-to-reported ratios. It should be noted that small losses exhibit significantly higher paid-to-reported ratios than those shown for the retentions above.

The following ISO excess of $25,000 loss development data was available beyond 63 months for loss and ALAE combined.

OL&T BI Ratios

(1) Months	(2) Paid to Reported	(3) Outstanding to Reported	(4) Ratio of (3) to Prior Value of (3)
63	.5710	.4290	—
75	.6809	.3191	.7438
87	.7768	.2232	.6995
99	.8717	.1283	.5748

M&C BI Ratios

(1) Months	(2) Paid to Reported	(3) Outstanding to Reported	(4) Ratio of (3) to Prior Value of (3)
63	.5660	.4340	—
75	.7091	.2909	.6703
87	.8019	.1981	.6810
99	.8680	.1320	.6663

In light of the column (4) ratios, and the fact that the outstanding to reported ratio will ultimately reach zero, a factor of .67 was selected judgmentally to be repeatedly applied to the outstanding to reported ratios at 63 months. The resulting patterns of paid to reported excess loss and ALAE are shown on Exhibit 10.

EXHIBIT 10

ISO Excess of $25,000 Loss Development Data
Ratios of Paid to Reported Excess Loss and ALAE

OL&T BI		M&C BI	
Valuation	Ratio	Valuation	Ratio
27	.1616	27	.1425
39	.3217	39	.2358
51	.4634	51	.4069
63	.5964	63	.5294
75	.7296	75	.6847
87	.8188	87	.7887
99	.8786	99	.8585
111	.9187	111	.9052
123	.9455	123	.9365
135	.9635	135	.9574
147	.9755	147	.9715
159	.9836	159	.9809
171	.9890	171	.9872
183	.9926	183	.9914
Ult.	1.0000	Ult.	1.0000

Excess paid to reported ratios have been used thus far since they vary less by retention and valuation than paid development factors. Also, they allow for the use of the more extensive reported data in estimating paid development. Excess paid loss and ALAE development factors can be determined simply by multiplying each reported loss development factor linking two valuations by the quotient of the paid to reported ratios for the later and earlier valuations. For example, the estimated paid loss development factors for loss and ALAE excess of $100,000 are as follows (see Exhibits 5 and 6).

OL&T BI		M&C BI	
27– 39	2.7817	27– 39	2.8682
39– 51	1.8190	39– 51	2.3121
51– 63	1.5623	51– 63	1.5146
63– 75	1.4056	63– 75	1.4264
75– 87	1.2322	75– 87	1.2351
87– 99	1.1437	87– 99	1.1470
99–111	1.0940	99–111	1.0985
111–123	1.0641	111–123	1.0692
123–135	1.0454	123–135	1.0504
135–147	1.0331	135–147	1.0379
147–159	1.0249	147–159	1.0293
159–171	1.0192	159–171	1.0232
171–183	1.0152	171–183	1.0188
183–Ult.	1.0872	183–Ult.	1.1152

4. RELATION OF RESULTS TO THE SINGLE PARAMETER PARETO DISTRIBUTION

It has been seen that excess loss development increases as the retention increases. A perspective on this relationship, and excess loss development in general, can be obtained by considering a model that illustrates the two influences underlying loss development:

1) The reporting pattern of claims over time.
2) The changing characteristics of the size of loss distribution at successive reports.

Without the latter influence, the development factors for losses excess of different retentions would be identical.

It has been noted, by both Philbrick [1] and Reichle and Yonkunas [2], that the single parameter Pareto distribution fits the tail of casualty loss distributions fairly well (at least if the interval of loss sizes is not too long), and that the parameter tends to decrease at successive stages of development. This motivated our use of the curve ax^b to fit loss development factors as a function of the retention x, as explained below.

If a series of Pareto distributions with parameters that are decreasing and greater than one were to perfectly represent a series of actual tails of loss distributions at successive development stages, the excess loss development factor from any stage m to stage $m + n$ $(n > 0)$ for retention x (where x is big enough to be included in the tail) would increase as x increased, since it equals ax^b for some fixed $a > 0$ and $b > 0$. The proof follows.

If k is the lower bound of the tail which is represented by a Pareto distribution with parameter q, and x represents the size of loss divided by k, then the density function $qx^{-(q+1)}$, as x ranges from one to infinity, represents the "normalized" (i.e., divided by k) loss distribution. The probability of a loss greater than k being between ak and bk equals $\int_a^b qx^{-(q+1)}dx$, and the losses excess of a retention ck are $nk\int_c^\infty (x - c)qx^{-(q+1)}dx$, where n is the number of losses greater than k. If the distribution of losses greater than k at i^{th} report is represented by a Pareto with parameter q_i, and at j^{th} report $(j > i)$ by a Pareto with parameter q_j, and the numbers of losses greater than k at i^{th} and j^{th} report are n_i and n_j, then the development factor for losses excess of ck from i^{th} to j^{th} report equals

$$\frac{n_j}{n_i}\left(\frac{q_i - 1}{q_j - 1}\right) c^{q_i - q_j}.$$

Therefore, if d is the development factor from i^{th} to j^{th} report for losses excess of k, then $dy^{q_i - q_j}$ is the development factor for losses excess of yk (for $y > 1$).

The development factor for losses excess of x, where $x > k$, is thus

$d\left(\frac{x}{k}\right)^{q_i - q_j}$, which equals $\frac{d}{k^{q_i - q_j}}x^{q_i - q_j}$, and $\frac{d}{k^{q_i - q_j}} > 0$ and $q_i - q_j > 0$.

This completes the proof.

The term $\frac{n_j}{n_i}$ in the expression $\frac{n_j}{n_i}\left(\frac{q_i - 1}{q_j - 1}\right) c^{q_i - q_j}$ represents the development due to additional reportings greater than k. The term $(q_i - 1)/(q_j - 1)$ represents the development arising from the change in the average excess loss above ck for occurrences greater than ck. The term $c^{q_i - q_j}$ reflects the development arising from the increased proportion of occurrences greater than k which are also greater than ck, resulting from the changing shape of the distribution. It can be seen that $c^{q_i - q_j}$ is the only term affected by a change in the retention.

As an example, let:

k = the lower bound of the tail = 25,000;
x = the primary retention = 100,000;
q_1 = the Pareto parameter for 1st report tail losses = 1.75;
q_{10} = the Pareto parameter for 10th report tail losses = 1.25; and,
d = the 1st to 10th development factor for losses excess of \$25,000 = 2.5.

Then the 1st to 10th development factor for losses excess of 100,000 is given by the formula

$$d\left(\frac{x}{k}\right)^{q_i - q_j}, \text{ i.e., } 2.5\,(4)^{.5} = 5.0.$$

Philbrick [1] and Reiche and Yonkunas [2] also noted that when a Pareto is fitted to a distribution of casualty losses greater than some amount k, the tail of the Pareto is thicker than the tail of the empirical loss distribution at very large loss sizes. Nevertheless, the effect of this error is lessened in using a ratio to estimate a development factor if the error is similar in the numerator and denominator.

5. DEVELOPMENT FACTORS BY LAYER, EXCESS LOSS RATIOS, AND INCREASED LIMITS FACTORS

The following method is used to produce development factors by layer, where the layer of losses from a to b is defined as the total of the portions between a and b of every loss. By applying the excess age-to-ultimate loss development factors to the latest available excess losses for each retention for each policy year, we get projected ultimate excess losses for each retention for each policy year. We also have "ground up" development factors, based on the same data, with which we project ultimate ground up losses for each policy year. The ground up age-to-ultimate factors are derived by fitting a curve $1 + ax^b$ to the factors through 99 months.

By taking weighted averages of the ratios of ultimate excess losses to ultimate ground up losses for all policy years for the retentions (in 000's) 10, 25, 50, 100, 250, 500, and 1000, we get ratios that we call $f(10)$, $f(25)$, $f(50)$, $f(100)$, $f(250)$, $f(500)$ and $f(1000)$. An exponential curve could then be fit between any two successive data points to get intermediate values of $f(x)$. This curve gives estimates of the ratios of ultimate excess losses to ultimate ground up losses for each retention. In order to produce the n^{th}-to-ultimate development factor for the layer from c to d, we first divide the curve values $f(c)$ and $f(d)$ by the n^{th} to ultimate development factors for losses excess of c and d, respectively, to get estimates $e_{c,n}$ and $e_{d,n}$ of the ratios of n^{th} report excess losses, for retentions c and d, to ultimate ground up losses.

We then let the development from n^{th}-to-ultimate for the layer from c to d equal $(f(c) - f(d)) \div (e_{c,n} - e_{d,n})$, i.e., the estimated ultimate excess losses in the layer divided by the n^{th} report excess losses in the layer. The n^{th} to

$(n + 1)^{th}$ development factor for a layer is produced by dividing the n^{th} to ultimate factor by the $(n + 1)^{th}$-to-ultimate factor.

The values of $f(x)$ (x is in \$000's) given by the data and derived development factors for losses and ALAE are:

	OL&T BI	M&C BI	Products BI
$f(10)$	.677	.802	.835
$f(25)$	.579	.755	.735
$f(50)$	.484	.674	.617
$f(100)$	.372	.543	.463
$f(250)$	.240	.319	.243
$f(500)$	.144	.148	.125
$f(1{,}000)$	.076	.041	.032

The OL&T development factors for 27 months to ultimate for retentions of (in 000's) 50, 100, 250, 500 and 1,000 are 3.150, 3.668, 4.485, 5.223, and 6.081, respectively. The factors for the layers 50–100, 50–250, 50–500, and 50–1,000, using the above method, follow:

Layer (in \$000's)	Method and Development Factor
50–1,000	$(.484 - .076) \div ((.484 \div 3.150) - (.076 \div 6.081)) = 2.891$
50– 500	$(.484 - .144) \div ((.484 \div 3.150) - (.144 \div 5.223)) = 2.697$
50– 250	$(.484 - .240) \div ((.484 \div 3.150) - (.240 \div 4.485)) = 2.437$
50– 100	$(.484 - .372) \div ((.484 \div 3.150) - (.372 \div 3.668)) = 2.144$

As with our unlimited development factors by retention, these factors for layers are somewhat lower than the factors would be for losses uncensored by policy limits. (See Appendix B.) Since about 80% of the losses are not censored by policy limits below \$500,000, the factors produced by the above method are more accurate for layers whose upper bound does not exceed \$500,000. The techniques of producing different development factors by retention or layer and projecting development to ultimate could be useful in estimating ultimate uncensored excess loss ratios, which are important in reinsurance pricing. The techniques could also be used in producing increased limits factors, which are an important part of primary insurance pricing. The actual development factors and data from this study concerning excess losses by layer could provide estimates of increased limits factors up to \$100,000 or possibly \$250,000 limits, since the policy limits in effect have little effect on the layer up to $100,000, or even $250,000. We do not present such estimates, however.

6. SUMMARY

The results that have been produced indicate clearly that loss and ALAE development varies significantly by retention. Accordingly, pricing and reserving estimates incorporating development factors may be substantially in error if this is not taken into account. As this applies to paid as well as reported loss development, recognition of retention is also a major factor in estimating discounted losses using paid development factors.

The protracted development of excess losses and the data limitations inherent in this study suggest a need for further study of development factors beyond 99 months. It would also be beneficial to review development by retention for other lines of business such as medical malpractice and workers' compensation.[2]

The results are closely related to the decrease in the Pareto parameter in successive reports and its relationship to loss development by retention. The principles employed would have relevance for other lines for which the Pareto provides a good fit.

With sufficient data, it would be very worthwhile to study excess development for uncensored losses and for higher retentions than those examined here.

[2] Study of New York data appears in Taylor and Lattanzio [4].

REFERENCES

[1] Stephen W. Philbrick, "A Practical Guide to the Single Parameter Pareto Distribution," *PCAS* LXXII, 1985, p. 44.

[2] Kurt A. Reichle and John P. Yonkunas, Discussion of "A Practical Guide to the Single Parameter Pareto Distribution," *PCAS* LXXII, 1985, p. 85.

[3] Richard E. Sherman, "Extrapolating, Smoothing, and Interpolating Development Factors," *PCAS* LXXI, 1984, p. 123.

[4] Frank C. Taylor and Francis Lattanzio, Discussion of "Accident Limitations for Retrospective Rating," *PCAS* LXIV, 1977, p. 96.

APPENDIX A
TREATMENT OF ALAE IN ESTIMATING DEVELOPMENT FACTORS

The type of occurrence excess coverage that is most common in casualty treaty reinsurance covers the amount of the loss and allocated loss adjustment expense combined in excess of the retention for each occurrence. The method of estimating the development factors for this type of reinsurance, however, was based on the development of the amount of the loss and allocated loss adjustment expense combined in excess of the retention for only those occurrences for which the pure loss exceeded the retention.

The error involved in using this approach is relatively small, since the amount in excess of any retention that is produced by the losses plus ALAE for all occurrences for which the losses alone are less than the retention is small compared to the total losses plus ALAE in excess of the retention. In other words, only a small portion of the excess is missing from our development factors.

Suppose, for example, that for every occurrence, the ratio of the loss to the loss plus ALAE is a. If the tail of the "normalized" (see Section 4) loss distribution is represented by the Pareto density function $qx^{-(q+1)}$, with $q > 1$, then the portion of the total losses plus ALAE in excess of the retention x_0 that is produced by occurrences for which the pure loss is greater than the retention equals

$$\int_{x_0}^{\infty} qx^{-(q+1)} \left(\frac{x}{a} - x_0\right) \mathrm{dx} \div \int_{ax_0}^{\infty} qx^{-(q+1)} \left(\frac{x}{a} - x_0\right) \mathrm{dx},$$

which equals $(q + a - qa)/(a^{1-q})$. If $q = 1.5$ and $a = .87$, for example, then the above expression equals .993.

If $q = 1.5$ and $a = .87$ at first report and $q = 1.3$ and $a = .85$ at ultimate report, then the expression changes from .993 to .995. In this case, the estimate of the first to ultimate development factor would be 1.002 times the development that would be computed using a precise treatment of ALAE.

This problem does not apply to the development factors for losses plus prorated ALAE, since occurrences with pure losses below the retention are not covered by reinsurance arrangements with prorated ALAE. Those factors involve a different estimate—use of losses excess of a retention divided by total losses for the occurrences greater than the retention—as a multiplier for the ALAE. To be precise, the ALAE for each occurrence should be multiplied by the loss excess of the retention divided by the total loss for that occurrence.

The distortion in development factors should be small, even in the product of all the development factors. For each loss and corresponding ALAE and each retention, prorated ALAE = (excess loss ÷ loss) ALAE, so prorated ALAE ÷ excess loss = ALAE ÷ loss for each loss. Since the data indicated that ALAE ÷ loss is about .15 on the average, whatever distortion there is in the estimate of the prorated ALAE would cause less than .15 times as much distortion in losses plus prorated ALAE.

APPENDIX B
EFFECT OF POLICY LIMITS ON DEVELOPMENT FACTORS

The general liability sublines studied had policy limits distributions based on policy year 1982 and policy year 1983 data.

Distribution of Premium

Policy Limit (in $000's)	OL&T BI	M&C BI	Products BI
25	.0043	.0034	.0018
50	.0069	.0031	.0042
100	.0366	.0347	.0248
200	.0022	.0010	.0000
250	.0013	.0032	.0025
300	.1351	.1367	.1792
500	.4161	.5334	.6464
1,000	.3609	.2464	.1354
1,500	.0043	.0027	.0005
2,000	.0191	.0136	.0019
3,000	.0132	.0218	.0033
Total	1.0000	1.0000	1.0000

As an illustration of the approximate effect of these policy limits on excess loss development factors, consider the following example of their effect on an unlimited (no policy limits) loss distribution. Let $10,000 be the lower bound of a tail of unlimited losses for which the "normalized" (divided by 10,000) loss distribution is represented by the Pareto density function $qx^{-(q+1)}$.

Let $q = 1.6$ for a policy year as of 27 months and 1.3 for a policy year at ultimate development, and let a represent the development factor from 27 months to ultimate for losses excess of $10,000.

Since $b^{(1-q)} \div (q - 1)$ is the formula for the losses excess of bk, normalized at k and divided by the number of occurrences greater than k, the unlimited losses excess of \$10,000, \$100,000, \$300,000, \$500,000 and \$1,000,000 at 27 months and at ultimate development can be represented as:

Retention	Excess at 27 months	Excess at Ultimate
\$ 10,000	x	ax
100,000	$.251x$	$.501ax$
300,000	$.130x$	$.360ax$
500,000	$.096x$	$.309ax$
\$1,000,000	$.063x$	$.251ax$

From this, the excess losses can be divided into the following layers, by subtracting from each excess amount the amount directly below it:

Layers (in \$000's)	Amount at 27 months	Amount at Ultimate
100–300	$.121x$	$.141ax$
300–500	$.034x$	$.051ax$
500–1000	$.033x$	$.058ax$
over 1000	$.063x$	$.251ax$

Now suppose that the policy limits earned premium distribution corresponding to the time period of the losses is 20% at \$300,000 (per occurrence), 60% at \$500,000, and 20% at \$1,000,000, instead of the losses being unlimited.

The development of the unlimited losses excess of \$100,000 from 27 months to ultimate $= (.501\ ax) \div (.251\ x) = 1.996\ a$, whereas the development of the limited losses $= (.141\ ax + .8(.051\ ax) + .2(.058\ ax)) \div (.121x + .8(.034x) + .2(.033x)) = 1.252a$. This is a big difference, but we should consider that the development factor for the losses limited only by \$500,000 limits $= (.141ax + .051ax) \div (.121x + .034x) = 1.239a$ and that the development factor for the losses limited only by \$1,000,000 limits $= (.141ax + .051ax + .058ax) \div (.121x + .034x + .033x) = 1.330a$. Thus, the limited development is not that different from the development of losses limited only at \$500,000 or only at \$1,000,000. If $a = 3$, which is not unreasonable, then $1.252a = 3.756$, $1.239a = 3.717$, and $1.330a = 3.990$. For retentions less than \$100,000, the difference between these types of development factors is less, since the portion below \$100,000 is not affected by the limits. Similarly, the development factors for losses excess of \$300,000 from 27 months to ultimate for unlimited losses, limited losses, losses limited only at \$500,000 and losses limited only at

\$1,000,000 are 2.769*a*, 1.559*a*, 1.500*a*, and 1.627*a*, respectively. The development factors for losses excess of \$500,000 are the same for the given policy limit distribution as for losses limited only at \$1,000,000.

For simplicity, we have considered only one policy year rather than a series of policy years with inflation operating on both average cost per occurrence and the average policy limit. But it seems probable that the development factors for retentions up to amounts corresponding to \$500,000 on a 1982 cost level, using actual limited losses for any policy year prior to 1982, are similar to development factors for losses limited only by any single limit which is between amounts corresponding to \$500,000 and \$1,000,000 on a policy year 1982 level. The development factors for limited losses are considerably different from unlimited development factors, but only a small portion of premium is written at policy limits over \$1,000,000, so development factors for limited losses are very useful. Also, the substantial disparity between limited and unlimited losses would be expected given the excessive thickness of the Pareto tail at extremely large loss amounts.