# A BAYESIAN CREDIBILITY FORMULA FOR IBNR COUNTS

DR. IRA ROBBIN

*Abstract*

*A formula for IBNR counts is derived as the credibility weighted average of three standard actuarial estimates:*

| *Estimate* | *IBNR Formula* |
|---|---|
| *Pegged* | *Initial Estimate of Ultimate* − *Reported to Date* |
| LDF | *(Reported to Date)* × (LDF − *1)* |
| *Bornhuetter-Ferguson* | *Initial Estimate of Ultimate* × *(1* − *1*/LDF*)* |

*Here* LDF *denotes the age-to-ultimate development factor. The credibility weights vary by age of development in a methodical fashion reflecting prior belief in the reporting pattern and the estimate of ultimate.*

*To derive the formula, IBNR is modelled as a parametrically dependent random variable. Bayes Theorem leads to a natural revision of the prior distribution of the parameters based on the data to date. Using the best least squares linear approximation to the true Bayesian estimate, and performing some algebraic manipulations, the credibility formula is obtained. While the formula could be applied in many ways, for demonstration purposes a fully automatic procedure is applied to three hypothetical triangles of data.*

## I. INTRODUCTION

This paper will present a formula which estimates IBNR (Incurred But Not Reported) claim counts in terms of a credibility weighted average of more traditional actuarial estimates. The formula will be derived from a theoretical foundation using Bayesian analysis methods applied to claim count development models.

Before presenting the formula, it is instructive to review the traditional actuarial estimates under discussion. In the usual context, we are estimating IBNR counts for an exposure period at a certain stage of development. We are given, or can obtain some preliminary estimate of ultimate counts that does not depend on the count data reported to date. For instance, the preliminary estimate could be the product of expected frequency times exposures, where the expected frequency is calculated with data from prior exposure periods. We also have count data reported to date and a set of expected age-to-ultimate count loss development factors (*LDF*). With all this information, three different IBNR count estimates may be obtained for the exposure period in question at its current stage of development.

1. Pegged Method
   IBNR = Preliminary Estimate of Ultimate Counts
   − Counts Reported to Date
2. Loss Development Factor Method
   IBNR = Counts Reported to Date × (*LDF* − 1)
3. Bornhuetter-Ferguson Method
   IBNR = Preliminary Estimate of Ultimate Counts × (1 − 1/LDF)

To decide amongst these, the actuary has heretofore been forced to rely on qualitative reasoning. Such "actuarial judgement" is not necessarily the arbitrary Delphic process one might suppose. For instance, if the actuary knows from long experience that reporting patterns are generally stable, the *LDF* method would be preferred. If reporting patterns have characteristically been erratic and the preliminary estimate of ultimate counts is generally near the mark, the pegged estimate would be favored. Such qualitative reasoning involves implicit non-quantified assumptions regarding the stochastic variability of ultimate claim counts and reporting patterns. It also reflects the degree of confidence in the prelimary estimate of expected ultimate counts and in the expected *LDF*.

By constructing an explicitly stochastic claims development model, and making Bayesian prior assumptions on the parameters defining the model, one advances the art of reserving beyond the realm of qualitative guesswork. Theoretically, Bayes Theorem leads to revised IBNR estimates reflecting prior belief appropriately modified by the data to date. Unfortunately, the mathematics often becomes intractable. Thus, one is led to considering linear estimators with least squared error.

The simplest general estimator one obtains can be expressed as a credibility weighted average of the three traditional estimates. The credibility weights vary with the stage of development, so that, for instance, the pegged estimate might receive the most weight initially, the Bornhuetter-Ferguson estimate might predominate for a few subsequent periods, and the loss development estimate could have the most weight thereafter. This methodical evolution of credibility weights is perhaps the key practical advantage of the Bayesian approach. Based on our initial beliefs, we are able to decide when to give each method credence.

The object of this paper is to present the formula and demonstrate one method of applying it to a triangle of data. The method of application uses the data to approximate needed parameters, so that, in the end, one has an automated procedure for estimating IBNR counts. Other methods of application are possible.

Finally, it should be noted that the theory leads naturally to an estimate of the variance of the IBNR counts. This variance reflects both process and parameter uncertainty.

## II. BAYESIAN ANALYSIS OF COUNT DEVELOPMENT MODELS

Let $N$ denote the ultimate number of claims for a fixed set of exposures and write $N_j$ for the counts reported in the $j^{\text{th}}$ development period. Set $M_j = N_1 + \ldots + N_j$ so that $M_j$ denotes the counts reported to date as of the end of the $j^{\text{th}}$ period. Define the IBNR count as of the end of the $j^{\text{th}}$ period as $R_j$. Thus, $R_j$ can be written as the sum, $N_{j+1} + N_{j+2} + \ldots + N_u$, where $u$ is the number of periods until ultimate, or one can write $R_j = N - M_j$.

Assume the $N_j$ are (conditionally) independent Poisson random variables whose parameters we denote as $n_j$. It follows that $N$, $M_j$, and $R_j$ are also Poisson distributed, since the sum of independent Poisson variables is Poisson. Let $n = n_1 + \ldots + n_u$ and define $p_j = n_j/n$. Thus, the sum of the $p_j$ is unity. Also, set $q_j = p_{j+1} + \ldots + p_u$. We summarize the random variables thus far defined:

### II.1. *Conditional Poisson Random Variables*

| Variable | Description | Poisson Parameter |
|---|---|---|
| $N_j$ | Counts Reported During Period $j$ | $n_j = np_j$ |
| $M_j$ | Counts Reported as of Period $j$ | $m_j = n(1 - q_j)$ |
| $R_j$ | IBNR Counts as of Period $j$ | $r_j = nq_j$ |
| $N$ | Ultimate Counts | $n$ |

subject to constraints

(*i*) $0 \leq p_j \leq 1$

(*ii*) $p_1 + p_2 + \ldots + p_u = 1$

Next we define $LDF_j = N/M_j$ when $M_j$ is strictly positive. Though not strictly true mathematically, we may from time to time estimate $\mathrm{E}(LDF_j)$ as $1/(1 - q_j)$.

It should be further noted that the parameter $p_j$ is distinct from, but related to, the ratio random variable, $N_j/N$. Maintaining the assumption that the parameters $n$ and $p_j$ are fixed, one can show:

II.2. *Relation of $p_j$ to $N_j/N$*

$$p_j = \mathrm{E}(N_j/N \mid N > 0)$$

Proof

See Appendix A.

Next, we allow the parameters $n$ and $p_j$ to vary according to some prior distribution whose density we write as $f(n,p)$. Unconditional expectation and variance formulas for $N$, $N_j$, $M_j$, and $R_j$ can then be derived in terms of expectations and variances involving $n$, $p_j$, and $q_j$.

II.3. *Expectation and Variance Formulas*

(*i*) $N$

$$\mathrm{E}(N) = \mathrm{E}(n)$$
$$\mathrm{Var}(N) = \mathrm{E}(n) + \mathrm{Var}(n)$$

(*ii*) $N_j$

$$\mathrm{E}(N_j) = \mathrm{E}(p_jn)$$
$$\mathrm{Var}(N_j) = \mathrm{E}(p_jn) + \mathrm{Var}(p_jn)$$

(*iii*) $M_j$

$$\mathrm{E}(M_j) = \mathrm{E}[(1 - q_j)n]$$
$$\mathrm{Var}(M_j) = \mathrm{E}[1 - q_j)n] + \mathrm{Var}((1 - q_j)n)$$

(*iv*) $R_j$

$$\mathrm{E}(R_j) = \mathrm{E}(q_jn)$$
$$\mathrm{Var}(R_j) = \mathrm{E}(q_jn) + \mathrm{Var}(q_jn)$$

Proof

We prove only *(ii)* and leave the rest as an exercise for the reader. Consider

$$\begin{aligned}
E(N_j) &= E_{n,p}(E(N_j/n,p)) \\
&= E_{n,p}(np_j) = E(np_j) \\
E(N_j^2) &= E_{n,p}(E(N_j^2/n,p)) \\
&= E((np_j)^2) + E(np_j)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathrm{Var}(N_j) &= E(N_j^2) - (E(N_j))^2 \\
&= E((np_j)^2) + E(np_j) - (E(np_j))^2 \\
&= \mathrm{Var}(p_j n) + E(p_j n)
\end{aligned}$$

Before providing a simple example demonstrating these concepts, it should be noted that in writing $f(n,p)$ we have implicitly incorporated the constraints on the $p$ parameters. In applications, these restrictions must be explicitly reflected. One way to do this is to define the $p_j$ as functions of some other parameters in such a way that the constraints are automatically satisfied. Letting $g$ denote these generating parameters, we may write $f(n,p(g))$ or $f(n,g)$.

Now, for a simple example to demonstrate these concepts suppose:

II.4. *Assumptions for Example*

(i) The prior distribution for $n$ is a gamma with a mean of 1,000 and a variance of 10,000.

$$f(n) = \left(\frac{1}{10}\right)^{100} \frac{1}{99!} n^{99} \quad e^{-n/10}$$

$$E(n) = 1{,}000 \qquad E(n^2) = 1{,}010{,}000$$

(ii) (a) $\bar{p}$ and $\bar{q}$ are given via:

$$\begin{aligned}
p_1 &= 1 - g_1 & q_1 &= g_1 \\
p_2 &= g_1(1 - g_2) & q_2 &= g_1 g_2 \\
p_3 &= g_1 g_2 & q_3 &= 0
\end{aligned}$$

where $g_j \in (0,1)$. (Observe that the constraints on the $p_j$ are automatically satisfied.)

(b) The prior joint distribution for $g_1$ and $g_2$ is

$$f(g_1,g_2) = 2(1 - g_2)$$

We compute the first and second moments of the $p$, $1 - q$, and $q$ variables:

II.5. *First and Second Moments of $p$, $1 - q$, and $q$ Variables in Example*

| | First Moments | | | Second Moments | | |
|---|---|---|---|---|---|---|
| $j$ | $E(p_j)$ | $E(1 - q_j)$ | $E(q_j)$ | $E(p_j^2)$ | $E((1 - q)^2)$ | $E(q_j^2)$ |
| 1 | 1/2 | 1/2 | 1/2 | 1/3 | 1/3 | 1/3 |
| 2 | 1/3 | 5/6 | 1/6 | 1/6 | 13/18 | 1/18 |
| 3 | 1/6 | 1 | 0 | 1/18 | 1 | 0 |

To show how these figures were obtained, we calculate $E(p_2^2)$ in detail.

$$
\begin{aligned}
E(p_2^2) &= \int_0^1 \int_0^1 g_1^2(1 - g_2)^2 2(1 - g_2)\, dg_1\, dg_2 \\
&= \left(g_1^3/3\Big|_0^1\right)\left(-2(1 - g_2)^4/4\Big|_0^1\right) \\
&= (1/3)(1/2) = 1/6
\end{aligned}
$$

We are now in a position to compute the means, variances, and standard deviations of the various count random variables.

II.6. *Means, Variances, and Standard Deviations of $N$, $N_j$, $M_j$, and $R_j$ in Example*

Means

$E(N) = 1{,}000$

| $j$ | $E(N_j)$ | $E(M_j)$ | $E(R_j)$ |
|---|---|---|---|
| 1 | 500 | 500 | 500 |
| 2 | 333 | 833 | 167 |
| 3 | 167 | 1,000 | 0 |

Variances

$\mathrm{Var}(N) = 11{,}000$

| $j$ | $\mathrm{Var}(N_j)$ | $\mathrm{Var}(M_j)$ | $\mathrm{Var}(R_j)$ |
|---|---|---|---|
| 1 | 87,167 | 87,167 | 87,167 |
| 2 | 57,558 | 35,889 | 28,502 |
| 3 | 28,502 | 11,000 | 0 |

Standard Deviations

$\mathrm{Var}^{1/2}(N) = 105$

| $j$ | $\mathrm{Var}^{1/2}(N_j)$ | $\mathrm{Var}^{1/2}(M_j)$ | $\mathrm{Var}^{1/2}(R_j)$ |
|---|---|---|---|
| 1 | 295 | 295 | 295 |
| 2 | 240 | 189 | 169 |
| 3 | 169 | 105 | 0 |

Again demonstrating one of the calculations in more detail, we compute:

$$\begin{aligned}\mathrm{Var}(N_2) &= \mathrm{Var}(np_2) + \mathrm{E}(np_2)\\ &= \mathrm{E}(n^2)\mathrm{E}(p_2^2) - \mathrm{E}(n)\mathrm{E}(p_2)^2 + \mathrm{E}(np_2)\\ &= (1{,}010{,}000)(1/6) - (333)^2 + 333 = 57{,}556\end{aligned}$$

We return now to the general presentation and follow the Bayesian approach by modifying our beliefs about the parameter distribution, $f(n,p)$, as more data becomes available. Let $f^{(0)}$ denote the prior density before any development has occurred, and let $f^{(j)}$ denote the revised density as of the end of the $j^{\text{th}}$ period of development. Given development data $(N_1 = x_1, N_2 = x_2, \ldots, N_j = x_j)$, Bayes Theorem allows one to derive the modified belief density, $f^{(j)}$, in sequential fashion.

II.7 *Bayes Revised Belief Density*

$$f^{(j)}(n,p) = c\ \mathrm{Prob}(N_j = x_j/n,p)\, f^{(j-1)}(n,p)$$

where $c$ is a normalization constant, and

$$\mathrm{Prob}(N_j = x/n,p) = \exp(-np_j)(np_j)^x/x!$$

Equivalently, one can write

$$f^{(j)}(n,p) = c\ L(n,p/x_1, x_2, \ldots, x_j)f^{(0)}(n,p)$$

where $c$ is some normalization constant and $L$ is the likelihood function,

$$L = \prod_{i=1}^{j} \text{Prob}(N_i = x_i/n,p)$$

The revised belief density yields revised IBNR count estimates via II.3.

Thus, the IBNR count estimation problem is theoretically solved. Further, the variance equation in II.3 (iv) could be used to calculate the standard deviation of the IBNR estimate. This deviation would reflect both process and parameter uncertainty.

Returning to our example, our prior density is:

$$f^{(0)}(n,g_1,g_2) = 2(1 - g_2)\ \frac{1}{10^{100}}\ \frac{1}{99!}\ n^{99}e^{-n/10}$$

If we observe $N_1 = 400$, the revised parameter density would be

$$f^{(1)}(n,\ g_1,\ g_2) = c\ e^{-n(1-g_1)}\ (n(1 - g_1))^{400}\ (1 - g_2)\ n^{99}\ e^{-n/10}$$

where $c$ is a normalizing constant. This density is rather inconvenient to work with.

Such difficulties are not peculiar to this example. Indeed, the computations become intractable in most interesting models. Thus, the formulas are difficult to apply and consequently of limited practical use. As is usually the case in Bayesian analysis, one is led to consider linear estimators.

## III. LINEAR APPROXIMATION OF THE BAYESIAN ESTIMATOR

We first recall some general results of Bayesian credibility theory. Let $X$ and $Y$ be (possibly vector-valued) random variables, each parameterized by a common (vector) parameter. Assume the distribution of the parameter is governed by some underlying structure function. We consider linear estimators of $Y$ given results for $X$. It is known that the linear estimator, $Y^*$, with least mean square error (against the Bayesian estimator) is given via:

III.1. *General Least Squares Linear Approximation*

$$Y^* = \mathrm{E}(Y) + C(X,Y')\mathrm{V}(X)^{-1}(X - \mathrm{E}(X))$$

where

$X$ and $Y$ are column vectors

$X'$ = transpose of $X$

$C(X,Y) = \mathrm{Cov}(X,Y) = \mathrm{E}(XY') - \mathrm{E}(X)\mathrm{E}(Y')$

$\mathrm{V}(X) = C(X,X')$

Applying this result with $X = (N_1, \ldots, N_j)'$ and $Y = R_j$, we obtain:

III.2. *General Linear IBNR Count Estimator*

$$R_j^* = \mathrm{E}(R_j) +$$

$$(C(N_1, R_j), \ldots, C(N_j, R_j)) \begin{bmatrix} C(N_1,N_1) & \ldots & C(N_1,N_j) \\ & \ldots & \\ C(N_j, N_1) & \ldots & C(N_j, N_j) \end{bmatrix}^{-1} \begin{bmatrix} N_1 - \mathrm{E}(N_1) \\ \ldots \\ N_j - \mathrm{E}(N_j) \end{bmatrix}$$

The quantities in the above equation can be expressed in terms of expectations, variances, and covariances of the $n$, $p$, and $q$.

III.3. *Expectation Variance and Covariance Formulas*

(i) $\mathrm{E}(R_i) = \mathrm{E}(n)\,\mathrm{E}(q_i)$

(ii) For $i \geq j$
$C(N_j, R_i) = \mathrm{E}(p_j)\,\mathrm{E}(q_i)\,\mathrm{V}(n) + \mathrm{E}(n^2)\,C(p_j,q_i)$

(iii) $C(N_i, N_j) = \mathrm{E}(n^2)\,C(p_i, p_j) + \mathrm{E}(p_i)\,\mathrm{E}(p_j)\,\mathrm{V}(n) + \delta_{ij}\,\mathrm{E}(n)\,\mathrm{E}(p_i)$

where $\mathrm{V}(X) = \mathrm{Var}\,(X)$

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Formula III.2 is thus reasonable to apply in practice and there is no necessity for further simplification due to computational considerations. However, with one additional simplification, we achieve a formula expressing the estimator as a credibility weighted average of the traditional actuarial estimators as discussed in the introduction.

Applying III.1 with $X = M_j$ and $Y = R_j$, and grouping terms appropriately (as shown in Appendix B), we obtain

III.4. *Credibility Weighting Formula for IBNR Counts*

$$R_j^* = Z_{nj}(\mathrm{E}(n) - M_j) + Z_{pj}\frac{M_j\mathrm{E}(q_j)}{1 - \mathrm{E}(q_j)} + (1 - Z_{nj} - Z_{pj})\,\mathrm{E}(n)\mathrm{E}(q_j)$$

where

$$Z_{nj} = \mathrm{E}(n^2)\mathrm{V}(1 - q_j)/D_j$$

$$Z_{pj} = \mathrm{E}(1 - q_j)^2\mathrm{V}(n)/D_j$$

and

$$D_j = \mathrm{E}(n^2)\mathrm{V}(1 - q_j) + \mathrm{E}(1 - q_j)^2\mathrm{V}(n) + \mathrm{E}(n)\mathrm{E}(1 - q_j)$$

Approximating $\mathrm{E}(q_j)$ via $(1 - 1/LDF_j)$, we have:

III.5. *Credibility Weighting Formula for IBNR Counts - LDF Notation*

$$\begin{aligned} R_j^* = {} & Z_{nj}\,(\mathrm{E}(n) - (M_j)) + Z_{pj}\,(M_j)\,(LDF_j - 1) \\ & + (1 - Z_{nj} - Z_{pj})\,\mathrm{E}(n)\,(1 - 1/LDF_j) \end{aligned}$$

This is the formula promised at the outset since in this notation the traditional estimates may be expressed as:

| IBNR Estimate | Expression |
|---|---|
| Pegged | $\mathrm{E}(n) - M_j$ |
| *LDF* | $M_j(LDF_j - 1)$ |
| Bornhuetter-Ferguson | $\mathrm{E}(n)\,(1 - 1/LDF_j)$ |

There are several qualitative conclusions that can be drawn from the formula. First, if there is no parameter uncertainty with respect to both ultimate counts and reporting patterns, then the data to date is given no credibility. In that case, the formula reduces to a Bornhuetter-Ferguson type estimate.

If there is some parameter uncertainty regarding counts, but none regarding reporting patterns, then the formula become a weighted average of loss development factor and Bornhuetter-Ferguson estimates. As the count parameter uncertainty increases, the formula approaches a loss development factor estimate. Finally, if there is some parameter uncertainty about reporting patterns, but none regarding counts, then the formula becomes a weighted average of pegged and Bornhuetter-Ferguson estimates.

## IV. APPLICATION

In this section, the formula will be applied to three triangles of hypothetical data. The first triangle was constructed so that the Bornhuetter-Ferguson method will work almost exactly. The second triangle was generated to have nearly constant age-to-age factors. The last triangle is obtained by averaging the counts from the original triangles.

The formula could be applied in many different ways. For instance, a pure Bayesian approach would entail making explicit assumptions for the forms and parameters of the prior distributions. The resulting system would then require actuarial judgement in setting the parameters appropriately each time it was run. While this would be the most theoretically pure method of application, it might be regarded as somewhat impractical.

In order to provide a reasonably convincing demonstration that the formula is of practical use, we proceed now to present a fully automatic method of application. Under this particular approach, we let the data dictate parameter values to the degree possible. We introduce explicit forms for prior distributions if needed, but let the data determine the parameters of the priors.

To begin the application in detail, assume that a triangle of data is given. Let $N_{ij}$ denote the counts reported in the $j^{\text{th}}$ development period for the $i^{\text{th}}$ accident period, where $i = 1, 2, \ldots, u$ and $j = 1, 2, \ldots, u - i + 1$. Define $M_{ij}$ and $R_{ij}$ in a fashion analogous to the definitions of $M_j$ and $R_j$ in II.

Assume $N_{ij}$ is (conditionally) Poisson distributed with parameter $n_{ij} = B_i w_{ij}$, where $B_i$ denotes the exposures for the $i^{\text{th}}$ accident year.

Define:

$$n_i = \sum_j n_{ij}$$

$$w_i = \sum_j w_{ij}$$

$$p_{ij} = w_{ij}/w_i$$

so that $n_{ij} = B_i w_i p_{ij}$

Now assume that each of the frequency parameters, $w_i$, is, in effect, drawn from a common distribution. Thus, a priori, we have $E(w_i) = E(w)$. Similar assumptions are made for the set of $p_{ij}$ and the set of $q_{ij}$ when $i$ is fixed. Thus, we may write $E(p_{ij}) = E(p_j)$ and $E(q_{ij}) = E(q_j)$.

We next find maximum likelihood estimators, $w_i^*$ and $p_j^*$, for $w_i$ and $p_j$. The likelihood function is:

IV.1. *Likelihood Function*

$$L(\bar{p}, w/\bar{N}) = \prod_{i=1}^{u} \prod_{j=1}^{u-i+1} e^{-B_i w_i p_j}(B_i w_i p_j)^{N_{ij}}/N_{ij}!$$

subject to $p_1 + p_2 + \ldots + p_u = 1$

We maximize as usual by taking the natural log and then the necessary partial derivatives.

IV.2. *"Log Likelihood" and Partials*

$$\ln L = \sum_{i=1}^{u} \sum_{j=1}^{u-i+1} - B_i w_i p_j + N_{ij} \ln(w_i p_j)$$

$$+ \text{ independent terms of } w_i \text{ and } p_j$$

$$\frac{\partial \ln L}{\partial w_i} = \sum_{j=1}^{u-i+1} - B_i p_j + N_{ij}/w_i$$

$$\frac{\partial \ln L}{\partial p_j} = \sum_{i=1}^{u-j+1} - B_i w_i + N_{ij}/p_j$$

Utilizing the constraint, we solve the equations via numerical iteration to obtain $w_i^*$ and $p_j^*$ which satisfy:

IV.3. *Maximum Likelihood Estimates*

$$w_i^* = M_{i,u-i+1}/B_i(1 - q_{u-i+1}^*)$$

$$p_j^* = \left(\sum_{i=1}^{u-j+1} N_{ij}\right) \Big/ \left(\sum_{i=1}^{u-j+1} w_i^* B_i\right)$$

Using the maximum likelihood estimates just obtained, we approximate the frequency mean and frequency variance.

IV.4. *Frequency Mean and Variance Estimators*

$$E(w) \approx \bar{w} = \left[\frac{\Sigma B_i w_i^*(1 - q_{u-i+1}^*)}{\Sigma B_i(1 - q_{u-i+1}^*)}\right]$$

$$\text{Var}(w) \approx S_w^2 = \left[\frac{\Sigma B_i(1 - q_{u-i+1}^*)(w_i^* - \bar{w})^2}{\Sigma B_i(1 - q_{u-i+1}^*)}\right]$$

While this seems intuitively reasonable, the properties of this variance estimator need further investigation in the future. Perhaps it is biased.

To estimate the required second moments of the reporting pattern parameters, we assume that $p_{ij}$ is Beta distributed with parameters $(Hp_j^*, H(1 - p_j^*))$. We further have that $q_{ij}$ is Beta distributed with parameters $(Hq_j^*, H(1 - q_j^*))$. Note the use of the maximum likelihood estimates in defining the parameters of these Betas. Under these assumptions, we can obtain convenient expressions for the mean and variance of the reporting pattern parameters.

IV.5. *Mean and Variance of* $p_{ij}$ *and* $q_{ij}$

$$E(p_{ij}) = p_j^* \qquad \mathrm{Var}(p_{ij}) = p_j^*(1 - p_j^*)/(1 + H)$$

$$E(q_{ij}) = q_j^* \qquad \mathrm{Var}(q_{ij}) = q_j^*(1 - q_j^*)/(1 + H)$$

Observe that the parameters of the reporting pattern have variances inversely proportional to $H$. To use the data to solve for $H$, we first estimate $p_{ij}$ via: $\hat{p}_{ij} = N_{ij}/(M_{i,u-i+1} + B_i w_i^* q_j^*)$ and define

IV.6. *Estimator For Variance of Reporting Pattern Parameters*

$$S_p^2 = \left[\sum_{i=1}^{u} \sum_{j=1}^{u-i+1} B_i(\hat{p}_{ij} - p_j^*)^2\right] \Big/ \sum_i \sum_j B_i$$

Plugging the $\mathrm{Var}(p_{ij})$ formula of IV.5 in place of $(\hat{p}_{ij} - p_j^*)^2$, we obtain the approximation

$$E(S_p^2) = \sum_{ij} B_i p_j^*(1 - p_j^*) / \sum_{ij} B_i(1 + H).$$

Thus we derive an estimator for $H$:

IV.7. *Estimator for H*

$$H^* = \frac{\sum_{ij} B_i\, p_j^*\, (1 - p_j^*)}{S_p^2 \sum_{ij} B_i} - 1$$

As before, the author must caution that the theoretical vices or virtues of this estimator have not been investigated. It is probably biased toward overstating $H$ and thus understating $\mathrm{Var}(1 - q_j)$. This will tend to give too much credibility to the *LDF* method.

At this point, we have enough to estimate all the terms required in the credibility formulas.

IV.8. *Estimators for Terms in Credibility Formulas*

| Notation Used in Chapter II | Notation Used in Chapter IV | Estimator |
|---|---|---|
| $E(n)$ | $E(n_i)$ | $B_i\bar{w}$ |
| $Var(n)$ | $Var(n_i)$ | $B_i^2S_w^2$ |
| $E(n^2)$ | $E(n_i^2)$ | $B_i^2S_w^2 + B_i^2\bar{w}^2$ |
| $E(1 - q_j)$ | $E(1 - q_{ij})$ | $1 - q_j^*$ |
| $Var(1 - q_j)$ | $Var(1 - q_{ij})$ | $(1 - q_j^*)q_j^*/(H^* + 1)$ |

These were used to obtain the Bayesian credibility IBNR estimates shown in the attached exhibits. While the credibilities are not 100% for the "right" method in the "pure" cases, they nonetheless show that the application methodology is at least somewhat responsive. The credibility estimated IBNR is in all cases reasonably close to the correct answer. Further, the correct answer is well within one standard deviation of the estimate. Finally, considered over all three examples, the credibility formula approach appears to perform better than any one of the methods alone. The reader will, of course, arrive at his or her own judgement.

## V. CONCLUSION

To conclude, it is hoped that the proposed IBNR count formula will not only advance reserving theory, but will also prove of practical use. It settles old arguments about which of three traditional actuarial estimates should be employed by showing how they may be credibility weighted in a methodical fashion to obtain a final estimate. The credibility weights differ depending on the development period. Thus, the Bayesian credibility approach provides a far more subtle method than simply picking one set of credibility weights which would apply at every development period. The formula could be applied in many ways, but at least one practical application has been demonstrated with fairly good results.

## APPENDIX A

Let $N_1$ and $N_2$ be two independent Poisson random variables with parameters $n_1$ and $n_2$, respectively. Set $n = n_1 + n_2$ and $p = n_1/n$. We consider the ratio random variable $N_1/(N_1 + N_2)$.

A.1. *Proposition on Expectation*

$$E(N_1/(N_1 + N_2)/N_1 + N_2 > 0) = p$$

*Proof*

$$E(N_1/(N_1 + N_2)/N_1 + N_2 > 0) \text{ Prob } (N_1 + N_2 > 0)$$

$$= e^{-n} \sum_{x=1}^{\infty} \sum_{y=0}^{\infty} (x/(x + y))n_1^x n_2^y/(x!y!)$$

$$= e^{-n} \sum_{x=1}^{\infty} \sum_{z=x}^{\infty} (x/z)n_1^x n_2^{z-x}/(x!(z - x)!)$$

$$= e^{-n} \sum_{z=1}^{\infty} z^{-1}(z!)^{-1}(n)^z \sum_{x=1}^{z} \binom{z}{x} x\, p^x(1 - p)^{z-x}$$

$$= e^{-n} \sum_{z=1}^{\infty} z^{-1}(z!)(n)^z z_p = pe^{-n}(e^n - 1)$$

$$= p(1 - e^{-n})$$

The result follows since

$$\text{Prob } (N_1 + N_2 > 0) = 1 - e^{-n}$$

## APPENDIX B
## DERIVATION OF CREDIBILITY WEIGHTING FORMULA FROM GENERAL LINEAR LEAST SQUARE ERROR BAYESIAN APPROXIMATION

Applying the general formula yields

*B.1.*

$$R_j^* = \mathrm{E}(R_j) + C(M_j,R_j)C(M_j)^{-1}(M_j - \mathrm{E}(M_j))$$

Expressing the terms of B.1 using terms involving $n$ and $q_j$,

*B.2.*

$$\begin{aligned}
\mathrm{E}(M_j) &= \mathrm{E}(n)\mathrm{E}(1 - q_j) \\
\mathrm{E}(R_j) &= \mathrm{E}(n)\mathrm{E}(q_j) \\
C(M_j,R_j) &= \mathrm{E}(n^2)\ \mathrm{E}((1 - q_j)q_j) - \mathrm{E}(n)^2\mathrm{E}(1 - q_j)\mathrm{E}(q_j) \\
C(M_j) &= \mathrm{E}(n^2)\ \mathrm{E}((1 - q_j)^2) + \mathrm{E}(n)\mathrm{E}(1 - q_j) - \mathrm{E}(n)^2\mathrm{E}(1 - q_j)^2
\end{aligned}$$

Simplify the second order terms as follows

*B.3.*

$$\begin{aligned}
\text{(i)}\ C(M_j,R_j) &= \mathrm{E}(n^2)\ \mathrm{E}((1 - q_j)q_j) - \mathrm{E}(n^2)\ \mathrm{E}(1 - q_j)\mathrm{E}(q_j) \\
&\quad + \mathrm{E}(n^2)\ \mathrm{E}(1 - q_j)\ \mathrm{E}(q_j) - \mathrm{E}(n)^2\mathrm{E}(1 - q_j)\mathrm{E}(q_j) \\
&= \mathrm{E}(n^2)\ \mathrm{E}(1 - q_j)^2 - \mathrm{E}(n^2)\ \mathrm{E}((1 - q_j)^2) \\
&\quad + \mathrm{V}(n)\mathrm{E}(1 - q_j)\ \mathrm{E}(q_j) \\
&= -\mathrm{E}(n^2)\ \mathrm{V}(1 - q_j) + \mathrm{V}(n)\mathrm{E}(1 - q_j)\mathrm{E}(q_j) \\
\text{(ii)}\qquad C(M_j) &= \mathrm{E}(n^2)\ \mathrm{E}((1 - q_j)^2) - \mathrm{E}(n^2)\mathrm{E}(1 - q_j)^2 \\
&\quad + \mathrm{E}(n^2)\mathrm{E}(1 - q_j)^2 - \mathrm{E}(n)^2\mathrm{E}(1 - q_j)^2 + \mathrm{E}(n)\mathrm{E}(1 - q_j) \\
&= \mathrm{E}(n^2)\mathrm{V}(1 - qj) + \mathrm{E}(1 - q_j)^2\mathrm{V}(n) + \mathrm{E}(n)\mathrm{E}(1 - q_j)
\end{aligned}$$

Plugging into B.1 one finds

*B.4.*

$$R_j^* = \mathrm{E}(n)\mathrm{E}(q_j) +$$

$$\frac{\mathrm{V}(n)\mathrm{E}(q_j)\mathrm{E}(1-q_j) - \mathrm{E}(n^2)\mathrm{V}(1-q_j)}{\mathrm{E}(n^2)\mathrm{V}(1-q_j) + \mathrm{E}(1-q_j)^2\mathrm{V}(n) + \mathrm{E}(n)\mathrm{E}(1-q_j)}\,(M_j - \mathrm{E}(n)\mathrm{E}(1-q_j))$$

$$\begin{aligned}
&= (\mathrm{E}(n) - M_j)\,(\mathrm{E}(n^2)\mathrm{V}(1-q_j))\,/\,D + (M_j\,\mathrm{V}(n)\,\mathrm{E}(q_j)\,\mathrm{E}(1-q_j))\,/\,D \\
&\quad - (\mathrm{E}(n)\,\mathrm{V}(n)\,\mathrm{E}(q_j)\,\mathrm{E}(1-q_j))\,/\,D \\
&\quad + \mathrm{E}(n)\mathrm{E}(q_j)\,(1 + (\mathrm{V}(n)\mathrm{E}(q)\mathrm{E}(1-q_j) \\
&\quad - \mathrm{E}(n^2)\mathrm{V}(1-q_j))\,/\,D)
\end{aligned}$$

$$\begin{aligned}
&= (\mathrm{E}(n) - M_j)(\mathrm{E}(n^2)\mathrm{V}(1-q_j)\,/\,D) + M_j\,\frac{\mathrm{E}(q_j)}{\mathrm{E}(1-q_j)} \times \frac{\mathrm{V}(n)\mathrm{E}(1-q_j)^2}{D} \\
&\quad + \mathrm{E}(n)\mathrm{E}(q_j)\,(1 - (\mathrm{V}(n)\mathrm{E}(1-q_j)\,(\mathrm{E}(q_j) - 1) - \mathrm{E}(n^2)\mathrm{V}(1-q_j))/D)
\end{aligned}$$

which simplifies immediately to III.4.

## REFERENCES


F. DeVylder, "Estimation of IBNR Claims by Credibility Theory," *Insurance: Mathematics and Economics*, Vol. 1, January 1982, North-Holland.

P. M. Kahn, (Editor), *Credibility Theory and Applications*, 1975, Academic Press.

R. Norberg, "The Credibility Approach to Experience Rating," *Scandinavian Actuarial Journal*, 1979, pp. 131–142, Almqvist and Wiksell.

R. Norberg, "Empirical Bayes Credibility," *Scandanavian Actuarial Journal*, 1980, pp. 177–194, Almqvist and Wiksell.

H. Buhlmann, "Experience Rating and Credibility," *Astin Bulletin*, 4, 1967, pp. 199–207, Tieto Ltd.

H. Buhlmann and E. Straub, "Credibility for Loss Ratios," Actuarial Research Clearing House, 1972.

R. L. Bornhuetter and R. E. Ferguson, "The Actuary and IBNR," *PCAS* LIX, 1972, p. 181.

E. W. Weissner, "Estimation of the Distribution of Report Lags By the Method of Maximum Likelihood," *PCAS* LXV, 1978, p. 1.

# EXHIBIT 1

## SHEET 1
BORNHUETTER-FERGUSON DATA

BAYESIAN CREDIBILITY FORMULA
IBNR ESTIMATION
HYPOTHETICAL DATA

*N(I,J)*
COUNTS REPORTED DURING
DEVELOPMENT PERIOD *J*

| ACCIDENT YEAR (*I*) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 150 | 450 | 225 | 100 | 50 | 25 | 5 |
| 2 | 100 | 25 | 150 | 450 | 225 | 100 | 50 | 25 | |
| 3 | 100 | 75 | 150 | 450 | 225 | 100 | 50 | | |
| 4 | 100 | 15 | 150 | 450 | 225 | 100 | | | |
| 5 | 100 | 50 | 150 | 450 | 225 | | | | |
| 6 | 100 | 25 | 150 | 450 | | | | | |
| 7 | 100 | 75 | 150 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

*M(I,J)*
COUNTS REPORTED TO DATE
DEVELOPMENT PERIOD *J*

| ACCIDENT YEAR (*I*) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 200 | 650 | 875 | 975 | 1,025 | 1,050 | 1,055 |
| 2 | 100 | 25 | 175 | 625 | 850 | 950 | 1,000 | 1,025 | |
| 3 | 100 | 75 | 225 | 675 | 900 | 1,000 | 1,050 | | |
| 4 | 100 | 15 | 165 | 615 | 840 | 940 | | | |
| 5 | 100 | 50 | 200 | 650 | 875 | | | | |
| 6 | 100 | 25 | 175 | 625 | | | | | |
| 7 | 100 | 75 | 225 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

# EXHIBIT 1

## SHEET 2

### BORNHUETTER-FERGUSON DATA

$N(I,J)/B(I)$

DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.500 | 1.500 | 4.500 | 2.250 | 1.000 | 0.500 | 0.250 | 0.050 |
| 2 | 0.250 | 1.500 | 4.500 | 2.250 | 1.000 | 0.500 | 0.250 | |
| 3 | 0.750 | 1.500 | 4.500 | 2.250 | 1.000 | 0.500 | | |
| 4 | 0.150 | 1.500 | 4.500 | 2.250 | 1.000 | | | |
| 5 | 0.500 | 1.500 | 4.500 | 2.250 | | | | |
| 6 | 0.250 | 1.500 | 4.500 | | | | | |
| 7 | 0.750 | 1.500 | | | | | | |
| 8 | 0.150 | | | | | | | |

### AGE-TO-AGE FACTORS

DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1–2 | 2–3 | 3–4 | 4–5 | 5–6 | 6–7 | 7–8 |
|---|---|---|---|---|---|---|---|
| 1 | 4.000 | 3.250 | 1.346 | 1.114 | 1.051 | 1.024 | 1.005 |
| 2 | 7.000 | 3.571 | 1.360 | 1.118 | 1.053 | 1.025 | |
| 3 | 3.000 | 3.000 | 1.333 | 1.111 | 1.050 | | |
| 4 | 11.000 | 3.727 | 1.366 | 1.119 | | | |
| 5 | 4.000 | 3.250 | 1.346 | | | | |
| 6 | 7.000 | 3.571 | | | | | |
| 7 | 3.000 | | | | | | |

## EXHIBIT 1

## SHEET 3
### BORNHUETTER-FERGUSON DATA

### IBNR ESTIMATES

| ACCIDENT YEAR | REPORT PERIOD | REPORTED TO DATE | PEGGED METHOD | *LDF* METHOD | BORNHUETTER-FERGUSON METHOD | BAYESIAN CREDIBILITY METHOD | STANDARD DEV. OF BAYESIAN CRED. IBNR |
|---|---|---|---|---|---|---|---|
| 1 | 8 | 1,055 | −10 | 0 | 0 | 0 | 0 |
| 2 | 7 | 1,025 | 20 | 5 | 5 | 5 | 3 |
| 3 | 6 | 1,050 | −5 | 31 | 30 | 31 | 8 |
| 4 | 5 | 940 | 105 | 77 | 80 | 78 | 13 |
| 5 | 4 | 875 | 170 | 181 | 179 | 181 | 22 |
| 6 | 3 | 625 | 420 | 393 | 404 | 398 | 38 |
| 7 | 2 | 225 | 820 | 1,009 | 855 | 897 | 67 |
| 8 | 1 | 15 | 1,030 | 341 | 1,001 | 948 | 76 |
| TOTAL | | 5,810 | 2,551 | 2,038 | 2,553 | 2,537 | |

# EXHIBIT 1

## SHEET 4
## BORNHUETTER-FERGUSON DATA

### ESTIMATES OF ULTIMATE

| ACCIDENT YEAR | PEGGED METHOD | *LDF* METHOD | BORNHUETTER-FERGUSON METHOD | BAYESIAN CREDIBILITY METHOD |
|---|---|---|---|---|
| 1 | 1,045 | 1,055 | 1,055 | 1,055 |
| 2 | 1,045 | 1,030 | 1,030 | 1,030 |
| 3 | 1,045 | 1,081 | 1,080 | 1,081 |
| 4 | 1,045 | 1,017 | 1,020 | 1,018 |
| 5 | 1,045 | 1,056 | 1,054 | 1,056 |
| 6 | 1,045 | 1,018 | 1,029 | 1,023 |
| 7 | 1,045 | 1,234 | 1,080 | 1,122 |
| 8 | 1,045 | 356 | 1,016 | 963 |

### CREDIBILITIES

| REPORT PERIOD | PEGGED | *LDF* | B-F |
|---|---|---|---|
| 1 | 0.43193 | 0.09885 | 0.46923 |
| 2 | 0.29120 | 0.33820 | 0.37060 |
| 3 | 0.08355 | 0.69136 | 0.22509 |
| 4 | 0.03106 | 0.78064 | 0.18830 |
| 5 | 0.01283 | 0.81165 | 0.17552 |
| 6 | 0.00468 | 0.82550 | 0.16981 |
| 7 | 0.00076 | 0.83218 | 0.16706 |
| 8 | 0.00000 | 0.83347 | 0.16653 |

# EXHIBIT 1

## SHEET 5
## BORNHUETTER-FERGUSON DATA

### MAXIMUM LIKELIHOOD ESTIMATES

| ACCIDENT YEAR | MLE FREQUENCY $W(I)$ | EXPOSURES $B(I)$ | INITIAL ESTIMATED COUNT PARAMETER $(B(I) \times W(I)$* |
|---|---|---|---|
| 1 | 10.550 | 100 | 1,055 |
| 2 | 10.299 | 100 | 1,030 |
| 3 | 10.810 | 100 | 1,081 |
| 4 | 10.173 | 100 | 1,017 |
| 5 | 10.560 | 100 | 1,056 |
| 6 | 10.175 | 100 | 1,017 |
| 7 | 12.274 | 100 | 1,227 |

| | |
|---|---|
| ESTIMATED FREQUENCY MEAN | 10.45106 |
| ESTIMATED FREQUENCY VARIANCE | .52307 |

| REPORT PERIOD | MLE $P(J)$ | PERCENT REPORTED $E(P_j)$ | PERCENT REPORTED TO DATE $E(1 - Q_j)$ | PERCENT UNREPORTED $E(Q_j)$ | AGE-TO-AGE FACTORS | FACTORS TO ULTIMATE |
|---|---|---|---|---|---|---|
| 1 | 0.042 | 4.2 | 4.2 | 95.8 | 4.332 | 23.759 |
| 2 | 0.140 | 14.0 | 18.2 | 81.8 | 3.366 | 5.484 |
| 3 | 0.431 | 43.1 | 61.4 | 38.6 | 1.350 | 1.629 |
| 4 | 0.215 | 21.5 | 82.8 | 17.2 | 1.115 | 1.207 |
| 5 | 0.096 | 9.6 | 92.4 | 7.6 | 1.051 | 1.082 |
| 6 | 0.047 | 4.7 | 97.1 | 2.9 | 1.025 | 1.030 |
| 7 | 0.024 | 2.4 | 99.5 | 0.5 | 1.005 | 1.005 |
| 8 | 0.005 | 0.5 | 100.0 | 0.0 | 1.000 | 1.000 |
| TOTAL | 1.000 | | | | | |

# EXHIBIT 1

## SHEET 6

## Bornhuetter-Ferguson Data

## Report Pattern Parameters

| Report Period | Beta Param (A) | Beta Param (B) | Beta Mean A ÷ (A + B) | Expected Pct Reported During Period | Expected Pct Reported to Date $E(1 - Q_j)$ | Expected Pct Unrep. $E(Q_j)$ | Expected Pct Unrep. Squared $E(Q_j^2)$ | Pct Unrep. Var. $Var(Q_j)$ | Pct Unrep. Stand Dev SD | Pct Reported to Date CV SD ÷ Mean | Pct Unrep. CV SD ÷ Mean |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 45.95 | 1,045.83 | 0.042 | 4.2 | 4.2 | 95.8 | 91.8 | 0.004 | 0.607 | 14.431 | 0.634 |
| 2 | 153.11 | 938.67 | 0.140 | 14.0 | 18.2 | 81.8 | 66.9 | 0.014 | 1.168 | 6.406 | 1.428 |
| 3 | 470.96 | 620.82 | 0.431 | 43.1 | 61.4 | 38.6 | 14.9 | 0.022 | 1.473 | 2.400 | 3.813 |
| 4 | 234.34 | 857.44 | 0.215 | 21.5 | 82.8 | 17.2 | 3.0 | 0.013 | 1.141 | 1.377 | 6.645 |
| 5 | 104.36 | 987.43 | 0.096 | 9.6 | 92.4 | 7.6 | 0.6 | 0.006 | 0.802 | 0.868 | 10.542 |
| 6 | 51.71 | 1,040.07 | 0.047 | 4.7 | 97.1 | 2.9 | 0.1 | 0.003 | 0.505 | 0.520 | 17.595 |
| 7 | 26.17 | 1,065.61 | 0.024 | 2.4 | 99.5 | 0.5 | 0.0 | 0.000 | 0.208 | 0.209 | 43.846 |
| 8 | 5.17 | 1,086.61 | 0.005 | 0.5 | 100.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | — |

$H = 1{,}091.8$

# EXHIBIT 2

## SHEET 1
## *LDF* DATA

### BAYESIAN CREDIBILITY FORMULA
### IBNR ESTIMATION
### HYPOTHETICAL DATA

$N(I,J)$

COUNTS REPORTED DURING
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 217 | 730 | 986 | 1,100 | 1,128 | 1,134 | 1,139 |
| 2 | 100 | 25 | 109 | 366 | 494 | 551 | 565 | 568 | |
| 3 | 100 | 75 | 325 | 1,094 | 1,477 | 1,647 | 1,689 | | |
| 4 | 100 | 15 | 65 | 219 | 296 | 330 | | | |
| 5 | 100 | 50 | 217 | 730 | 986 | | | | |
| 6 | 100 | 25 | 109 | 366 | | | | | |
| 7 | 100 | 75 | 325 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

$M(I,J)$

COUNTS REPORTED TO DATE
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 167 | 513 | 256 | 114 | 28 | 6 | 5 |
| 2 | 100 | 25 | 84 | 257 | 128 | 57 | 14 | 3 | |
| 3 | 100 | 75 | 250 | 769 | 383 | 170 | 42 | | |
| 4 | 100 | 15 | 50 | 154 | 77 | 34 | | | |
| 5 | 100 | 50 | 167 | 513 | 256 | | | | |
| 6 | 100 | 25 | 84 | 257 | | | | | |
| 7 | 100 | 75 | 250 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

# EXHIBIT 2

## SHEET 2
## *LDF* DATA

$N(I,J)/B(I)$
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.500 | 1.670 | 5.130 | 2.560 | 1.140 | 0.280 | 0.060 | 0.050 |
| 2 | 0.250 | 0.840 | 2.570 | 1.280 | 0.570 | 0.140 | 0.030 | |
| 3 | 0.750 | 2.500 | 7.690 | 3.830 | 1.700 | 0.420 | | |
| 4 | 0.150 | 0.500 | 1.540 | 0.770 | 0.340 | | | |
| 5 | 0.500 | 1.670 | 5.130 | 2.560 | | | | |
| 6 | 0.250 | 0.840 | 2.570 | | | | | |
| 7 | 0.750 | 2.500 | | | | | | |
| 8 | 0.150 | | | | | | | |

AGE-TO-AGE FACTORS
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1–2 | 2–3 | 3–4 | 4–5 | 5–6 | 6–7 | 7–8 |
|---|---|---|---|---|---|---|---|
| 1 | 4.340 | 3.364 | 1.351 | 1.116 | 1.025 | 1.005 | 1.004 |
| 2 | 4.360 | 3.358 | 1.350 | 1.115 | 1.025 | 1.005 | |
| 3 | 4.333 | 3.366 | 1.350 | 1.115 | 1.026 | | |
| 4 | 4.333 | 3.369 | 1.352 | 1.115 | | | |
| 5 | 4.340 | 3.364 | 1.351 | | | | |
| 6 | 4.360 | 3.358 | | | | | |
| 7 | 4.333 | | | | | | |

# EXHIBIT 2

## SHEET 3
## *LDF* DATA

### IBNR ESTIMATES

| ACCIDENT YEAR | REPORT PERIOD | REPORTED TO DATE | PEGGED METHOD | *LDF* METHOD | BORNHUETTER-FERGUSON METHOD | BAYESIAN CREDIBILITY METHOD | STANDARD DEV. OF BAYESIAN CRED. IBNR |
|---|---|---|---|---|---|---|---|
| 1 | 8 | 1,139 | −187 | 0 | 0 | 0 | 0 |
| 2 | 7 | 568 | 384 | 3 | 4 | 3 | 3 |
| 3 | 6 | 1,689 | −737 | 16 | 9 | 16 | 6 |
| 4 | 5 | 330 | 622 | 12 | 33 | 12 | 18 |
| 5 | 4 | 986 | −34 | 153 | 128 | 153 | 66 |
| 6 | 3 | 366 | 586 | 205 | 341 | 206 | 176 |
| 7 | 2 | 325 | 627 | 1,380 | 770 | 1,368 | 395 |
| 8 | 1 | 15 | 937 | 327 | 910 | 375 | 467 |
| TOTAL | | 5,418 | 2,196 | 2,095 | 2,195 | 2,132 | |

# EXHIBIT 2

## SHEET 4
## *LDF* DATA

### ESTIMATES OF ULTIMATE

| ACCIDENT YEAR | PEGGED METHOD | *LDF* METHOD | BORNHUETTER-FERGUSON METHOD | BAYESIAN CREDIBILITY METHOD |
|---|---|---|---|---|
| 1 | 952 | 1,139 | 1,139 | 1,139 |
| 2 | 952 | 571 | 572 | 571 |
| 3 | 952 | 1,705 | 1,698 | 1,705 |
| 4 | 952 | 342 | 363 | 342 |
| 5 | 952 | 1,139 | 1,114 | 1,139 |
| 6 | 952 | 571 | 707 | 572 |
| 7 | 952 | 1,705 | 1,095 | 1,693 |
| 8 | 952 | 342 | 925 | 390 |

### CREDIBILITIES

| REPORT PERIOD | PEGGED | *LDF* | B-F |
|---|---|---|---|
| 1 | 0.00004 | 0.91622 | 0.08374 |
| 2 | 0.00001 | 0.97936 | 0.02063 |
| 3 | 0.00000 | 0.99378 | 0.00622 |
| 4 | 0.00000 | 0.99539 | 0.00461 |
| 5 | 0.00000 | 0.99586 | 0.00414 |
| 6 | 0.00000 | 0.99596 | 0.00404 |
| 7 | 0.00000 | 0.99598 | 0.00402 |
| 8 | 0.00000 | 0.99600 | 0.00400 |

# EXHIBIT 2

## SHEET 5
## *LDF* DATA

### MAXIMUM LIKELIHOOD ESTIMATES

| ACCIDENT YEAR | MLE FREQUENCY $W(I)$ | EXPOSURES $B(I)$ | INITIAL ESTIMATED COUNT PARAM $B(I) \times W(I)$* |
|---|---|---|---|
| 1 | 11.390 | 100 | 1,139 |
| 2 | 5.705 | 100 | 571 |
| 3 | 17.055 | 100 | 1,705 |
| 4 | 3.417 | 100 | 342 |
| 5 | 11.390 | 100 | 1,139 |
| 6 | 5.711 | 100 | 571 |
| 7 | 17.059 | 100 | 1,706 |

| | |
|---|---|
| ESTIMATED FREQUENCY MEAN | 9.51743 |
| ESTIMATED FREQUENCY VARIANCE | 23.70887 |

| REPORT PERIOD | MLE $P(J)$ | PERCENT REPORTED $E(P_j)$ | PERCENT REPORTED TO DATE $E(1 - Q_j)$ | PERCENT UNREPORTED $E(Q_j)$ | AGE-TO-AGE FACTORS | FACTORS TO ULTIMATE |
|---|---|---|---|---|---|---|
| 1 | 0.044 | 4.4 | 4.4 | 95.6 | 4.340 | 22.768 |
| 2 | 0.147 | 14.7 | 19.1 | 80.9 | 3.364 | 5.246 |
| 3 | 0.451 | 45.1 | 64.1 | 35.9 | 1.350 | 1.560 |
| 4 | 0.225 | 22.5 | 86.6 | 13.4 | 1.115 | 1.155 |
| 5 | 0.100 | 10.0 | 96.6 | 3.4 | 1.025 | 1.035 |
| 6 | 0.025 | 2.5 | 99.0 | 1.0 | 1.005 | 1.010 |
| 7 | 0.005 | 0.5 | 99.6 | 0.4 | 1.004 | 1.004 |
| 8 | 0.004 | 0.4 | 100.0 | 0.0 | 1.000 | 1.000 |
| TOTAL | 1.000 | | | | | |

# EXHIBIT 2

## SHEET 6
## *LDF* DATA

### REPORT PATTERN PARAMETERS

| REPORT PERIOD | BETA PARAM (A) | BETA PARAM (B) | BETA MEAN A ÷ (A + B) | EXPECTED PCT REPORTED DURING PERIOD | EXPECTED PCT REPORTED TO DATE $E(1 - Q_j)$ | EXPECTED PCT UNREP. $E(Q_j)$ | EXPECTED PCT UNREP. SQUARED $E(Q_j^2)$ | PCT UNREP. VAR. $Var(Q_j)$ | PCT UNREP. STAND DEV SD | PCT REPORTED TO DATE CV SD ÷ MEAN | PCT UNREP. CV SD ÷ MEAN |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 97,715.43 | 2,127,084.44 | 0.044 | 4.4 | 4.4 | 95.6 | 91.4 | 0.000 | 0.014 | 0.313 | $1.4 \times 10^{-7}$ |
| 2 | 326,344.90 | 1,898,454.97 | 0.147 | 14.7 | 19.1 | 80.9 | 65.5 | 0.000 | 0.026 | 0.138 | $6.2 \times 10^{-8}$ |
| 3 | 1,002,477.70 | 1,222,322.16 | 0.451 | 45.1 | 64.1 | 35.9 | 12.9 | 0.000 | 0.032 | 0.050 | $2.3 \times 10^{-8}$ |
| 4 | 499,940.66 | 1,724,859.21 | 0.225 | 22.5 | 86.6 | 13.4 | 1.8 | 0.000 | 0.023 | 0.026 | $1.2 \times 10^{-8}$ |
| 5 | 222,108.92 | 2,002,690.95 | 0.100 | 10.0 | 96.6 | 3.4 | 0.1 | 0.000 | 0.012 | 0.013 | $5.7 \times 10^{-9}$ |
| 6 | 54,730.59 | 2,170,069.28 | 0.025 | 2.5 | 99.0 | 1.0 | 0.0 | 0.000 | 0.007 | 0.007 | $3.0 \times 10^{-9}$ |
| 7 | 11,714.15 | 2,213,085.72 | 0.005 | 0.5 | 99.6 | 0.4 | 0.0 | 0.000 | 0.004 | 0.004 | $1.9 \times 10^{-9}$ |
| 8 | 9,767.53 | 2,215,032.34 | 0.004 | 0.4 | 100.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | |

$H = 2,224,799.9$

# EXHIBIT 3

## SHEET 1
MIXED DATA

BAYESIAN CREDIBILITY FORMULA
IBNR ESTIMATION
HYPOTHETICAL DATA

*N(I,J)*
COUNTS REPORTED DURING
DEVELOPMENT PERIOD *J*

| ACCIDENT YEAR (*I*) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 159 | 482 | 241 | 107 | 39 | 16 | 5 |
| 2 | 100 | 25 | 117 | 354 | 177 | 79 | 32 | 14 | |
| 3 | 100 | 75 | 200 | 610 | 304 | 135 | 46 | | |
| 4 | 100 | 15 | 100 | 302 | 151 | 67 | | | |
| 5 | 100 | 50 | 159 | 482 | 241 | | | | |
| 6 | 100 | 25 | 117 | 354 | | | | | |
| 7 | 100 | 75 | 200 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

*M(I,J)*
COUNTS REPORTED TO DATE
DEVELOPMENT PERIOD *J*

| ACCIDENT YEAR (*I*) | EXPOSURES | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 100 | 50 | 209 | 691 | 932 | 1,039 | 1,078 | 1,094 | 1,099 |
| 2 | 100 | 25 | 142 | 496 | 673 | 752 | 784 | 798 | |
| 3 | 100 | 75 | 275 | 885 | 1,189 | 1,324 | 1,370 | | |
| 4 | 100 | 15 | 115 | 417 | 568 | 635 | | | |
| 5 | 100 | 50 | 209 | 691 | 932 | | | | |
| 6 | 100 | 25 | 142 | 496 | | | | | |
| 7 | 100 | 75 | 275 | | | | | | |
| 8 | 100 | 15 | | | | | | | |

# EXHIBIT 3

## SHEET 2
MIXED DATA

$N(I,J)/B(I)$
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.500 | 1.590 | 4.820 | 2.410 | 1.070 | 0.390 | 0.160 | 0.050 |
| 2 | 0.250 | 1.170 | 3.540 | 1.770 | 0.790 | 0.320 | 0.140 | |
| 3 | 0.750 | 2.000 | 6.100 | 3.040 | 1.350 | 0.460 | | |
| 4 | 0.150 | 1.000 | 3.020 | 1.510 | 0.670 | | | |
| 5 | 0.500 | 1.590 | 4.820 | 2.410 | | | | |
| 6 | 0.250 | 1.170 | 3.540 | | | | | |
| 7 | 0.750 | 2.000 | | | | | | |
| 8 | 0.150 | | | | | | | |

AGE-TO-AGE FACTORS
DEVELOPMENT PERIOD $J$

| ACCIDENT YEAR ($I$) | 1–2 | 2–3 | 3–4 | 4–5 | 5–6 | 6–7 | 7–8 |
|---|---|---|---|---|---|---|---|
| 1 | 4.180 | 3.306 | 1.349 | 1.115 | 1.038 | 1.015 | 1.005 |
| 2 | 5.680 | 3.493 | 1.357 | 1.117 | 1.043 | 1.018 | |
| 3 | 3.667 | 3.218 | 1.344 | 1.114 | 1.035 | | |
| 4 | 7.667 | 3.626 | 1.362 | 1.118 | | | |
| 5 | 4.180 | 3.306 | 1.349 | | | | |
| 6 | 5.680 | 3.493 | | | | | |
| 7 | 3.667 | | | | | | |

## EXHIBIT 3

### SHEET 3
### Mixed Data

### IBNR Estimates

| Accident Year | Report Period | Reported to Date | Pegged Method | *LDF* Method | Bornhuetter-Ferguson Method | Bayesian Credibility Method | Standard Dev. of Bayesian Cred. IBNR |
|---|---|---|---|---|---|---|---|
| 1 | 8 | 1,099 | −100 | 0 | 0 | 0 | 0 |
| 2 | 7 | 798 | 201 | 4 | 5 | 4 | 3 |
| 3 | 6 | 1,370 | −371 | 28 | 20 | 28 | 8 |
| 4 | 5 | 635 | 364 | 38 | 56 | 38 | 17 |
| 5 | 4 | 932 | 67 | 169 | 153 | 169 | 43 |
| 6 | 3 | 496 | 503 | 295 | 373 | 297 | 102 |
| 7 | 2 | 275 | 724 | 1,201 | 813 | 1,165 | 219 |
| 8 | 1 | 15 | 984 | 334 | 956 | 522 | 258 |
| TOTAL | | 5,620 | 2,375 | 2,069 | 2,376 | 2,224 | |

# EXHIBIT 3

## SHEET 4
## MIXED DATA

### ESTIMATES OF ULTIMATE

| ACCIDENT YEAR | PEGGED METHOD | *LDF* METHOD | BORNHUETTER-FERGUSON METHOD | BAYESIAN CREDIBILITY METHOD |
|---|---|---|---|---|
| 1 | 999 | 1,099 | 1,099 | 1,099 |
| 2 | 999 | 802 | 803 | 802 |
| 3 | 999 | 1,398 | 1,390 | 1,398 |
| 4 | 999 | 673 | 691 | 673 |
| 5 | 999 | 1,101 | 1,085 | 1,101 |
| 6 | 999 | 791 | 869 | 793 |
| 7 | 999 | 1,476 | 1,088 | 1,440 |
| 8 | 999 | 349 | 971 | 537 |

### CREDIBILITIES

| REPORT PERIOD | PEGGED | *LDF* | B-F |
|---|---|---|---|
| 1 | 0.07101 | 0.70066 | 0.22833 |
| 2 | 0.01814 | 0.91327 | 0.06859 |
| 3 | 0.00264 | 0.97558 | 0.02178 |
| 4 | 0.00081 | 0.98294 | 0.01625 |
| 5 | 0.00026 | 0.98513 | 0.01460 |
| 6 | 0.00009 | 0.98582 | 0.01408 |
| 7 | 0.00002 | 0.98611 | 0.01386 |
| 8 | 0.00000 | 0.98620 | 0.01380 |

# EXHIBIT 3

## SHEET 5
### Mixed Data

### Maximum Likelihood Estimates

| Accident Year | MLE Frequency $W(I)$ | Exposures $B(I)$ | Initial Estimated Count Param $B(I) \times W(I)$* |
|---|---|---|---|
| 1 | 10.990 | 100 | 1,099 |
| 2 | 8.016 | 100 | 802 |
| 3 | 13.984 | 100 | 1,398 |
| 4 | 6.725 | 100 | 672 |
| 5 | 11.007 | 100 | 1,101 |
| 6 | 7.907 | 100 | 791 |
| 7 | 14.711 | 100 | 1,471 |

| | |
|---|---|
| Estimated Frequency Mean | 9.99352 |
| Estimated Frequency Variance | 7.14026 |

| Report Period | MLE $P(J)$ | Percent Reported $E(P_i)$ | Percent Reported to Date $E(1 - Q_i)$ | Percent Unreported $E(Q_i)$ | Age-to-Age Factors | Factors to Ultimate |
|---|---|---|---|---|---|---|
| 1 | 0.043 | 4.3 | 4.3 | 95.7 | 4.339 | 23.284 |
| 2 | 0.143 | 14.3 | 18.6 | 81.4 | 3.364 | 5.366 |
| 3 | 0.441 | 44.1 | 62.7 | 37.3 | 1.350 | 1.595 |
| 4 | 0.220 | 22.0 | 84.7 | 15.3 | 1.115 | 1.181 |
| 5 | 0.098 | 9.8 | 94.4 | 5.6 | 1.038 | 1.059 |
| 6 | 0.035 | 3.5 | 98.0 | 2.0 | 1.016 | 1.021 |
| 7 | 0.016 | 1.6 | 99.5 | 0.5 | 1.005 | 1.005 |
| 8 | 0.005 | 0.5 | 100.0 | 0.0 | 1.000 | 1.000 |
| TOTAL | 1.000 | | | | | |

# EXHIBIT 3

## SHEET 6

## MIXED DATA

### REPORT PATTERN PARAMETERS

| Report Period | Beta Param (A) | Beta Param (B) | Beta Mean A ÷ (A + B) | Expected Pct Reported During Period | Expected Pct Reported to Date $E(1 - Q_j)$ | Expected Pct Unrep. $E(Q_j)$ | Expected Pct Unrep. Squared $E(Q_j^2)$ | Pct Unrep. Var. $Var(Q_j)$ | Pct Unrep. Stand Dev SD | Pct Reported to Date CV SD ÷ Mean | Pct Unrep. CV SD ÷ Mean |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 141.48 | 3,152.57 | 0.043 | 4.3 | 4.3 | 95.7 | 91.6 | 0.001 | 0.353 | 8.224 | 0.369 |
| 2 | 472.38 | 2,821.66 | 0.143 | 14.3 | 18.6 | 81.4 | 66.2 | 0.005 | 0.678 | 3.640 | 0.834 |
| 3 | 1,451.43 | 1,842.62 | 0.441 | 44.1 | 62.7 | 37.3 | 13.9 | 0.007 | 0.842 | 1.344 | 2.259 |
| 4 | 723.27 | 2,570.77 | 0.220 | 22.0 | 84.7 | 15.3 | 2.4 | 0.004 | 0.628 | 0.742 | 4.092 |
| 5 | 321.73 | 2,972.31 | 0.098 | 9.8 | 94.4 | 5.6 | 0.3 | 0.002 | 0.400 | 0.423 | 7.167 |
| 6 | 116.79 | 3,177.25 | 0.035 | 3.5 | 98.0 | 2.0 | 0.0 | 0.001 | 0.246 | 0.251 | 12.094 |
| 7 | 51.98 | 3,242.06 | 0.016 | 1.6 | 99.5 | 0.5 | 0.0 | 0.000 | 0.117 | 0.118 | 25.772 |
| 8 | 14.98 | 3,279.06 | 0.005 | 0.5 | 100.0 | 0.0 | 0.0 | 0.000 | 0.000 | 0.000 | — |

$H = 3,294.0$