THE COST OF MIXING REINSURANCE

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Abstract

Excess and surplus lines underwriters, and others, rely heavily on facultative reinsurance support as an important part of their underwriting function. Individual risks are often subject to multiple reinsurance transactions as a result of the underwriting process. The net retained by the underwriters for the company's account is then subject to the overall company reinsurance treaty. As a result, the final company net position has been layered in a complicated fashion. It is management's task to provide guidelines for the proper use of facultative proportional and excess reinsurance that achieves corporate risk and profitability objectives under such conditions.

This paper investigates the impact on profitability of a common reinsurance mixing situation. The impact on the stability function of excess reinsurance is quantified. General rules to guide practical use and evaluation of mixed situations are developed.

These results are equally applicable to property as well as casualty risks. The implications are valid for facultative reinsurance underwriters, and others that make heavy use of facultative proportional reinsurance arrangements.

INTRODUCTION

Many underwriters rely heavily on facultative reinsurance support as an important part of their underwriting function. This is especially the case in the excess and surplus lines and commercial property lines. Individual risks are often subject to multiple reinsurance transactions as a result of the initial underwriting process. The net exposure retained by the underwriters for the company's account is then subject to the overall company reinsurance treaty. As a result, the final company net retention has been layered in a complicated fashion. This complicated net position can lead to unexpected net loss ratio and combined ratio results.

The purpose of this paper is to investigate the consequences of one such reinsurance situation—the application of an excess of loss reinsurance treaty after the placement of proportional reinsurance on the same risk—and to investigate ways of managing this situation. We will take the viewpoint of the ceding company, although the subject is also of interest to the excess reinsurer. We will assume that, in general, the mixed reinsurance situation comes about through the application of proportional facultative reinsurance on individual risks, and the retained amounts are then subject to a corporate excess of loss treaty. In the case of a portfolio of risks, we assume the aggregate effect of individual facultative cessions can be adequately modeled by an average proportional retention applying to the entire portfolio.

The consequences of this mixed reinsurance situation are twofold:

Magnitude of net loss ratio: The application of proportional reinsurance below an excess of loss layer reduces the excess reinsurer's loss ratio and raises the ceding company's loss ratio. The expected loss ratio on the pro rata reinsurance is unchanged; it will always be the same as the gross loss ratio.

Stability of net loss ratio: While the purpose of excess of loss reinsurance is to provide stability to the net retained loss ratio, the application of proportional reinsurance under the excess of loss cover actually decreases the stability of the net loss ratio.

A heuristic argument can show that each of these effects is intuitively plausible. Actual examples will show the mechanics of both the magnitude and the stability effects. Beyond the examples, it is demonstrated that these are not isolated instances, but the effects can be mathematically shown to hold always. We will use the term "mixing reinsurance" or "mixing" to denote this scenario of applying an excess of loss reinsurance treaty *after* a proportional transaction.

Reasons for Mixing

As we investigate the implications of mixing proportional and excess reinsurance, we need to keep in mind the purpose for the particular mixing situations. Since all instances of mixing will penalize the net loss ratio to different extents, management must carefully evaluate whether the cost of mixing is justified by the advantage gained. Generally, senior management is heavily involved in negotiating and placing the major treaties of the company. Historically, lower levels of management have directed the use of facultative reinsurance. Often, the individual desk underwriter places quota share facultative reinsurance on a risk as he writes it.

The premise of this paper is that the *total* corporate reinsurance program (not just the major corporate treaties) must be actively managed to assure that corporate objectives are met. The interaction between proportional and excess reinsurance in the mixed case can be very significant. Management must institute guidelines and controls for use of proportional reinsurance which assure the objectives intended by placement of the corporate excess treaties are met. These objectives will generally be stated in the form of expected net loss ratio, or cost of reinsurance, and protection from large swings in net loss ratio (stability).

Some common reasons for the occurrence of mixed reinsurance situations are:

a) capacity;
b) net premium targets;
c) protecting the treaty;
d) sharing of layers; and,
e) commission overrides.

Capacity: An individual risk is too large to be retained net by the insurer. A proportion of the risk may be ceded on a quota share or surplus share basis to reduce its size. This is common on property risks. A mixed situation exists if the corporate property treaty is on an excess of loss basis.

Net Premium Targets: A corporate plan may call for a certain net premium increase that must be strictly adhered to (for instance, because of statutory income or surplus restrictions). If more gross premium is written than planned, the net target may be achieved by increased use of facultative proportional reinsurance. This strategy should be evaluated in light of the penalty imposed on the net loss ratio position.

Protecting the Treaty: If the rate on the excess treaty is clearly insufficient to absorb the exposure from a risk the insurer wishes to write, the excess loss potential can be scaled down by a facultative quota share placement to fit the treaty pricing. This comes about because proportional reinsurance changes the frequency and severity characteristics of the excess loss exposure. This is one case where mixing reinsurance may be the prescribed course of action to achieve the corporate objective of excess treaty perpetuation at a reasonable price.

Sharing of Layers: For any of the reasons above, the underwriter may substitute the direct writing of a proportional share of a risk in place of acceptance of the entire risk followed by a facultative quota share reinsurance transaction. This is, in fact, a disguised mixed reinsurance situation and is fully equivalent in its effect on net loss ratio and stability. The popularity of sharing layers increases as the facultative reinsurance market tightens. The normal operating procedure of the facultative reinsurance underwriter or the brokered treaty underwriter is to accept proportional shares of an excess layer. This is also a mixed reinsurance situation if an excess of loss treaty protects the reinsurer's net position.

Commission Overrides: In most cases, the proportional facultative reinsurer pays a ceding commission to the ceding company. This ceding commission is meant to cover direct commission costs, plus an additional "override" commission to cover the cedent's non-commission costs. The override has the effect of reducing the net expense ratio, and can even cause a negative net commission expense in some cases. A company, or an individual underwriter, may cede large amounts of facultative proportional reinsurance to obtain this override relief to the commission expense ratio.

A Simple Example: The magnitude effect can be demonstrated by inspecting a very simple situation. Suppose a ceding company has a size of loss distribution that allows only claim sizes of either $10,000 or $90,000, with equal probability. With an expected claim frequency of 48 claims per year, and an average claim size of $50,000, we have annual expected losses of $2,400,000 annually. If the company carries an excess of loss treaty with a $40,000 retention, the treaty reinsurer will have expected losses of $1,200,000 per year (24 claims at $50,000 each). Assuming an 80% expected loss ratio for both companies, the excess of loss reinsurer will expect a treaty rate of 50% of subject premium.

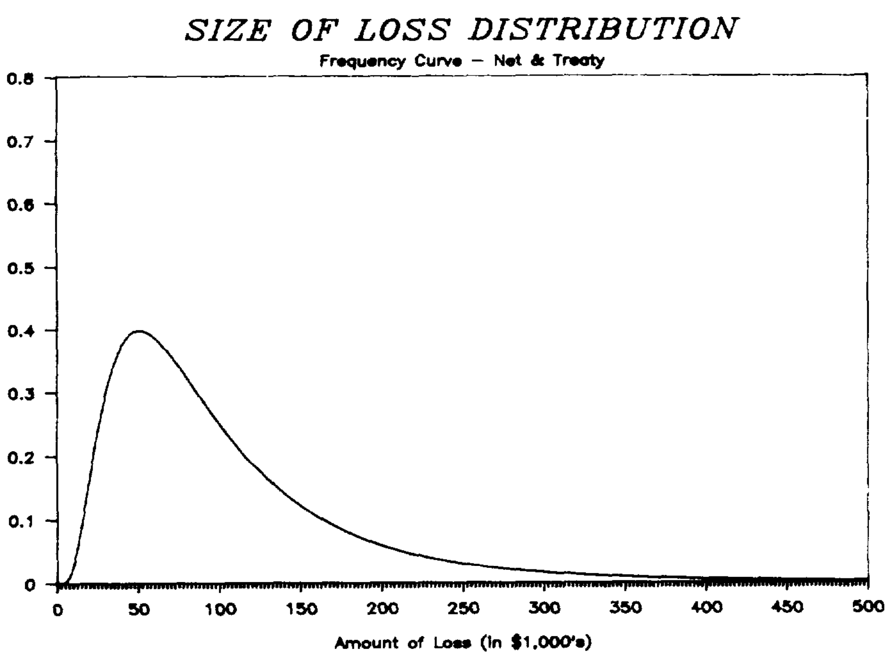
Now assume the underwriters writing this portfolio for the company place 50% quota share facultative reinsurance on every policy as they write it. The ceding company will retain 25% of gross premium, or $750,000, after paying for treaty and facultative reinsurance. The facultative reinsurer will pay half of every loss while the excess reinsurance only responds when the ceding company's 50% share of each loss penetrates the $40,000 retention. Since there are only 24 of these large losses expected, and after the proportional reinsurance they are $45,000 each, the excess reinsurer will have an expected incurred loss of $120,000. This will give it an expected loss ratio of 16% on the $750,000 of treaty premium. The ceding company will retain $1,080,000 of expected losses, for a loss ratio of 144% on its net retained premium of $750,000.

In this simplified example the two reinsurance negotiations have a combined unfavorable effect on the company. The treaty rate was correct for placement of 100% of each risk into the treaty. Because the underwriters did not tailor the facultative cessions to coordinate with the treaty rating, the company has suffered a penalty of 64 loss ratio points. Even though the direct business was correctly priced and evaluated, the net result is a totally unacceptable combined ratio. While the example is constructed to illustrate a point, actual variations on this situation can easily occur. In fact, every instance of an excess of loss reinsurance contract placed over proportional reinsurance works to the disadvantage of the net position, and thus the ceding company.

THE ROLE OF THE SIZE OF LOSS DISTRIBUTION

An inspection of a typical size of loss distribution indicates the underlying cause of mixing effects. Consider a size of loss frequency distribution of the amount of a single claim, as shown in Figure 1. The amount of loss can be read from the horizontal scale, and the relative frequency of such a loss amount

FIGURE 1

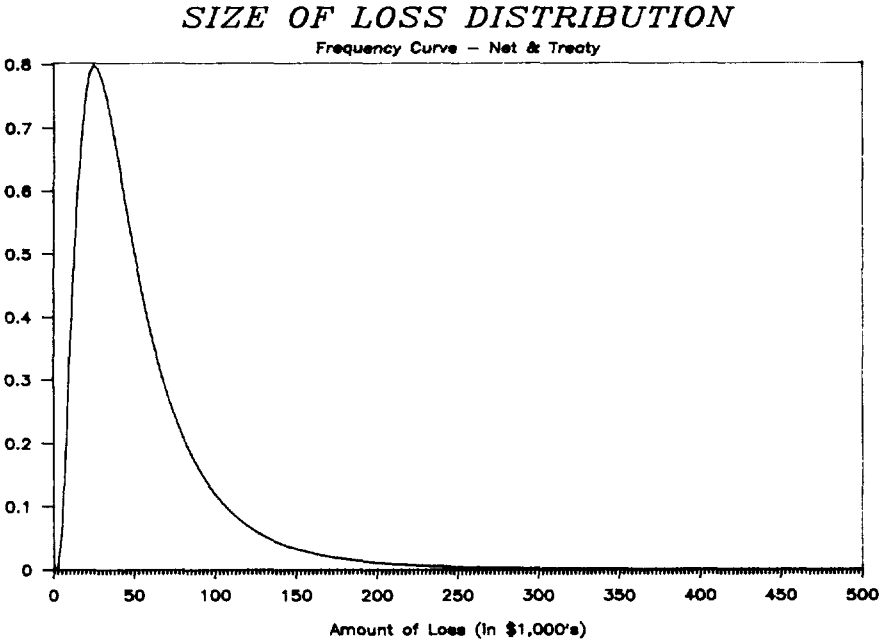


from the vertical scale. Figure 1 can also be used to determine the percent of total claim counts due to claims in a given range of amounts. For instance, we can see that loss over \$150,000 will represent 20% of the claims arising from this particular loss distribution. This is because the area under the size of loss curve above \$150,000 represents 20% of the total area under the curve.

The application of a 50% quota share reinsurance to this size of loss distribution essentially "shrinks" the curve horizontally, while maintaining its relative "shape," as shown in Figure 2.

Now consider the area of the "tail" of this new distribution over \$150,000. This area represented 20% of the total number of claims of the original loss distribution of Figure 1. The tail area of the "shrunken" distribution (Figure 2) over \$150,000, however, accounts for only 3.4% of total claims counts—much less than half of the original gross loss size distribution.

FIGURE 2



Thus, after the proportional "shrinking," the excess reinsurer will receive 50% of the premium that would have been received before proportional reinsurance was placed, but will experience much less penetration of its coverage layer than would have been expected in a situation without proportional reinsurance. In fact, the frequency of loss for the excess reinsurer after the 50% proportional reinsurance will be 17% (3.4% / 20%) of its original excess frequency. As a result, the excess reinsurer's expected net loss ratio after proportional reinsurance is now substantially improved over the experience before the proportional transactions.

Of course, this is simply a consequence of the nonlinear nature of the size of loss distribution. It is another way of stating that for large loss activity, a loss double a given size is experienced much less than half of the time.

Note also that the area under the curve of Figure 2 beyond \$150,000 is the same as the area under the curve of Figure 1 beyond \$300,000 (\$150,000 / 50%). Thus the excess rate over \$150,000, after a 50% quota share placement, should be the same as the excess rate for a \$300,000 retention with no quota share, ignoring risk charge and expense components, and the effect of the upper limit on the excess layer.

In understanding the impact of proportional reinsurance on the net position and the excess reinsurer, the fundamental relationship is the simple idea illustrated above. An excess retention of M after a proportional reinsurance retention of $100a\%$, is equivalent to an excess retention of M/a without proportional reinsurance. This result is shown as the Mixing Price Rule below.

This relationship is key in understanding how mixed reinsurance destabilizes net results. It seems intuitive, and can be shown mathematically (see the Appendix), that net aggregate loss results will show more stability (i.e., a lower coefficient of variation) under a \$150,000 retention than under a \$300,000 retention. In general, if an entire portfolio is proportionally reinsured to retain $100a\%$ of the total risk, with an excess of loss treaty with retention M, the stability of the portfolio's results will be identical to that of the same portfolio without proportional reinsurance and an excess loss limit of M/a. This result is shown as the Mixing Stability Rule below.

It is worth noting that the application of proportional reinsurance *after* the application of an excess of loss treaty does not change the magnitude of stability of the net loss ratio position. Hence, the order of application of reinsurance is extremely important.

Some simple examples will be instructive, and show situations where a disadvantageous net position can result in the ordinary course of business through mixing of reinsurance. This will be especially apparent if we consider the process of underwriting a single risk.

LOSS RATIO MAGNITUDE EFFECTS

A Casualty Example: Suppose an insurer is operating under an excess of loss treaty with \$2,000,000 limits, excess of a retention of \$250,000. The premium for this cover will be 30% of the subject premium that remains available for net and treaty, i.e., remaining after facultative placements.

The primary company underwriter writes an excess liability policy with limits of \$1,000,000, excess of a self-insured retention of \$100,000. He prices this at \$400,000, expecting a loss ratio of 60%. He pays a commission of 15%, and his internal expenses will account for another 10% of the gross premium. This leaves him with 15% (\$60,000) for profit and contingency load on this risk. This allows a 25% load on expected losses as a fluctuation margin. That is, the underwriter could suffer losses of up to \$300,000, or 125% of expected losses, before he has to dip into his surplus funds.

Next, he wishes to reduce his net and treaty exposure to this risk, so he arranges a facultative quota share placement of 50% of the risk. Thus, he is left with a \$500,000 exposure, net and treaty, and a subject premium for purposes of the excess treaty of \$200,000.

Generally, the cedent will receive a ceding commission that will cover his direct ceding commission costs (15% in this example), plus an "override" that is meant to cover the cedent's non-commission, or fixed, expenses. The override for this example will be 10%, which is identical to the ceding reinsurer's other expense ratio.

One can analyze the underwriter's net position before his facultative quota share placement. Assume that a lognormal distribution is an adequate model (Benckert [1]) for size of loss on this risk, with a mean claim size of \$30,000 and a coefficient of variation (CV) of 5.0. The following analysis of direct, reinsurance, and net results is summarized in Exhibit 1, the Mixing Cost Worksheet for this risk. Calculations on this exhibit are discussed below.

The size of loss assumption implies an average first-dollar claim severity of \$270,190 in the layer of interest, hence, an excess policy claim severity of \$170,190. Recall that this is the expected severity for all claims greater than

EXHIBIT 1

MIXING COST WORKSHEET

Policy: a casualty example without mixing

Input parameters:

Direct premium		$400,000
Policy limits		$1,000,000
Underlying retention		$100,000
Expected loss ratio		60.0%
Commission ratio		15.0%
Other expense ratio		10.0%
Reinsurance:		
Percent proportional		0.0%
Ceding commission		25.0%
Excess retention		$250,000
Excess limits		$2,000,000
Excess rate		30.0%
Ceding commission		0.0%
Loss distribution:	mean	$30,000
Lognormal	CV	5

Net results:

	Gross	Proportional	Excess	Net
Loss ratio	60.0%	NA	71.0%	55.3%
Expense ratio	25.0	NA	5.0	35.7
Combined ratio	85.0%	NA	76.0%	91.0%
Net underwriting profit				$25,144
Cost of Reinsurance:				
with mixing	$0	$0	$34,856	$34,856
Pure excess	0	0	34,856	34,856
Additional cost of reinsurance	$0	$0	$0	$0
Cost of Mixing Calculation:				
Actual cost of excess reinsurance			$34,856	
Cost based on subject premium			34,856	
Cost of mixing			$0	

$100,000, but with a maximum ceding carrier liability of $1,000,000 on those claims that are greater than $1,100,000 first-dollar. Expected losses of $240,000, (60% × $400,000) imply an expected claim frequency of 1.41 claims per annum on this risk for the excess carrier ($240,000/$170,190). This analysis is displayed on Exhibit 1.1.

Now the excess of loss reinsurer would assume all loss amounts over $350,000 first-dollar, up to a maximum policy limit loss of $1,100,000 first-dollar. Thus the excess of loss reinsurer will be providing the coverage for the layer from $350,000 first-dollar to $1,100,000 first-dollar for its $120,000 premium. Since 582 losses out of 10,000 exceed $100,000 first-dollar, and 118 losses out of 10,000 exceed $350,000 first-dollar, the excess of loss reinsurer's frequency will be 20% (118/582) of the direct reinsurer's frequency. Then, the reinsurer should expect 0.286 claims (1.41 × 20.3%) at an average severity of about $298,000 in the layer from $350,000 to $1,100,000 first-dollar. This implies a pure premium (expected losses) of about $85,000 (0.286 claims at $298,113 each), and an expected loss ratio of 71% for the excess of loss reinsurer. This analysis of the excess carrier's frequency and severity is displayed on Exhibit 1.3.

The primary company underwriter retains an expected loss cost of $155,000 and a net premium of $280,000, for an expected loss ratio of 55%. This would leave $25,000 for profit and contingency load on the net position, giving a 16% loading of expected losses for a fluctuation margin.

Thus, the primary company has paid 30% of its direct premium to the excess reinsurer. In return, its maximum exposure to loss from any one claim has been reduced from $1,000,000 to $250,000. The margin in the premium that is available to absorb fluctuations in results, however, has also decreased from 25% to 16%. In light of this reduction in the fluctuation loading, it is not immediately obvious whether the insurer is in a better position in terms of protection from random variation of results after this excess reinsurance transaction. As will be demonstrated below, however, excess of loss reinsurance decreases the probability of large aggregate losses to such a significant extent that this 16% risk margin actually reflects more safety than the gross position with its 25% margin.

On Exhibit 1 we have also calculated the cost of reinsurance. Of course, this is the *expected* cost of the reinsurance transaction. The actual cost in retrospect will vary considerably from year to year. The cost of reinsurance is

EXHIBIT 1.1

MIXING COST WORKSHEET
Casualty Example
Allocation of Layer Costs &
Determination of Net Position

Policy Parameters:	(a) Gross	(b) Proportional	(c) Excess	(d) Net
1. Premium	$400,000	$0	$120,000	$280,000
2. Commission	60,000	0	0	60,000
3. Other expenses	40,000	0	6,000	40,000
4. Expected losses	240,000	0	85,144	154,856
5. Profit/risk charge	60,000	0	28,856	
6. Retention	$100,000	NA	$250,000	$100,000
7. First-$ equivalent*	100,000	NA	350,000	100,000
8. Nominal layer width	1,000,000	0	2,000,000	250,000
9. First-$ equivalent*	1,100,000	NA	1,100,000	350,000
10. Effective layer width	1,000,000	0	750,000	250,000
11. First-$ equivalent*	1,100,000	NA	1,100,000	350,000
12. Claim severity	$170,192	$0	$298,113	$109,814
13. Claim frequency	1.410	1.410	0.286	1.410
14. Commission ratio	15.0%	25.0%	0.0%	21.4%
15. Other expense ratio	10.0%	3.0%	5.0%	14.3%
16. Premium rate	100.0%	0.0%	30.0%	70.0%
17. Fluctuation loading	25.0%	NA	33.9%	16.2%
18. Expected loss ratio	60.0%	NA	71.0%	55.3%
19. Combined ratio	85.0%	NA	76.0%	91.0%
20. Cost of reinsurance	$0	$0	$34,856	$34,856

* First-dollar equivalent is the amount of first dollar loss needed to hit this limit.

EXHIBIT 1.2

LOSS DISTRIBUTION TABLE

	Loss Amount x	Number Distribution $f\#(x)$	Amount Distribution $f\$(x)$
Primary retention	\$100,000	0.9417370	0.4069118
Reinsured's retention	350,000	0.9881997	0.6767204
Primary policy limit	1,100,000	0.9981221	0.8627949
Effective excess limit	1,100,000	0.9981221	0.8627949

Distribution type: lognormal
Distribution parameters:

mean =	\$30,000	μ =	8.6799043
CV =	5	σ =	1.8050198

EXHIBIT 1.3

DERIVATION OF LOSS CHARACTERISTICS FOR EXCESS TREATY

	(a) Amounts	(b) $f\#(x)$	(c) $f\$(x)$
1. Primary frequency	1.410		
First dollar equivalents:			
2. Primary retention	\$100,000	0.94173699	0.4069118
3. Primary policy limit	\$1,100,000	0.99812207	0.8627949
4. Reinsured's retention	\$350,000	0.98819966	0.6767204
5. Effective reinsurer limit	\$1,100,000	0.99812207	0.8627949
6. Ratio of excess carrier's frequency to primary frequency {1.0 − (4b)} / {1.0 − (2b)}	20.3%		
7. Excess layer frequency Expected claims per policy term (6) × (1)	0.286		
Severity calculations:			
8. Mean loss (SOL)	\$30,000		
9. Layer loss cost {(5c) − (4c)} × (8)	\$5,582		
10. Limit loss cost (5a) × {1 − (5b)}	\$2,066		
11. Number of layer losses (5b) − (4b)	0.992%		
12. Number of limit losses 1.0 − (5b)	0.188%		
13. Average severity of reinsured losses {(9) + (10)} / {(11) + (12)}	\$648,113		
14. Less: effective retention	\$350,000		
15. Excess layer severity (13) − (14)	\$298,113		
16. Percent pro rata reinsurance	0.0%		
17. Excess reinsurer's severity (15) × {1 − (16)}	\$298,113		

simply defined as the reinsurance premium paid, less the sum of ceding commissions received and expected reinsurance recoveries. Note that since reinsurance is a service that provides value to the cedent, we should expect a positive cost of reinsurance to be the hallmark of any long term reinsurance relationship. This definition of cost of reinsurance ignores investment income lost by the ceding carrier. This component may be required, however, to get realistic cost estimates.

The cost of excess reinsurance in this case is $34,856, which can be expressed as a cost of $87.14 per $1,000 of premium subject to the excess treaty.

The Effect of a Proportional Cession: Now consider the net position of the ceding underwriter after a 50% proportional reinsurance transaction on this policy. As shown in Exhibits 2-2.3, $200,000 net and treaty premium remains, of which $60,000 must go to the excess of loss reinsurer. Since all losses are 50% shared before application of this excess of loss treaty, a first-dollar loss of at least $600,000 is needed before the excess of loss reinsurance responds. Since such a loss occurs for only 52 claims out of every 10,000, the excess of loss reinsurer's frequency has been cut to 9% of the reinsured's frequency by use of the proportional reinsurance (Exhibit 2.3).

The average severity of losses greater than $600,000 limited at $1,100,000 is $900,586. These losses are 50% quota shared above $100,000, so the pro rata reinsurer and the reinsured evenly split the layer $500,000 excess of $100,000. The pro rata reinsurer and the excess reinsurer split the next $500,000 loss layer evenly. This leaves the excess of loss reinsurer with an average claim severity of $150,293 in its layer. With a claim frequency of 0.126 claims in the excess reinsurance layer, the excess reinsurer has an expected loss cost of only about $19,000. The reinsurer, however, has received $60,000 of premium for the excess reinsurance, so it has now improved its expected loss ratio position to 31.4%.

Who pays for this improvement of the excess reinsurer's loss ratio? Consider the proportional reinsurer's position. For 50% of the premium, the proportional reinsurer shares in all the gross losses equally. Thus, the expected losses of the proportional reinsurer are $120,000. This indicates an expected loss ratio of 60% for the pro rata reinsurer, the same as the gross loss ratio. In fact, the expected loss ratio of the quota share reinsurer will always be identical to that of the gross position.

EXHIBIT 2

MIXING COST WORKSHEET

Policy: a casualty example with mixing

Input parameters:

Direct premium		$400,000
Policy limits		$1,000,000
Underlying retention		$100,000
Expected loss ratio		60.0%
Commission ratio		15.0%
Other expense ratio		10.0%
Reinsurance:		
Percent proportional		50.0%
Ceding commission		25.0%
Excess retention		$250,000
Excess limits		$2,000,000
Excess rate		30.0%
Ceding commission		0.0%
Loss distribution:	mean	$30,000
Lognormal	CV	5

Net results:

	Gross	Proportional	Excess	Net
Loss ratio	60.0%	60.0%	31.5%	72.2%
Expense ratio	25.0	28.0	5.0	35.7
Combined ratio	85.0%	88.0%	36.5%	107.9%
Net underwriting profit				($11,081)
Cost of Reinsurance:				
with mixing	$0	$30,000	$41,081	$71,081
Pure excess	0	0	34,856	34,856
Additional cost of reinsurance	$0	$30,000	$6,225	$36,225
Cost of Mixing Calculation:				
Actual cost of excess reinsurance			$41,081	
Cost based on subject premium			17,428	
Cost of mixing			$23,653	

EXHIBIT 2.1

MIXING COST WORKSHEET

Casualty Example
Allocation of Layer Costs &
Determination of Net Position

Policy Parameters:	(a) Gross	(b) Proportional	(c) Excess	(d) Net
1. Premium	$400,000	$200,000	$60,000	$140,000
2. Commission	60,000	50,000	0	10,000
3. Other expenses	40,000	6,000	3,000	40,000
4. Expected losses	240,000	120,000	18,919	101,081
5. Profit/risk charge	60,000	24,000	38,081	(11,081)
6. Retention	$100,000	NA	$250,000	$100,000
7. First-$ equivalent*	100,000	NA	600,000	100,000
8. Nominal layer width	1,000,000	500,000	2,000,000	250,000
9. First-$ equivalent*	1,100,000	NA	1,100,000	350,000
10. Effective layer width	1,000,000	500,000	1,000,000	250,000
11. First-$ equivalent*	1,100,000	NA	1,100,000	350,000
12. Claim severity	$170,192	$85,096	$150,293	$71,680
13. Claim frequency	1.410	1.410	0.126	1.410
14. Commission ratio	15.0%	25.0%	0.0%	7.1%
15. Other expense ratio	10.0%	3.0%	5.0%	28.6%
16. Premium rate	100.0%	50.0%	30.0%	35.0%
17. Fluctuation loading	25.0%	20.0%	201.3%	−11.0%
18. Expected loss ratio	60.0%	60.0%	31.5%	72.2%
19. Combined ratio	85.0%	88.0%	36.5%	107.9%
20. Cost of reinsurance	$0	$30,000	$41,081	$71,081

* First-dollar equivalent is the amount of first dollar loss needed to hit this limit.

EXHIBIT 2.2

LOSS DISTRIBUTION TABLE

	Loss Amount x	Number Distribution $f\#(x)$	Amount Distribution $f\$(x)$
Primary retention	\$100,000	0.9417370	0.4069118
Reinsured's retention	600,000	0.9947991	0.7755223
Primary policy limit	1,100,000	0.9981221	0.8627949
Effective excess limit	1,100,000	0.9981221	0.8627949

Distribution type: lognormal

Distribution parameters:

mean =	\$30,000	μ = 8.6799043
CV =	5	σ = 1.8050198

EXHIBIT 2.3

DERIVATION OF LOSS CHARACTERISTICS FOR EXCESS TREATY

	(a) Amounts	(b) $f\#(x)$	(c) $f\$(x)$
1. Primary frequency	1.410		
First dollar equivalents:			
2. Primary retention	$100,000	0.94173699	0.4069118
3. Primary policy limit	$1,100,000	0.99812207	0.8627949
4. Reinsured's retention	$600,000	0.99479906	0.7755222
5. Effective reinsurer limit	$1,100,000	0.99812207	0.8627949
6. Ratio of excess carrier's frequency to primary frequency {1.0 − (4b)} / {1.0 − (2b)}	8.9%		
7. Excess layer frequency Expected claims per policy term (6) × (1)	0.126		
Severity calculations:			
8. Mean loss (SOL)	$30,000		
9. Layer loss cost {(5c) − (4c)} × (8)	$2,618		
10. Limit loss cost (5a) × {1 − (5b)}	$2,066		
11. Number of layer losses (5b) − (4b)	0.332%		
12. Number of limit losses 1.0 − (5b)	0.188%		
13. Average severity of reinsured losses {(9) + (10)} / {(11) + (12)}	$900,586		
14. Less: effective retention	$600,000		
15. Excess layer severity (13) − (14)	$300,586		
16. Percent pro rata reinsurance	50.0%		
17. Excess reinsurer's severity (15) × {1 − (16)}	$150,293		

Consider the net loss ratio, which was 60% gross and 55% net before any facultative placement. Of the total expected loss costs of $240,000, the proportional reinsurer takes $120,000 and the excess reinsurer assumes $19,000. This leaves $101,000 of expected losses for the reinsured's net position. Since $140,000 of premium remains net, the expected net loss ratio is now 72%. This is substantially worse (17 loss ratio points) than the net loss ratio without any facultative proportional reinsurance. In addition, there is now no premium margin available for profit and contingency loading, since we are now at a combined ratio of 108%. Thus we see that use of proportional reinsurance below an excess of loss treaty simply moves loss dollars out of the excess reinsurer's account into the ceding insurer's account, without affecting the proportional reinsurer.

The Cost of Mixing: Notice that on Exhibit 2 we have calculated the Cost of Mixing. Recall that in the absence of any proportional reinsurance we calculated a cost of reinsurance of $87.14 per $1,000 of subject premium for the excess treaty. If we regard this cost as the reinsurer's price for providing an excess cover for this policy, we will hold this cost constant for any fraction of the policy that is retained after proportional reinsurance. This rate on the $200,000 of subject premium implies a reinsurance cost of $17,428 should be expected. In this mixed case, however, the actual cost for the excess reinsurance is $41,081. We define the *Cost of Mixing* to be the difference of $23,653. Note that this Cost of Mixing is greater than the underwriting loss on the policy of $11,081. This implies that without the Cost of Mixing, this net position would have been profitable for the ceding company. The total cost of reinsurance in the mixed situation can also be decomposed as follows:

Cost of proportional reinsurance	$30,000
Cost of excess reinsurance	17,428
Cost of mixing	23,653
Cost of total reinsurance	$71,081

This example demonstrates a general principle that is independent of the choice of the size of loss distribution or policy parameters. A corollary of the Mixing Price Rule is that the net position after mixed reinsurance will always be worse than under a pure excess reinsurance. This rule states that the excess loss rate for an excess retention of M after a proportional retention of $100a\%$ must equal the loss rate for a pure excess retention of M/a.

The progressive deterioration of the loss ratio and combined ratio as the percent of proportional reinsurance increases can be seen in the table below. This table is for the casualty risk analyzed above, which has a gross expected loss ratio of 60%, with a gross combined ratio of 85%.

Percent Ceded	Net Loss Ratio	Expense Ratio	Combined Ratio
0%	55.3%	35.7%	91.0%
10	58.0	35.7	93.7
20	61.0	35.7	96.7
30	64.3	35.7	100.0
40	68.0	35.7	103.7
50	72.2	35.7	107.9
60	77.0	35.7	112.7
70	82.6	35.7	118.3
75	85.7	35.7	121.4
80	85.7	35.7	121.4
90	85.7	35.7	121.4

As the percent proportional ceded increases, losses are reduced for the excess reinsurer. These costs are shifted to the ceding company, and result in the increasing net loss ratio. Note that in the pure excess case, the loss ratio is reduced from 60% gross, to 55.3% net. The excess reinsurer, however, pays no ceding commission. This increases the expense ratio, and hence the net combined ratio.

When 75% of the risk is proportionally reinsured, no losses can penetrate the excess retention. This is simply because policy limits are $1,000,000, and the 25% of each loss retained net and treaty can never be greater then the $250,000 excess treaty retention. At this point, ceding larger shares of a risk no longer affects the net loss ratio.

THE MIXING PRICE RULE

The mean value of a random variable representing the size of claim after application of proportional reinsurance and excess of loss reinsurance can be expressed analytically. This allows the calculation of the loss cost portion of the excess reinsurance rate. The risk charge and expense load components of the reinsurance rate are ignored for the purposes of this demonstration.

Let $f(x)$ be the probability density function of X, the random variable representing the amount of one claim. We will assume $f(x)$ is appropriately truncated to reflect the policy limit issued by the ceding carrier. Let a be the fraction of each loss retained by the ceding insurer after proportional reinsurance, and M the retention under the excess reinsurance program. (This notation is identical to that used in Centeno [2].)

Then, if X is the gross claim size, the amount of claim after both reinsurances apply is given by

$X(a,M) = \text{Min }(aX,M)$.

First, we establish the expected value of X under each single reinsurance type alone.

If only excess reinsurance applies,

$E(\min(X,M)) = \int_0^M xf(x)dx + M \int_M^\infty f(x)dx$.

If only proportional reinsurance applies,

$E(aX) = a \int_0^\infty xf(x)dx$.

It will also be useful to have an explicit formulation of the probability density of claim size subject to a proportional reinsurance. Let g_a be the density of x subject to proportional reinsurance that retains $100a\%$ of each claim.

Then $g_a(x) = 1/a\, f(x/a)$ will yield the expected value above. (Note: This is a probability density function since

$\int g_a(x)dx = (1/a) \int f(x/a)dx$.

Let $y = ax$; then $dy = adx$. Now we can substitute to obtain:

$$\begin{aligned}\int g_a(x)dx &= (1/a) \int f(y)ady \\ &= \int f(y)dy = 1.)\end{aligned}$$

Then applying excess of loss reinsurance to a claim after proportional reinsurance yields an expected value of

$E(\min(aX,M)) = \int_0^M xg_a(x)dx + M \int_M^\infty g_a(x)dx$.

Again set $ay = x$, so that $dx = ady$ and $x = M$ if and only if $y = M/a$. Rewrite these integrals in terms of the variable y.

$$\begin{aligned} E(\min(aX,M)) &= \int_0^{M/a} (ay)(1/a)\, f(y)a\,dy + M \int_{M/a}^{\infty} (1/a)f(y)a\,dy \\ &= a \int_0^{M/a} yf(y)dy + M \int_{M/a}^{\infty} f(y)dy \\ &= a[\int_0^{M/a} yf(y)dy + (M/a) \int_{M/a}^{\infty} f(y)dy] \\ &= aE(min(X,M/a)) \end{aligned}$$

This means that the expected net value of the amount of a single loss subject to the combination of proportional reinsurance that retains $100a\%$ of each claim, and excess reinsurance that retains the first M amount of each claim, is equivalent to $100a\%$ of the expected value under an excess of loss reinsurance that retains that first M/a amount of each gross claim. This is a specific instance of the more general Mixing Moment Principle demonstrated below when we discuss stability.

Excess treaty premiums are usually calculated using a rate as a percent of subject premium.

Let *Rate* $XS(a,M)$ represent the excess rate for an excess retention M after a proportional retention of $100a\%$.

For purposes of simplifying the demonstration, recall that $f(x)$ reflects underlying primary policy limits and assume that the excess treaty limit extends above the primary policy limits. This allows us to ignore the truncation term due to the excess layer limit.

If we consider only the loss component of the excess premium rate, before any proportional reinsurance, the excess loss rate for limits of L over a retention of M will be

$$\textit{Rate } XS(1,M) = \frac{\int_M^{L+M} (x - M)\, f(x)dx + (L + M) \int_{L+M}^{\infty} f(x)dx}{\textit{Subject Premium}} ,$$

in the most general case.

This simplifies to $\textit{Rate } XS(1,M) = \dfrac{\int_M^{\infty} (x - M)\, f(x)dx}{\textit{Subject Premium}}$,

because of our assumptions.

After proportional reinsurance that retains $100a\%$ of each claim, let *Rate XS*(a,M) represent the rate. Then $100a\%$ of the prior subject premium is now subject premium for the excess treaty, and

$$Rate\ XS(a,M) = \frac{a[\int_{M/a}^{\infty} (x - M/a)f(x)dx]}{a(Subject\ Premium)}$$

$$= \frac{\int_{M/a}^{\infty} (x - M/a)f(x)dx}{Subject\ Premium} = Rate\ XS(1,M/a).$$

Thus, we can state the following:

Mixing Price Rule: The excess reinsurance loss rate for a retention M under a proportional reinsurance that retains $100a\%$ of each loss is identical to the excess loss rate over a retention of M/a, with no proportional reinsurance.

Note one simple implication of the Mixing Price Rule. The limited mean of a distribution F under limit M is given by

$$E_M(x) = \int_0^M x\ dF + M(1 - F(M))$$

and is the "complement" of the excess loss cost $\int_M^{\infty} (x - M)dF$.

Then the excess reinsurance loss rate under a mixed reinsurance case must be smaller than under pure excess if and only if the limited mean of the distribution limited at M/a is larger than the limited mean at M. Thus we have the following:

Mixing Loss Ratio Rule: If the limited mean of a loss distribution is a strictly increasing function of the limit, then the net loss ratio will always deteriorate under a mixed reinsurance case.

Only a most unusual loss distribution does not have the property of increasing limited means. Consider the following:

If $M_1 < M_2$ then

$$\begin{aligned}\int_{M_1}^{\infty} (x - M_1)dF &= \int_{M_1}^{M_2} (x - M_1)dF + \int_{M_2}^{\infty} (x - M_1)dF \\ &= \int_{M_1}^{M_2} (x - M_1)dF + \int_{M_2}^{\infty} (M_2 - M_1)dF + \\ &\qquad \int_{M_2}^{\infty} (x - M_2)dF \\ &> \int_{M_2}^{\infty} (x - M_2)dF,\end{aligned}$$

unless $\int_{M_1}^{M_2} (x - M)dF + \int_{M_2}^{\infty} (M_2 - M_1)dF = 0.$

The above sum of integrals is zero only if $dF = 0$ for $x \geq M_1$.

Thus if $M_1 < M_2$, then $\int_{M_1}^{\infty} (x - M_1)dF > \int_{M_2}^{\infty} (x - M_2)dF$; hence $E_{M1} \leq E_{M2}$ with equality only if $dF=0$ for $x \geq M_1$. In practice, equality will occur only when $f(x)$, the density associated with F, is truncated by policy limits.

We can write the full excess reinsurance rate as follows including the risk charge, $RC(a,M)$, and treaty expenses, Exp:

$$Rate\ XS(a,M) = \frac{a\int_{M/a}^{\infty}(x - M/a)f(x)dx + RC(a,M) + Exp}{a(Subject\ Premium)}$$

Without further information about the form of the risk charge, little more can be said about the excess rate. Note that Bühlmann [3] has identified four premium calculation principles based on the form of the risk charge. These principles calculate the risk charge on the expected value, standard deviation or variance of losses, or utility theory. If the premium calculation principle used in the excess rate is stated, then explicit calculations of equivalent excess rates in terms of the limit M/a are possible.

APPLICATIONS TO PROPERTY INSURANCE

The phenomenon described in the casualty example is due to the shape of the size of loss distribution. The same deterioration of net loss ratio due to mixed reinsurance situations will occur in property situations, if the underlying size of loss distributions follow any of the accepted probability models. A study of this subject done by Shpilberg [4] indicates that a loss distribution that falls between the lognormal and Pareto distributions in its tail behavior is an adequate model for fire insurance. The Mixing Price Rule discussion shows that if the limited mean is an increasing function of the limit M, any mixture of proportional and excess of loss reinsurance worsens the net loss ratio.

As we have seen, the limited mean condition is not very restrictive. Any reasonable choice of size of loss distribution, in particular the Pareto or lognormal, will satisfy this condition. Thus, the adverse consequences of mixing reinsurance will also hold for property risks.

There are, however, special characteristics of property risks that are notable. The policy limits of a property policy may be extremely large if there is a high Probable Maximum Loss level. The traditional approach to reducing this loss exposure to a level appropriate for an excess reinsurance treaty is the use of proportional reinsurance. Hence, a very high percentage of policy limits may be ceded before excess reinsurance.

Thus, property risks are a particularly fertile ground for finding examples of mixed reinsurance situations. The use of facultative reinsurance on large property risks is traditional and necessary to cut large policy limits down to net and treaty positions appropriate for the insurer's treaty capacity. This usage can have a substantial impact on the net loss ratio.

A property example will show net effects of proportional reinsurance similar to the casualty example already considered above.

Suppose the insurer has an excess of loss property treaty with $2,000,000 limits over a retention of $250,000, for this example. If a property risk requiring policy limits of $20 million is written, the underwriter must place $18 million of facultative reinsurance before he can place the remaining risk into his treaty. Most facultative property reinsurance has traditionally been on a proportional basis, resulting in a 90% cession to the facultative reinsurers.

If the gross premium for the risk is $500,000, we will cede $450,000 to the facultative reinsurers and retain $50,000 net as shown in Exhibit 3-3.4.

The results of the reinsurance can be quite different based on the type of property risk being underwritten. The differences we can attempt to model will be reflected in the Probable Maximum Loss (PML) potential, which should be closely related to the underlying size of loss distribution. The policy limits should also be based on the PML potential. For instance, if the risk consists of a single large warehouse, there is a potential probability of losing the entire insured value. For the purposes of this discussion we will model this by choosing a size of loss distribution with 1 chance in 10,000 of a $20,000,000 loss. A lognormal distribution with a mean of $67,500 and a coefficient of variation of 10 is used. The net expected loss ratio in this case is shown in Exhibit 3 as 74%, with a combined ratio of 110%.

As expected, this net position compares unfavorably to the gross position with an 85% combined ratio. Note that this example demonstrates a capacity problem, where facultative reinsurance *must* be used before the treaty can come into use. The use of excess of loss facultative reinsurance in place of proportional may improve these net positions, if such reinsurance is available at an appropriate price. If not, the only recourse to the underwriter is to price the gross risk appropriately to achieve his target 95% net combined ratio. A premium of $610,000 for this risk would be required to achieve a 95% combined ratio under this mixing situation with 90% proportional reinsurance. This would require pricing to a gross loss ratio of 49% and a gross combined ratio of 74% for the property. It is unlikely that the marketplace will allow such pricing.

EXHIBIT 3

MIXING COST WORKSHEET

Policy: a property example

Input parameters:

Direct premium		$500,000
Policy limits		$20,000,000
Underlying retention		$0
Expected loss ratio		60.0%
Commission ratio		15.0%
Other expense ratio		10.0%
Reinsurance:		
Percent proportional		90.0%
Ceding commission		25.0%
Excess retention		$250,000
Excess limits		$2,000,000
Excess rate		30.0%
Ceding commission		0.0%
Loss distribution:	mean	$67,500
Lognormal	CV	10

Net results:

	Gross	Proportional	Excess	Net
Loss ratio	60.0%	60.0%	27.8%	73.8%
Expense ratio	25.0	28.0	5.0	35.7
Combined ratio	85.0%	88.0%	32.8%	109.5%
Net underwriting profit				($3,336)
Cost of Reinsurance:				
with mixing	$0	$67,500	$10,836	$78,336
Pure excess	0	0	47,155	47,155
Additional cost of reinsurance	$0	$67,500	($36,319)	$31,181
Cost of Mixing Calculation:				
Actual cost of excess reinsurance			$10,836	
Cost based on subject premium			4,715	
Cost of mixing			$6,121	

EXHIBIT 3.1

MIXING COST WORKSHEET
Property Example
Allocation of Layer Costs &
Determination of Net Position

Policy Parameters:	(a) Gross	(b) Proportional	(c) Excess	(d) Net
1. Premium	$500,000	$450,000	$15,000	$35,000
2. Commission	75,000	112,500	0	(37,500)
3. Other expenses	50,000	13,500	750	50,000
4. Expected losses	300,000	270,000	4,164	25,836
5. Profit/risk charge	75,000	54,000	10,086	(3,336)
6. Retention	$0	NA	$250,000	$0
7. First-$ equivalent*	0	NA	2,500,000	0
8. Nominal layer width	20,000,000	18,000,000	2,000,000	250,000
9. First-$ equivalent*	20,000,000	NA	20,000,000	250,000
10. Effective layer width	20,000,000	18,000,000	20,000,000	250,000
11. First-$ equivalent*	20,000,000	NA	20,000,000	250,000
12. Claim severity	$65,577	$59,019	$310,572	$5,648
13. Claim frequency	4.575	4.575	0.013	4.575
14. Commission ratio	15.0%	25.0%	0.0%	−107.1%
15. Other expense ratio	10.0%	3.0%	5.0%	142.9%
16. Premium rate	100.0%	90.0%	30.0%	7.0%
17. Fluctuation loading	25.0%	20.0%	242.2%	−12.9%
18. Expected loss ratio	60.0%	60.0%	27.8%	73.8%
19. Combined ratio	85.0%	88.0%	32.8%	109.5%
20. Cost of reinsurance	$0	$67,500	$10,836	$78,336

* First-dollar equivalent is the amount of first dollar loss needed to hit this limit.

EXHIBIT 3.2

LOSS DISTRIBUTION TABLE

	Loss Amount x	Number Distribution $f\#(x)$	Amount Distribution $f\$(x)$
Primary retention	\$0	0.0000000	0.0000000
Reinsured's retention	2,500,000	0.9970693	0.7281287
Primary policy limit	20,000,000	0.9999017	0.9423854
Effective excess limit	20,000,000	0.9999017	0.9423854

Distribution type: lognormal

Distribution parameters:

mean=	\$67,500	$\mu = 8.8123226$
CV=	10	$\sigma = 2.1482831$

EXHIBIT 3.3

DERIVATION OF LOSS CHARACTERISTICS FOR EXCESS TREATY

	(a) Amounts	(b) $f\#(x)$	(c) $f\$(x)$
1. Primary frequency	4.575		
First dollar equivalents:			
2. Primary retention	\$0	0	0
3. Primary policy limit	\$20,000,000	0.99990169	0.9423854
4. Reinsured's retention	\$2,500,000	0.99706933	0.7281287
5. Effective reinsurer limit	\$20,000,000	0.99990169	0.9423854
6. Ratio of excess carrier's frequency to primary frequency {1.0 − (4b)} / {1.0 − (2b)}	0.3%		
7. Excess layer frequency Expected claims per policy term (6) × (1)	0.013		
Severity calculations:			
8. Mean loss (SOL)	\$67,500		
9. Layer loss cost {(5c) − (4c)} × (8)	\$14,462		
10. Limit loss cost (5a) × {1 − (5b)}	\$1,966		
11. Number of layer losses (5b) − (4b)	0.283%		
12. Number of limit losses 1.0 − (5b)	0.010%		
13. Average severity of reinsured losses {(9) + (10)} / {(11) + (12)}	\$5,605,719		
14. Less: effective retention	\$2,500,000		
15. Excess layer severity (13) − (14)	\$3,105,719		
16. Percent pro rata reinsurance	90.0%		
17. Excess reinsurer's severity (15) × {1 − (16)}	\$310,572		

EXHIBIT 3.4

DERIVATION OF LOSS CHARACTERISTICS FOR PRIMARY POLICY

	(a) Amounts	(b) $f\#(x)$	(c) $f\$(x)$
1. Expected losses	\$300,000		
First dollar equivalents:			
2. Primary retention	\$0	0	0
3. Primary policy limit	\$20,000,000	0.99990169	0.9423854
Severity calculations			
4. Mean loss (SOL)	\$67,500		
5. Layer loss cost {(3c) − (2c)} × (4)	\$63,611		
6. Limit loss cost (3a) × {1 − (3b)}	\$1,966		
7. Number of layer losses (3b) − (2b)	99.990%		
8. Number of limit losses 1.0 − (3b)	0.010%		
9. Average severity of primary losses {(5) + (6)} / {(7) + (8)}	\$65,577		
10. Less: retention	\$0		
11. Primary policy severity (9) − (10)	\$65,577		
12. Primary policy frequency Expected claims per policy term (1) / (11)	4.575		

Note one very important implication of this example. We can no longer assume the underwriter can price this risk on the basis of gross frequency and severity characteristics alone. In order to achieve combined ratio results that allow long-run survival of the ceding insurer, the gross price must be set based on gross frequency and severity, the excess reinsurance rate, the amount of proportional reinsurance needed for capacity, and the ceding commission structures.

The excess reinsurance rate must also anticipate some use of facultative reinsurance for capacity purposes. Specifically, for property risks the excess rate must be calculated anticipating a certain amount of use of proportional reinsurance. This will be the case if a loss rating approach using past experience is used to calculate the excess rate, and this past period reflects a similar use of proportional reinsurance as anticipated for the next treaty year.

OTHER MAGNITUDE EFFECT CONSIDERATIONS

The net results of the casualty and property examples are not only a function of the percentage of proportional reinsurance used. Both the excess reinsurance rate and the ceding commission structure have an effect on the final net position. A detailed treatment of these subjects is not possible here, but some issues that relate to the magnitude effect will be mentioned.

The Excess Reinsurance Rate: In the casualty example, an excess treaty was specified with a $2,000,000 limit over a $250,000 retention. Depending on the underlying size of loss distribution one might assume that a "correct" excess loss rate could simply be calculated from the distribution statistics. However, the policy subject to the excess reinsurance could be any one of the following.

A primary policy with policy limits of $2,250,000 that uses the entire reinsurance layer of $2,000,000.

If the primary policy limits are only $1,000,000 the rate should be substantially different.

If the $1,000,000 policy limits are excess of a self insured retention of $100,000, the appropriate rate for the excess reinsurance would also be different.

If the ceding company writes an excess policy for $1,000,000 limits over a primary policy with $500,000 limits, the correct excess reinsurance rate is again different from any of the above.

One can immediately see that with no change in the underlying risk's loss potential (as characterized by its size of loss distribution), several different but "correct" excess reinsurance rates are possible. It becomes apparent that one cannot speak of a proper excess reinsurance rate on a portfolio without some measure of the anticipated underlying distributions of retentions and policy limits in the portfolio. Thus, the excess reinsurance rate must be formulated in anticipation of a certain portfolio structure.

This point has practical implications that generate mixing situations. Suppose an excess reinsurance program has been negotiated, with the parameters agreed to for two years forward. At the time of the negotiation, management of the ceding carrier fully intended to write a book of small surplus lines SMP risks. An excess and surplus lines carrier is usually very responsive to market opportunities; hence, six months into the program, management modifies its original marketing plan because conditions are excellent for obtaining strong rates on small casualty umbrellas. Management wants to take advantage of this opportunity. The original excess reinsurance rate, however, contemplated the SMP book and carried a provisional rate of 10%. The same calculations based on a book of small umbrella business would yield a proper rate of 35% for the excess reinsurance.

An excess reinsurance program can easily have 10 to 20 participants and have taken months of effort to place. Renegotiating the treaty at every shift in portfolio composition is not a realistic option. Furthermore, the excess and surplus lines market depends heavily on the reinsurance market for capacity. Many such companies may cede out 50% or more of their gross writings. Thus, including this umbrella book in the treaty at an inadequate excess rate is not a viable option for management concerned about maintaining a long term presence in the market with consistent reinsurer support.

As a practical matter, the ceding underwriter has little real choice but to attempt to "protect the treaty." As we have seen, the ceding underwriter has great control over his treaty loss ratio, through his use of proportional facultative reinsurance. By altering the percent of proportional reinsurance placed on a risk, the size of loss characteristics of the net position can be fit into the treaty rate structure.

Consider the casualty example given above to be representative of a typical umbrella policy. At a 10% rate, the excess reinsurer would receive $40,000 of premium and would have an expected loss ratio of 210% ($85,114 / $40,000), if no proportional reinsurance were placed. After the 50% proportional cession, however, the excess reinsurer would receive \$20,000 of premium at the 10% rate. With expected losses of \$18,853, this would yield an expected loss ratio of 94%, much better than the original 210%. Under the original scenario presented for the casualty example, the placement of 50% proportional reinsurance was not warranted. Under this new scenario, however, the 50% proportional reinsurance should clearly be placed before the identical policy is placed into the excess treaty. The cost of mixing in this case should be paid to the excess reinsurer to bolster an inadequate treaty rate for a risk not contemplated in the original treaty price.

Thus, the situation is manageable but becoming exceedingly complex. The underwriter must ascertain a correct price for the risk insured on a gross basis. This is no different from any underwriting situation. In addition, we again see that an essential part of the direct company's underwriting and pricing process must be the correct placement of reinsurance to achieve an acceptable net result. Even this, however, is not enough. The underwriter must also balance out his net position against the results he is passing on to the excess reinsurer. He must be able to maintain long-term acceptable results for his excess reinsurance support, in the face of continuing shifts in his portfolio composition due to market conditions.

The calculations we have made in our examples are complex and assume knowledge of the size of loss distribution underlying the policy. This is clearly an area where actuarial expertise can be applied to produce general guidelines and specific pricing procedures that aid in determining the net underwriting position. Without such pricing analysis available, management will have no effective way of controlling and evaluating the proper, coordinated use of proportional and excess reinsurance.

The Gearing Factor: The existence of the override in the ceding commission has been remarked on above. The purpose of the override is to reimburse the ceding company for the non-commission expenses it incurred in writing the direct business. Unfortunately, in times of excessive reinsurance capacity the override is used as a competitive tool by reinsurers. Thus, the casualty example considered above may be entitled to a 10% override based on the expense structure of the ceding carrier; however, a particularly aggressive reinsurer may offer an override of 15%. This, of course, makes the determination of the net position even less straightforward, and offers a powerful incentive to cede larger proportional reinsurance amounts.

The excessive override will tend to improve the combined ratio while the mixing effect will act to worsen the combined ratio. Hence, it becomes even more imperative to calculate the net position before a risk is bound and facultative arrangements settled. For instance, the 50% proportional reinsurance on the casualty risk with a 15% override would yield the same net loss ratio of 72.2%, but an improved net combined ratio of 100.8%. The effect on the property example with 90% ceded proportional reinsurance is even more leveraged, with a net loss ratio of 73.8%, but a net combined ratio of 45.2%, much improved from the original 110%.

The combined effect of an excessive override and a large percent of proportional ceded reinsurance may not only cancel out the mixing penalty, but also produce a favorable net combined ratio even when the direct risk is severely underpriced. For example, if the property risk example of Exhibit 3 were priced at a 100% gross loss ratio, the premium would be $300,000. Net retention after a 90% proportional reinsurance cession only would be $30,000 of written premium and expected losses. Expenses before ceding commission total 25% of gross premium, or $75,000. The ceding commission at a 15% override would total 30% of the $270,000 ceded premium, or $81,000. Thus, after the proportional cession the insurer would have net premium income of $30,000 and net costs as follows:

Net incurred losses:	$30,000
Direct expenses:	75,000
Ceding commission:	(81,000)
Net incurred costs	$24,000

This is equivalent to a combined ratio of 80%, a substantial improvement over the direct combined ratio of 125% at which the risk was written direct. This aspect of the override in proportional reinsurance has been termed the "Gearing Factor" by Buchanan [5]. The existence of the gearing factor effect can overwhelm the unfavorable mixing effects in the transaction.

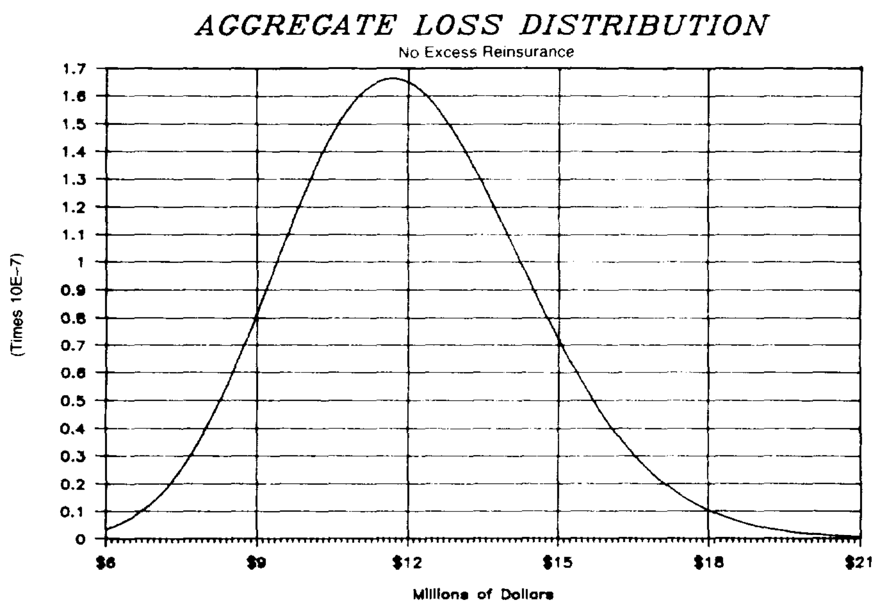
STABILITY EFFECTS

One of the less obvious effects of mixing proportional and excess of loss reinsurance types is the effect on the variation of the net loss ratio after reinsurance. The use of proportional reinsurance below an excess of loss treaty actually makes the resulting net aggregate loss costs more variable than would be the case under the excess treaty alone. This is significant because stability of net results is one of the most important benefits resulting from an excess reinsurance treaty. Any degradation of the stability "component" of the excess treaty "product" makes the treaty worth less.

We will use the casualty policy example to form a small portfolio that will allow us to investigate the impact on stability of mixing reinsurance. Assume we have a portfolio of 50 policies identical to the casualty example. Therefore, we have a book of excess casualty business that generates \$20 million of gross premium and an average of 70.5 claims annually (50 × 1.410). These claims follow the lognormal size of loss distribution specified earlier, i.e. with a mean of \$30,000 and a CV of 5.0. The expected loss ratios on this book of business are identical to those on the single policy—that is, 60% gross, 55% if only the excess treaty is applied but 72% in the mixed reinsurance case.

The aggregate loss distribution differs in the case of the portfolio and the single policy. As a simple demonstration, there is a substantial probability (24%) that the single policy will be loss-free. It is effectively impossible, however,

FIGURE 3

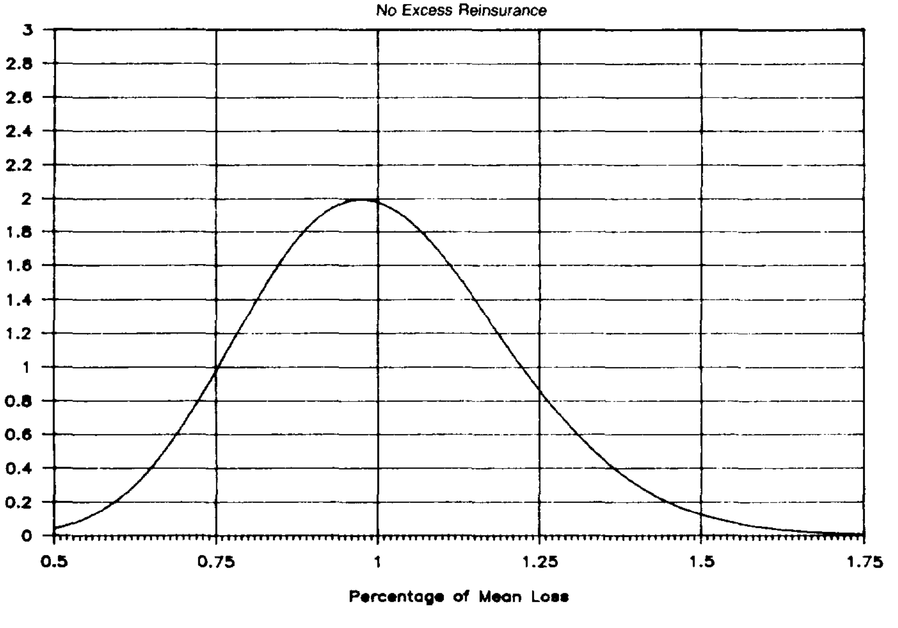


for the entire portfolio to be loss-free in any year (a probability of 2.4×10^{-31} of a loss-free year). The expected annual claim cost of the portfolio is \$12,000,000 (70.5 claims at \$170,200 each) and the aggregate losses of the portfolio are distributed as shown in Figure 3. All computations of aggregate loss distributions were made using the algorithm developed by Heckman and Meyers [6].

In order to make comparisons between aggregate loss distributions, we will normalize such distributions, by setting the mean aggregate loss to 100%, and present the probabilities of achieving various percentages of the mean. This maintains the relative shape of the distribution and facilitates the comparison of different distributions with various underlying aggregate loss means. The normalized aggregate distribution of the unreinsured portfolio above can be seen as Figure 4. This distribution has a coefficient of variation of 0.2.

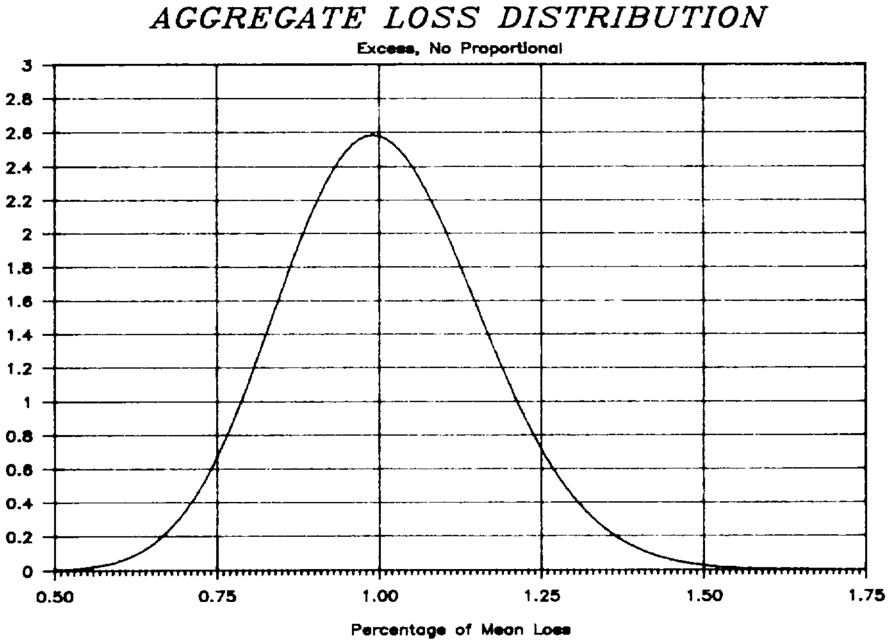
FIGURE 4

AGGREGATE LOSS DISTRIBUTION



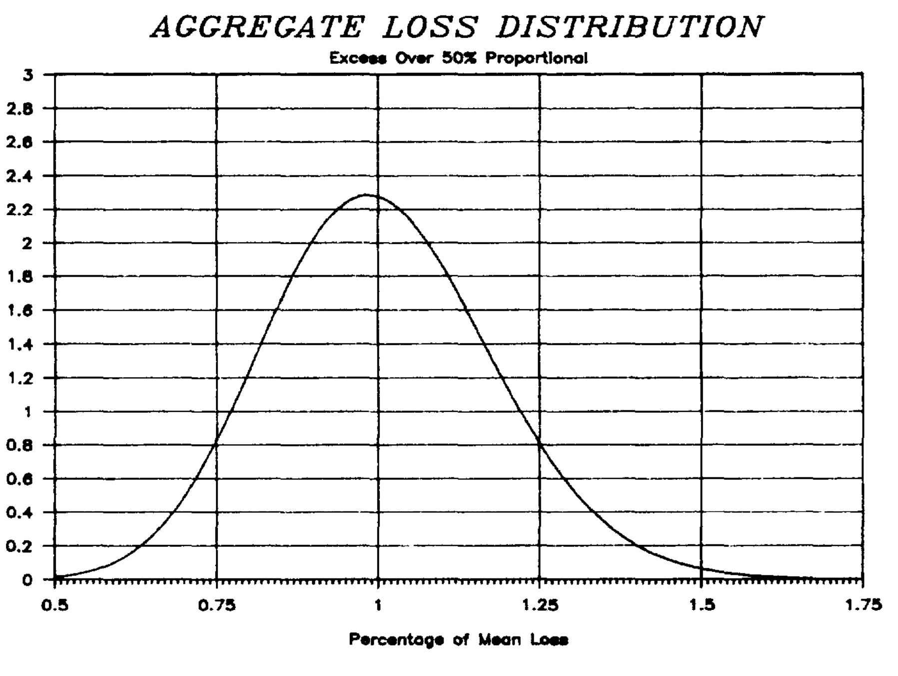
After placement of the excess treaty on this portfolio, the spread of the distribution is much reduced, as can be seen from Figure 5 below. Note that the probability of losses totalling over 150% of expected is substantially reduced by use of excess reinsurance, and the entire curve is distributed closer around its mean of 1.0. The coefficient of variation after excess reinsurance is reduced to 0.155.

FIGURE 5



Now, if the 50% proportional reinsurance is placed on each of the 50 policies in the portfolio, we obtain the aggregate loss distribution shown as Figure 6. This distribution clearly lies between the unlimited case and the pure excess case in its dispersion of possible loss amounts. Note the larger area under the curve over 150% of mean loss, for example, than under the pure excess treaty. The coefficient of variation has also increased to 0.175.

FIGURE 6



Since all aggregate distributions are normalized, they can be compared on the same scale as shown in Figure 7. This chart shows that the "spread" of possible results around the mean loss in the mixed case lies between the unlimited and pure net of excess distribution. In this sense, the stability paid for by purchase of excess reinsurance is "undone" by application of the proportional reinsurance.

Regarding the stability of the portfolio, we are most interested in the behavior of the aggregate loss distribution at the extreme right-hand tail. As shown in

FIGURE 7

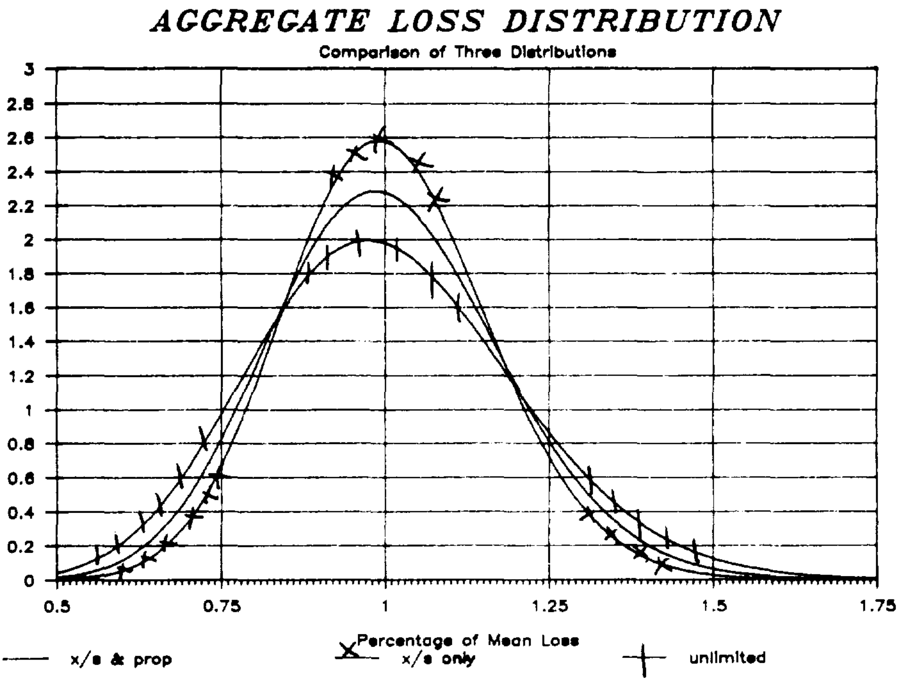
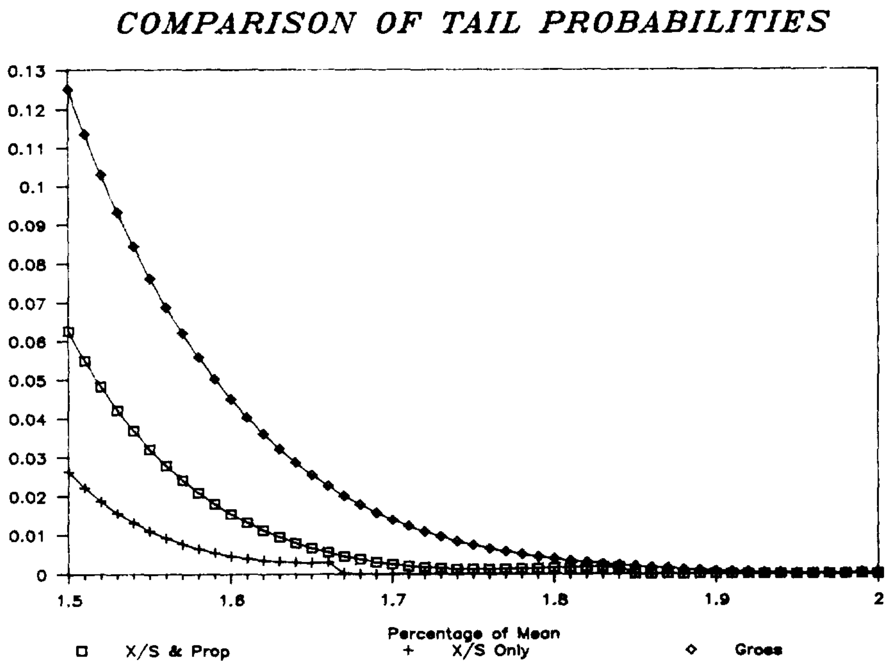


Figure 8, the tail behavior of the aggregate loss distribution in the mixed reinsurance case is substantially more severe than the pure excess treaty case.

The problem, of course, is that we are paying the same 30% rate of net and treaty premium for excess reinsurance protection in both the mixed reinsurance and pure excess cases. As Figure 8 shows, the protection from extreme fluctuations we receive for our 30% rate is substantially less in the mixed case.

While the normalized aggregate distributions are useful for comparing aggregate loss distributions with disparate means, it is also important to focus on

FIGURE 8



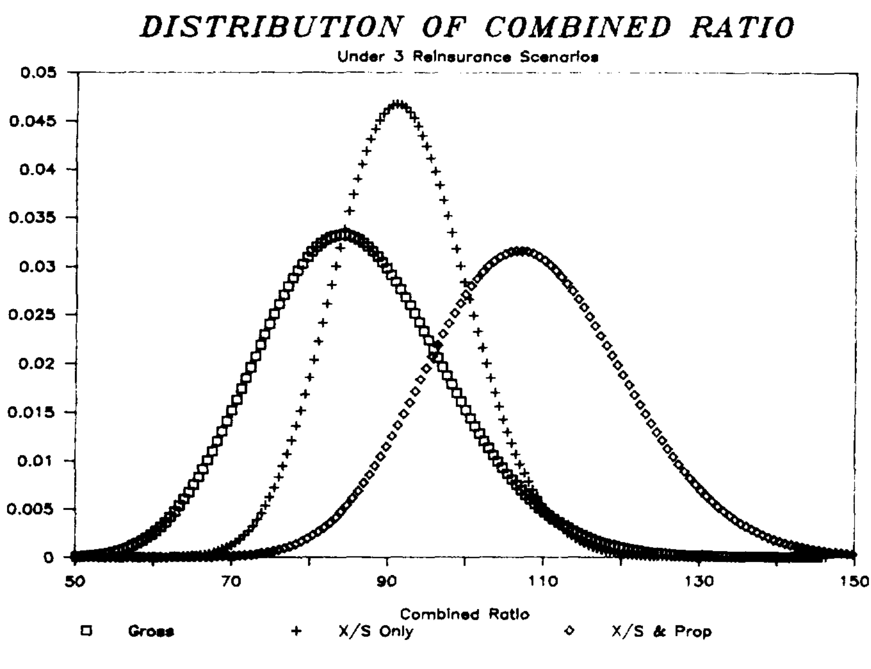
the bottom line—the distribution of combined ratios under the three different scenarios. The combined ratio becomes a random variable through the equation:

Combined Ratio = Expected Loss Ratio × Normalized Aggregate Loss Ratio + Expense Ratio.

Figure 9 shows the distribution of combined ratios for the three scenarios. Clearly, the range of alternatives under the mixed reinsurance scenario is the least desirable, not only in terms of its expected value, but also in terms of the probability of experiencing extremely adverse combined ratios. Note that there is little or no chance of a combined ratio over 120% in the case of the gross or pure excess case. The mixed case, however, leaves us exposed to a substantial probability that a combined ratio over 120% will be experienced.

Even the combined ratio comparison does not take the absolute scale into account. Dollar magnitudes are important, however, if we are to gauge the

FIGURE 9



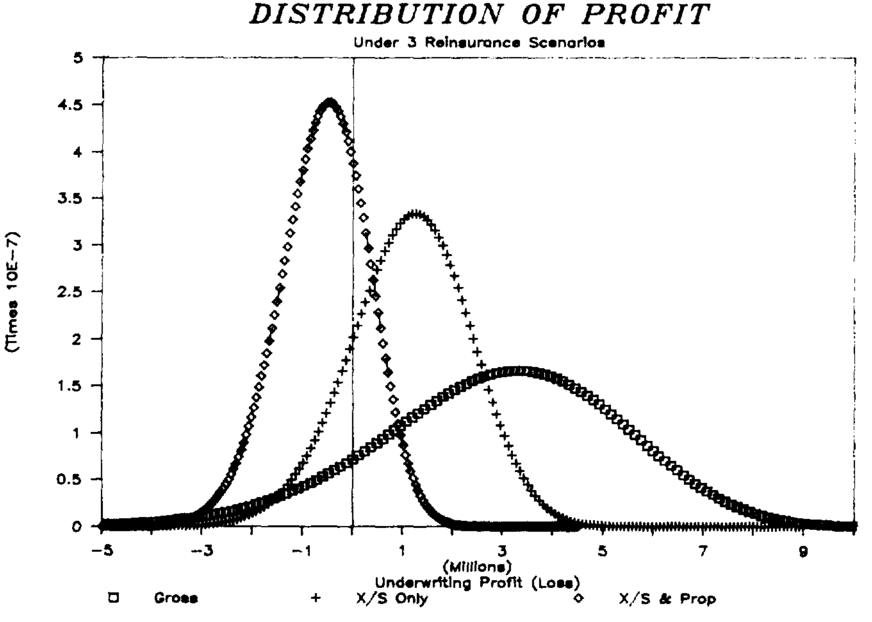
impact of the reinsurance programs on company surplus. An additional way of evaluating the bottom line is to simply review the distribution of statutory underwriting profit or loss. Profit can be represented as a random variable by:

Profit = Premium − Aggregate Losses − Expenses

where Aggregate Losses is the random variable we have been examining above, but not normalized. The resulting distribution is shown in Figure 10.

This chart is clearly of interest in evaluating ruin probabilities. Note that the gross loss distribution has a non-negligible probability of suffering an underwriting loss of over $4 million. The pure excess reinsurance makes a loss of over $3 million unlikely, and even the mixed case reduces the chance of suffering a $4 million underwriting loss significantly. The price that must be paid for this protection in the mixed case, however, is an expected underwriting loss. Thus the mixed case is clearly inferior to pure excess reinsurance in terms of both magnitude and stability of net underwriting results.

FIGURE 10



A usable table representing the tail probabilities for the three scenarios is presented below.

TAIL PROBABILITIES
Probabilities of Exceeding the Percent of Mean

Percent of Mean	Gross	Type of Reinsurance: Excess Over Proportional	Excess Only
125%	11.07%	8.15%	5.77%
130%	7.45	4.93	3.09
135%	4.85	2.84	1.55
140%	3.06	1.56	0.73
145%	1.87	0.82	0.32
150%	1.11	0.41	0.14
151%	1.00	0.36	0.11
152%	0.89	0.31	0.09
153%	0.80	0.27	0.08
154%	0.72	0.23	0.07
155%	0.64	0.20	0.05
Mean aggregate loss	$12,000,000	$5,054,050	$ 7,742,800
Net premium	20,000,000	7,000,000	14,000,000
Expenses	5,000,000	2,500,000	5,000,000
Expected U/W profit	$ 3,000,000	$ (554,050)	$ 1,257,200

Using this table it is possible to investigate alternate scenarios, using proportional only or excess of loss only, to achieve a desired risk level with net incurred loss. For instance, suppose that the 50% proportional reinsurance were placed in order to keep the probability of an extra $3,000,000 loss at about 1% or less. From the middle column, there is about a 1% probability of a loss over 142% of mean aggregate loss in the mixed reinsurance case. This corresponds to $2.1 million dollars of loss over the expected amount of $5,054,050. Taking expenses into account, about a 1% chance of suffering an underwriting loss of

$2.7 million is implied. Note that in order to achieve this protection, the company will have an expected underwriting loss of about $500,000.

Is there a more rewarding way to achieve the same risk position? There are at least two other reinsurance configurations that appear preferable. For instance, on a gross basis, there is a 1% probability of suffering a loss of $18,000,000 or higher. This is equivalent to a 1% chance of an underwriting loss of $3,000,000 or more. A 10% cession of this portfolio would reduce the 1% level of loss to $2.7 million, leaving an expected underwriting profit of $2.7 million. Even though the 90% proportional retention tail does not diminish as fast as the mixed case, the 1% level of risk is the same and expected profit is $3.2 million more.

Similarly, the 1% expected loss level for the excess of loss portfolio is 138% of the mean, or an underwriting loss of $1.7 million. Thus, the 1% loss level is much lower than the mixed reinsurance case, and the expected underwriting profit of $1.3 million is much higher than the mixed case.

To summarize, at the 1% probability of loss level we have inspected three alternatives, and the mixed case is the least desirable.

	90% Quota Share	$250,000 Excess Over 50% Proportional	$250,000 Excess Only
1% level of U/W loss	($2,700,000)	($2,700,000)	($1,700,000)
Expected profit	$2,700,000	($554,050)	$1,257,200

The simple calculations above hint at the complexity of the optimal reinsurance problem. Surprisingly, actuaries have studied this complex question extensively. See, for instance, Beard, Pentikainen, and Pesonen [7] for a bibliography. Three related results of interest are given:

1. For a fixed amount of reinsurance premium and ignoring risk loadings, aggregate stop loss is the optimum reinsurance to minimize the variance of net results [8].
2. With a risk load that increases with variance, proportional (quota-share) reinsurance is optimal to minimize the reinsurance cost for a given variance level [9].

Finally,

3. Allowing mixed reinsurance treaties and constraints on both mean and variance, in most cases pure excess of loss reinsurance is optimal to minimize the skewness of net aggregate losses [10].

THE MIXING STABILITY RULE

In a mixed reinsurance situation, a decrease in the amount retained after proportional reinsurance will decrease the stability of the net aggregate losses. In this sense proportional reinsurance will negate the major benefit of excess reinsurance.

As a measure of stability we will use the coefficient of variation of net aggregate loss results. Recall that if X is a random variable, we define

$$\text{CV}(X) = \frac{\text{Standard Deviation }(X)}{\text{Mean }(X)}$$

Let X be the random variable representing the amount of one claim, and N be the random variable representing the number of claims in the experience period. Let M be amount retained under an excess of loss treaty, and $100a\%$ be the percent retained under proportional reinsurance.

Let $X(a,M) = \min(aX,M)$ represent the net amount of one claim under both reinsurances. This is the random variable of claim amount under the mixed reinsurance situation.

Let λ_k be the kth moment of N, the number of losses, and β_k the kth moment of X, the amount of loss. Then for any compound process Y defined by

$$Y = \sum_{i=1}^{N} X_1 i,$$

we know that

$$E(Y) = \lambda_1 \beta_1 \text{ and,}$$

$$\text{Var}(Y) = \lambda_1 \text{ Var}(X) + \text{Var}(N)\, \beta_1^2 \text{ (see Miccolis [11]).}$$

Thus,

$$\text{Var}(Y) = \lambda_1 (\beta_2 - \beta_1^2) + (\lambda_2 - \lambda_1^2)\, \beta_1^2$$

in terms of central moments.

And, in general,

$$CV^2(Y) = \frac{\lambda_1\beta_2 + (\lambda_2 - \lambda_1 - \lambda_1^2)\,\beta_1^2}{(\lambda_1\beta_1)^2},$$

which simplies to

$$CV^2(Y) = \frac{\beta_2}{\lambda_1\beta_1^2} + \frac{\lambda_2 - \lambda_1 - \lambda_1^2}{\lambda_1^2}.$$

Both the mixing price and stability rules are essentially a result of the following relationship that holds for the kth central moment of $X(a,M)$, denoted by $\beta_k(a,M)$.

Mixing Moment Principle: $\beta_k(a,M) = a^k\beta_k(1,M/a)$
Proof: By definition,

$$\beta_k(a,M) = \int_0^M x^k g_a(x)dx + M^k \int_M^\infty g_a(x)dx,$$

where $g_a(x) = (1/a)f(x/a)$ is the probability density of x under proportional reinsurance. If we set $ay = x$, then $ady = dx$, and $x = M$ if and only if $y = M/a$. Now rewrite β_k in terms of y,

$$\begin{aligned}\beta_k(a,M) &= \int_0^{M/a}(ay)^k(1/a)f(y)ady + M^k \int_{M/a}^\infty(1/a)f(y)ady\\ &= a^k \int_0^{M/a} y^k f(y)dy + M^k \int_{M/a}^\infty f(y)dy,\\ \beta_k(a,M) &= a^k[\int_0^{M/a} y^k f(y)dy + (M/a)^k \int_{M/a}^\infty f(y)dy].\\ &= a^k\beta_k(1,M/a),\end{aligned}$$

which proves the result.

Following notation in Centeno [2], let $Y(a,M)$ represent net aggregate loss after application of both the proportional and excess reinsurance. Then

$$Y(a,M) = \Sigma_{i=1}^N \min(aX_i,M).$$

We are interested in the stability of $Y(a,M)$ as a decreases. The following rule characterizes the stability of Y as a changes.

Mixing Stability Rule: The stability (coefficient of variation) of net aggregate losses after retention of $100a\%$ under proportional reinsurance and retention of M under an excess of loss treaty is equivalent to the stability of net aggregate losses under an excess treaty with a retention of M/a.

Proof: Write the coefficient of variation in terms of λ_i and $\beta_i(a,M)$,

$$\mathrm{CV}(Y(a,M)) = \frac{[\lambda_1\beta_2(a,M) + (\lambda_2 - \lambda_1 - \lambda_1^2)\ \beta_1(a,M)^2]^{1/2}}{\lambda_1\beta_1(a,M)}$$

$$= \frac{[\lambda_1 a^2\beta_2(1,M/a) + (\lambda_2 - \lambda_1 - \lambda_1^2)a^2\beta_1(1,M/a)^2]^{1/2}}{\lambda_1 a\beta_1(1,M/a)}$$

$$= \frac{[\lambda_1\beta_2(1,M/a) + (\lambda_2 - \lambda_1 - \lambda_1^2)\beta_1(1,M/a)^2]^{1/2}}{\lambda_1\beta_1(1,M/a)}$$

$$= \mathrm{CV}\ (Y(1,M/a)),$$

which proves the result.

We would suspect that the stability of net losses decreases as the retention of the excess of loss treaty increases. This is indeed the case, as shown in the Appendix. Thus, we can conclude that, in general, as the percent retained under proportional reinsurance decreases, and the excess of loss retention M remains fixed, the stabilty of net results of the portfolio decreases.

This shows that the situation of Figure 7 is not the result of any fortuitous choice of distributions or parameters. For any compound process, represented in general by $Y(a,M)$, the distribution of net results after mixed reinsurance will show more "spread" than the pure excess reinsurance case but less than the gross position.

CONCLUSION

The application of an excess of loss treaty after a proportional reinsurance transaction on a policy has been shown to have a significant adverse impact on the net expected loss ratio. In addition, the stability of net results sought from the excess of loss reinsurance is also adversely affected. The Mixing Price Rule and Mixing Stability Rule allow us to evaluate these effects of the mixing situation. The Cost of Mixing Worksheet allows us to calculate the net position in a mixed reinsurance situation. These three tools should allow the underwriter to make appropriate evaluations of pricing and facultative reinsurance decisions in individual risk situations.

From a broader management perspective, the mixing of reinsurance at the individual risk level presents a difficult management control issue. In a worst case scenario, if company underwriters were to make facultative reinsurance arrangements without proper coordination and direction from management, a substantial loss ratio penalty on the entire book of business could be expected. Extremely adverse fluctuations in net results would also be possible. The challenge for management is to establish guidelines and controls enabling underwriters to understand the structure and objectives of overall corporate reinsurance. The underwriters will then be able to make decisions on individual risk facultative reinsurance placements that work with, not against, the excess treaty. It is hoped that the ideas developed here will give actuaries a start in attempting to explore this aspect of the underwriting and pricing process.

Pricing a risk at a profitable direct premium is not sufficient to assure a net profit when significant amounts of different reinsurances apply. As our examples show, one can price the risk perfectly on a direct basis, yet still have an unfavorable net combined ratio, due to facultative placements with high mixing costs.

On a corporate level, the more subtle concept of probability of ruin comes into play. We have shown that unanticipated large amounts of proportional placements can destabilize net results significantly. While most insurance organizations are large enough to make the probability of ruin of academic interest only, the chance of suffering extremely large combined ratios increases as the share retained on a proportional basis decreases. The protection in the excess treaty is negated by proportional reinsurance.

Finally, most of the discussion has been from the viewpoint of the ceding company. The mixing cost, however, can work both ways. The excess treaty rate is calculated anticipating a certain percent of the book will be ceded proportionally before the treaty applies. If the ceding company finds that it can only cede a smaller than anticipated portion of its business facultatively, it will be putting larger shares of each risk into the treaty. This will result in a highly leveraged adverse loss ratio and destabilization effect on the excess treaty. This is a sensitive issue for both the excess reinsurer and the ceding company.

Pricing actuaries on both sides of the excess reinsurance treaty transaction have an interest in the mixing effects. The more use a ceding company makes of proportional reinsurance prior to the treaty, the more important the mixing effect becomes. An increased awareness of the effects of mixing should decrease the likelihood of unexpected adverse consequences to both treaty partners.

APPENDIX

Theorem: As the fraction a retained under proportional reinsurance decreases, the stability of the net aggregate losses decreases.

Proof: We wish to prove that as a decreases, the quantity $CV(Y(a,M))$ decreases. From the Mixing Stability Rule, it suffices to prove that if $M_1 < M_2$, then,

$$CV(Y(1,M_1)) < CV(Y(1,M_2)).$$

This is the case if

$$(\delta/\delta M)\ CV(Y(1,M)) > 0,$$

which is equivalent to

$$(\delta/\delta M)\ CV^2(Y(1,M)) > 0, \text{ because } CV \geq 0.$$

Let β_k represent $\beta_k(1,M)$; then

$$CV^2(Y(1,M)) = \frac{\lambda_1\beta_2 + (\lambda_2 - \lambda_1^2 - \lambda_1)\beta_1^2}{\lambda_1^2\beta_1^2}$$

$$= \frac{\beta_2}{\lambda_1\beta_1^2} + \frac{(\lambda_2 - \lambda_1^2 - \lambda_1)}{\lambda_2^2}.$$

Since only β_k is a function of M,

$$(\delta/\delta M)\ CV^2(Y(1,M)) = \frac{\lambda_1\beta_1^2\beta_2' - 2\beta_2\lambda_1\beta_1'\beta_1}{(\lambda_1\beta_1^2)^2}$$

$$= \frac{\beta_1\beta_2' - 2\beta_2\beta_1'}{\lambda_1\beta_1^3}.$$

Thus, $(\delta/\delta M)\ (CV^2(Y(1,M)) > 0$ if and only if

$$\beta_1\beta_2' - 2\beta_2\beta_1' > 0.$$

Now compute β_1' and β_2',

$$(\delta/\delta M)\ \beta_1 = \delta/\delta M\ (\textstyle\int_0^M x\,dF + M(1 - F(M)))$$
$$= 1 - F(M), \text{ and}$$

$$(\delta/\delta M)\ \beta_2 = \delta/\delta M\ (\textstyle\int_0^M x^2\,dF + M^2(1 - F(M)))$$
$$= 2M(1 - F(M)).$$

$$\text{Let } I_1 = \int_0^M x\,dF \text{ and}$$
$$I_2 = \int_0^M x^2\,dF.$$

Then,
$$\beta_1\beta_2' = [I_1 + M(1 - F(M))]\,[2M(1 - F(M))], \text{ and}$$
$$2\beta_2\beta_1' = 2[I_2 + M^2(1 - F(M))]\,[1 - F(M)].$$

So,

$$\begin{aligned} \beta_1\beta_2' - 2\beta_2\beta_1' &= 2I_1M(1 - F(M)) - 2I_2(1 - F(M)) \\ &= 2(1 - F(M))\,(MI_1 - I_2) \\ &= 2(1 - F(M)) \int_0^M x(M - x)\,dF. \end{aligned}$$

Since $0 < x < M$, we know $M - x > 0$; hence, this integral is positive, and the result is proved.

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REFERENCES

[1] L. Benckert, "The Lognormal Model for the Distribution of One Claim," *ASTIN Bulletin*, Volume 2, Number 1, 1962.

[2] L. Centeno, "On Combining Quota-Share and Excess of Loss," *ASTIN Bulletin*, Volume 15, Number 1, 1985.

[3] H. Bühlmann, *Mathematical Methods in Risk Theory*, Springer-Verlag, 1970, p. 86.

[4] D. Shpilberg, "The Probability Distribution of Fire Loss Amount," The *Journal of Risk and Insurance*, 1977.

[5] N. Buchanan, "The Gearing Factor Explained," *Reinsurance*, February, 1985, p. 436.

[6] P. Heckman and G. Meyers, "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," *PCAS* LXX, 1983.

[7] R. Beard, T. Pentikäinen, and E. Pesonen, *Risk Theory*, Third Edition, Chapman and Hall, 1984.

[8] Beard, p. 172.

[9] Beard, p. 173.

[10] Centeno, p. 49.

[11] R. Miccolis, "On the Theory of Increased Limits and Excess of Loss Pricing," *PCAS* LXIV, 1977.