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AN ACTUARIAL NOTE ON CREDIBILITY PARAMETERS

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Abstract

In this paper the relationship between the Bayesian credibility parameter, k, *and the classical credibility standard for full credibility,* F, *is examined from a practical standpoint. A very useful "rule of thumb" is developed.*

For most practical applications one can determine the F *that roughly corresponds to* k, *and vice versa. First convert* k *to a number of claims, if necessary, by multiplying by an expected frequency. Then take* F *equal to approximately eight times* k.

A few other interesting results are also derived. Among them is the effect of misestimating the Bayesian credibility parameter k. *The results of using credibility are relatively insensitive to misestimates of* k.

INTRODUCTION

Credibility concepts and formulas are used in many actuarial applications. In this paper some practical questions concerning the use of credibility will be explored. While a few results of theoretical interest are derived, the emphasis is strictly on the practical impacts. This paper assumes that the reader is already generally familiar with credibility. For those interested in the theoretical questions, there are many fine papers, some of which are listed in the references at the end of this paper.

The first question explored is the practical impact of choosing between classical and Bayesian credibility. The answer depends on the parameters used in the two credibility formulas. For a certain simple relationship between the parameters, the choice between classical and Bayesian credibility makes only a relatively small difference. For many practical applications this difference is acceptable.[1]

The second question explored is what is the practical impact of misestimating the Bayesian credibility parameter. The credibilities are relatively insensitive to misestimating this parameter.

CLASSICAL CREDIBILITY FORMULA

This paper assumes the following formula for the "classical" credibility Z_C.

$$Z_C = \begin{cases} (n/F)^{.5} & 0 \leq n \leq F \\ 1 & n \geq F \end{cases} \tag{1}$$

where n is the number of claims, and F is the so-called standard for full credibility. This formula is discussed further in [1] and [2].

BAYESIAN CREDIBILITY FORMULA

This paper assumes the following formula for the "Bayesian" credibility Z_B.

$$Z_B = \frac{P}{P + k}$$

where P is some measure of exposure such as payroll, premium, number of claims, etc. This formula and methods of deriving a value for k are discussed further in [3], [4], [5], [6], and [7].[2]

In many cases P is the number of claims, for example, when we are trying to estimate the average claim cost by class. In those cases where P is an

[1] The degree of accuracy required depends on the particular application. The differences in credibility are given in this paper. The question of whether the resulting differences in the quantity to be estimated are large or small will have to be decided on a case by case basis.

[2] An example of where a more complicated formula holds is given in Meyers [8].

exposure unit other than claims, the formula for credibility can be approximated by multiplying P and k by an estimate of the expected claim frequency.[3] Then

$$Z_B \cong \frac{n}{n + k'}$$

where n is the number of claims and k' is in units of claims; k' equals k times the expected frequency.

For simplicity, hereafter, we will assume a claim-based form of the formula for credibility, such as

$$Z_B = \frac{n}{n + k} \tag{2}$$

where n is the number of claims.

COMPARISON OF THE TWO FORMULAS

The formulas (1) and (2) were derived from different points of view or different methods. A discussion of these differences is beyond the scope of this paper. In spite of these differences, the two formulas yield curves with very similar shapes, as stated in Longley-Cook [1]. This is illustrated in Exhibit 1.

The credibility given by formula (1) is equal to the credibility given by formula (2) when

$$\frac{n}{n + k} = \left(\frac{n}{F}\right)^{.5}$$

$$k = F\ (n/F)^{.5}[1 - (n/F)^{.5}]$$

$$k = FZ_C(1 - Z_C).$$

Since we specifically have $Z_C = Z_B$, this can be written as

$$k = FZ(1 - Z). \tag{3}$$

If we define $R = F/k$, equation (3) can be rewritten as $1/R = Z(1 - Z)$. In other words, the curves given by formula (1) and formula (2) will cross at the

[3] This estimate need not be very accurate since the credibility is not very sensitive to the value of k as shown in a later section of this paper. Therefore, one can usually use a larger body of data to estimate the expected claim frequency sufficiently well for this purpose.

two points where the credibility has the values Z and $1 - Z$, provided we have

$$R = \frac{1}{Z(1 - Z)} . \tag{4}$$

That is, selecting the credibilities Z at which the classical and Bayesian credibilities are to be the same, yields the factor R that is used to relate the credibility parameters. Or, alternatively, given Bayesian parameter k and classical parameter F, formula (4) indicates the points at which the two will yield equivalent credibilities.

Choosing the value of R determines the two credibility values at which the two curves intersect. To cross near the middle,[4] take $1/R = (.5)\ (1 - .5)$ or $R = 4$. To cross near the ends, take $1/R = .1\ (1-.1)$ or $R \cong 11$. In the former case, the two curves are relatively far apart near the end points. In the latter case, the two curves are relatively far apart near the middle.

We are interested in having the two curves be "close" over the entire range of possible values for the credibility. One useful criterion, to define the concept of how close the two curves are, would be the maximum difference between the curves.

Thus, one might want to minimize the maximum difference between the two curves. Taking $R = 6.75$ does so, producing a maximum difference of 13%[5], as illustrated numerically in Exhibits 1 and 2. This is a relatively small difference in credibility. For many practical applications, it will make relatively little difference which credibility formula is utilized, provided that $R \cong 7$.

MINIMIZING VARIANCE

In Bayesian credibility theory, the credibility is chosen so as to minimize the variance of the estimate around the true result.[6] See, for example, the ISO Credibility White Paper [3].

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[4] Actually, in this particular case the two curves are tangent at a single point, $Z = 50\%$.

[5] This problem reduces to the solution of a fifth-degree equation. The solution via numerical analysis is $R \cong 6.757$. The maximum difference of 12.89% occurs at $r = R$ and $r = 1.5401$.

[6] The estimate given by Bayesian credibility is the least squares linear unbiased estimate.

Appendix I shows that if we use in place of the Bayesian credibility, Z_B, a different estimate, $Z_B + \Delta Z$, then the variance increases. The variance is given by a parabola.[7] For small changes from the optimal credibility, there is only a very small increase in the variance. Thus, for most applications, it will make no practical difference if the credibilities used differ slightly from optimal. The use of credibilities other than the optimal one still usually leads to a substantial decrease in variance compared to not using credibility at all. The relative increase in variance is given by

$$\frac{\Delta \text{ Variance}}{\text{Variance}} = \frac{(\Delta Z)^2}{Z_B(1 - Z_B)} \,. \tag{5}$$

The full credibility standard that will produce the classical credibility curve with the smallest maximum relative increase in variance requires a choice of R that will minimize the maximum of

$$\frac{(Z_C - Z_B)^2}{Z_B(1 - Z_B)} \,.$$

The solution is $R = 8$. See Appendix II and Exhibit 3. The maximum increase in the variance in this case is only 12.5% = 1/8.

CHOOSING A RULE OF THUMB

A value of $R = 6.75$ minimizes the maximum difference between the classical and Bayesian credibility curves. However, taking $R = 8$ only increases this maximum difference from 13% to 17%. (See Exhibit 2.) On the other hand, taking $R = 6.75$ rather than $R = 8$, only increases the maximum variance to 1/6.75 = 14.8% from 1/8 = 12.5%. (See Appendix II.) Thus, either 7 or 8 would be equally good integral values of R for use as a general rule of thumb. They each have something to recommend themselves. The author is more concerned with the reduction in variance and thus prefers $R = 8$.

[7] This is the same result noted by Meyers [8]. Meyers' concept of efficiency is closely related to the variance of the estimate around the true result. One minus the efficiency is proportional to that variance.

EXAMPLES OF USES OF THE RULE OF THUMB

Example 1

You generally use Bayesian credibility methods to develop your territory relativities for private passenger automobile. However, you have to file for a rate change in one particular state where rates are tightly regulated. The insurance department refuses to accept anything but classical credibility methods.

Let's assume your Bayesian credibility parameter is 2500 car-years. Then, multiply this by the expected frequency and then by a factor of 8. If the expected frequency is 5%, then we get $2500 \times 5\% \times 8 = 1000$ claims. Thus you can use for your classical credibility standard roughly 1000 claims, for example, the traditional 1084. See Longley-Cook [1].

Example 2

You are computing estimated severities by classification for workers' compensation insurance, using an empirical Bayesian credibility method. When actually implementing the method, you find it is necessary to impose maximums and minimums on the computed values of k, the Bayesian credibility parameter. To aid you in choosing these values, you convert them to a classical credibility basis.

For example, $k = 350$ claims would correspond to a full credibility standard of $350 \times 8 = 2800$ claims. This could be thought of as a frequency standard of 1084, multiplied by a factor of 2.6 in order to convert it to a standard for severity. (2.6 can be thought of as the ratio of the variance of the severity to the square of the mean severity). See Longley-Cook [1].

THE EFFECT OF MISESTIMATING k

Quite often in the use of Bayesian credibility it is necessary to estimate k. For example, one might estimate k from the data as in either [3] or [7]. Fortunately, the results are not very sensitive to the value of k. Let $\tilde{k}$ be our estimate of the correct k.

Let $T = \tilde{k}/k$.

Then, as shown in Appendix III, the maximum difference in the credibility that results from $\tilde{k}$ as an estimate of k is

$$(\Delta Z)_{\text{Max}} = \frac{T - 1}{(1 + \sqrt{T})^2}. \tag{6}$$

For values of T near 1, this is relatively small. (See Exhibit 4.) For example, if $T = 1.25$ or .8, then it is 6%. Even if $T = 2$ or $T = .5$, then the maximum difference is only 17%.[8] In other words, even if the estimated k is wrong by a factor of 2, the estimated credibilities are off by at most 17%.[9] For many practical purposes this is an acceptable difference.

In Appendix IV it is shown that the maximum change in variance is given by:

$$\left(\frac{\Delta V}{V}\right)_{Max} = \frac{(T-1)^2}{4T} \tag{7}$$

For values of T near 1, this is relatively small. (See Exhibit 5.) For example, if $T = 1.5$ or 2/3, then it is 4%. Even if $T = 2$ or .5, then the maximum relative increase in the variance is only $1/8 \cong 13\%$.[10] Once again, even if the estimated k is wrong by a factor of 2 in either direction, for many practical purposes the result is still acceptable.

CONCLUSION

For most practical applications, one can determine the standard for full credibility F that roughly corresponds to the Bayesian credibility parameter k, and vice versa. First convert k to a number of claims, if necessary, by multiplying by an expected frequency. Then take F equal to approximately eight times k.

When estimating the Bayesian credibility parameter k, the estimate need not be extremely precise. For many practical applications, the estimate of k can be wrong by as much as a factor of two in either direction and still produce a fairly good estimate of the quantity, e.g., frequency, severity, pure premium, etc., that credibility is being used to estimate.

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[8] For $T = 2$, this maximum difference occurs when the correct credibility is 58.6% and the estimated credibility is 41.4%. For $T = .5$, the correct and estimated credibilities are reversed.

[9] Of course, if the estimated k is wrong by more than a factor of 2, the estimated credibilities can be off by more than 17%.

[10] For $T = 2$, this maximum relative increase in variance occurs when the correct credibility is 2/3. For $T = .5$, this occurs when the correct credibility is 1/3.

ACKNOWLEDGEMENTS

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[5] C. C. Hewitt, Jr., Discussion of Mayerson: "A Bayesian View of Credibility," *PCAS* LII, 1965, p. 121.

[6] C. C. Hewitt, Jr., "Credibility for Severity," *PCAS* LVII, 1970, p. 148.

[7] G. G. Meyers, "Empirical Bayesian Credibility for Workers' Compensation Classification Ratemaking," *PCAS* LXXI, 1984, p. 96.

[8] G. G. Meyers, "An Analysis of Experience Rating," *PCAS* LXXII, 1985, p. 278.

EXHIBIT 1
PART 1

ILLUSTRATIVE COMPARISON OF CREDIBILITIES
BAYESIAN CREDIBILITY WITH k = 200 VERSUS
CLASSICAL CREDIBILITY WITH VARIOUS VALUES OF F

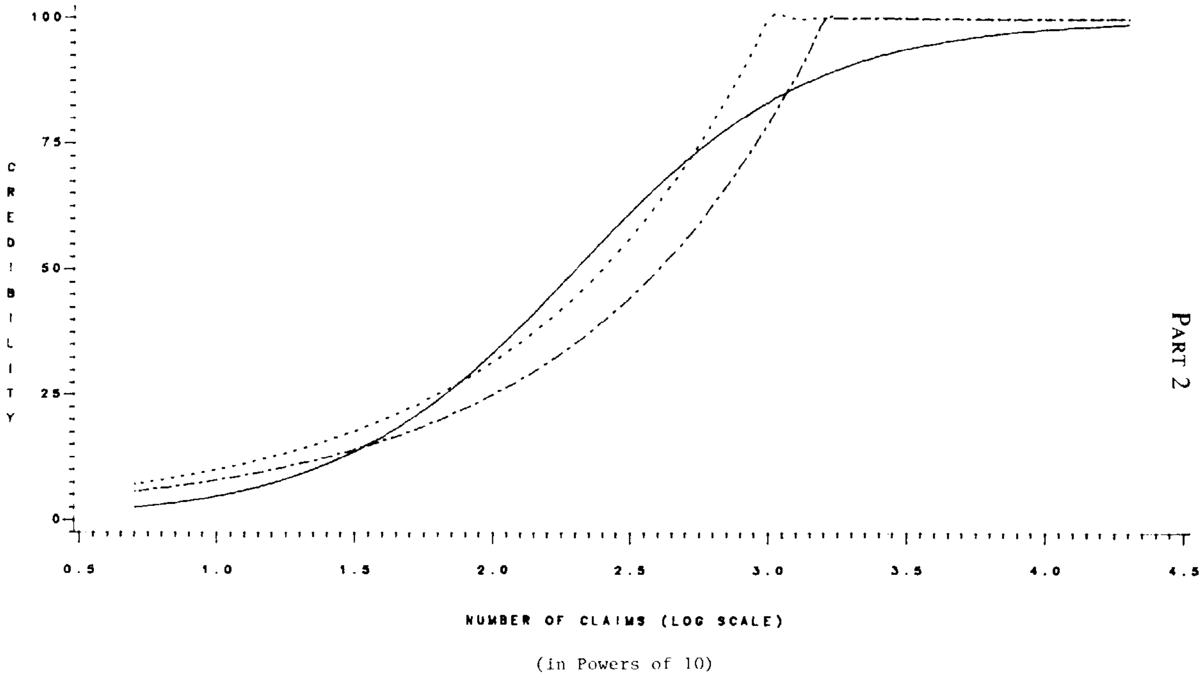
Claims	Bayesian Credibility k = 200	Classical Credibility F = 1000	F = 1200	F = 1350	F = 1400	F = 1600
5	2%	7%	6%	6%	6%	6%
10	5	10	9	9	8	8
20	9	14	13	12	12	11
30	13	17	16	15	15	14
40	17	20	18	17	17	16
50	20	22	20	19	19	18
60	23	24	22	21	21	19
70	26	26	24	23	22	21
80	29	28	26	24	24	22
90	31	30	27	26	25	24
100	33	32	29	27	27	25
125	38	35	32	30	30	28
150	43	39	35	33	33	31
175	47	42	38	36	35	33
200	50	45	41	38	38	35
250	56	50	46	43	42	40
300	60	55	50	47	46	43
350	64	59	54	51	50	47
400	67	63	58	54	53	50
450	69	67	61	58	57	53
500	71	71	65	61	60	56
600	75	77	71	67	65	61
700	78	84	76	72	71	66
800	80	89	82	77	76	71
900	82	95	87	82	80	75
1000	83	100	91	86	85	79
1200	86	100	100	94	93	87
1400	88	100	100	100	100	94
1600	89	100	100	100	100	100
1800	90	100	100	100	100	100
2000	91	100	100	100	100	100
3000	94	100	100	100	100	100
4000	95	100	100	100	100	100
5000	96	100	100	100	100	100
10000	98	100	100	100	100	100
20000	99	100	100	100	100	100

$$Z_B = \frac{n}{n+k}$$

$$Z_C = \begin{cases} (n/F)^{.5} & 0 \leqq n \leqq F \\ 1 & n \geqq F \end{cases}$$

EXHIBIT 1
PART 2

ILLUSTRATIVE COMPARISON OF CREDIBILITIES
BAYESIAN CREDIBILITY WITH K=200 VS. CLASSICAL CREDIBILITY
WITH VARIOUS VALUES OF F



DASHED LINE BASED ON CLASSICAL CREDIBILITY WITH CLAIMS = 1000.
VARIED LINE BASED ON CLASSICAL CREDIBILITY WITH CLAIMS = 1600.
CONTINUOUS LINE BASED ON A BAYESIAN CREDIBILITY WITH K = 200.

EXHIBIT 2

PART 1

CLASSICAL CREDIBILITY MINUS BAYESIAN CREDIBILITY

r = Claims ÷ k	$R = 5$	$R = 6$	$R = 6.75$	$R = 7$	$R = 8$
.025	5%	4%	4%	4%	3%
.05	5	4	4	4	3
.10	5	4	3	3	2
.15	4	3	2	2	1
.20	3	2	1	0	−1
.25	2	0	−1	−1	−2
.30	1	1	−2	−2	−4
.35	1	−2	3	−4	−5
.40	0	3	−4	−5	−6
.45	−1	−4	−5	−6	−7
.50	−2	−4	6	−7	−8
.625	−3	−6	8	−9	11
.75	−4	−8	−10	−10	−12
.875	−5	−8	−11	−11	−14
1.00	−5	−9	−11	−12	−15
1.25	−6	−10	12	−13	16
1.50	−5	−10	−13	−14	−17
1.75	−4	−10	−13	−14	−17
2.00	−3	−9	−13	13	−17
2.25	−2	−8	−12	−13	−16
2.50	−1	−7	−11	−12	−16
3.00	2	−4	8	−10	−14
3.50	6	−1	−6	−7	−12
4.00	9	2	−3	−4	−9
4.50	13	5	0	−2	−7
5.00	17	8	3	1	−4
6.00	14	14	9	7	1
7.00	13	13	13	13	6
8.00	11	11	11	11	11
9.00	10	10	10	10	10
10.00	9	9	9	9	9
15.00	6	6	6	6	6
20.00	5	5	5	5	5
25.00	4	4	4	4	4
50.00	2	2	2	2	2
100.00	1	1	1	1	1

$$R = \frac{F}{k}$$

$$Z_C - Z_B = \begin{cases} \left(\frac{r}{R}\right)^5 - \frac{r}{1+r} & 0 \leq r \leq R \\ \frac{1}{1+r} & r \geq R \end{cases}$$

EXHIBIT 2
PART 2

CLASSICAL MINUS BAYESIAN CREDIBILITY

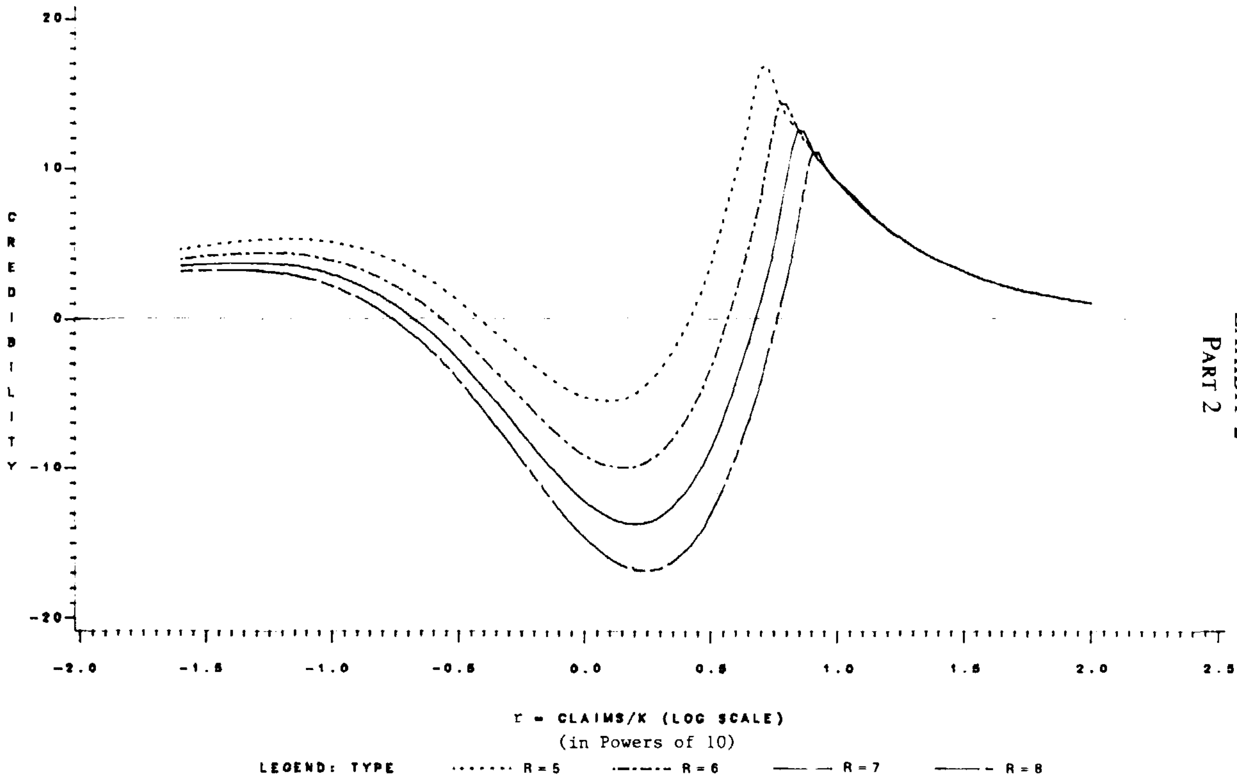


EXHIBIT 3
PART 1

INCREASE IN VARIANCE THROUGH USE OF CLASSICAL CREDIBILITY RATHER THAN BAYESIAN CREDIBILITY

r = Claims ÷ k	$R = 7$	$R = 8$	$R = 9$
0	14%	13%	11%
.0005	13	11	10
.001	12	10	9
.005	10	8	7
.010	8	7	6
.015	7	6	5
.02	6	5	4
.03	5	4	3
.04	4	3	2
.05	3	2	2
.10	1	1	0
.20	0	0	0
.30	0	1	1
.40	1	2	3
.50	2	3	4
.75	4	6	8
1.00	6	9	11
1.50	8	12	15
2.00	8	13	17
2.50	7	12	17
3.00	5	10	16
3.50	3	8	14
4.00	1	5	11
4.50	0	3	8
5.00	0	1	6
5.50	1	0	3
6.00	4	0	1
6.50	8	1	0
7.00	14	3	0
7.50	13	7	1
8.00	13	13	3
8.50	12	12	6
9.00	11	11	11
9.50	11	11	11
10.00	10	10	10
15.00	7	7	7
20.00	5	5	5
25.00	4	4	4
50.00	2	2	2
100.00	1	1	1

Note: The value given for $r = 0$ is actually the limit as $r \to 0$

$R = F/k$

$$\frac{\Delta V}{V} = \frac{(\Delta Z)^2}{Z_B(1 - Z_B)}$$ See Appendix II.

EXHIBIT 3
PART 2

INCREASE IN VARIANCE
THROUGH USE OF CLASSICAL CREDIBILITY
RATHER THAN BAYESIAN CREDIBILITY

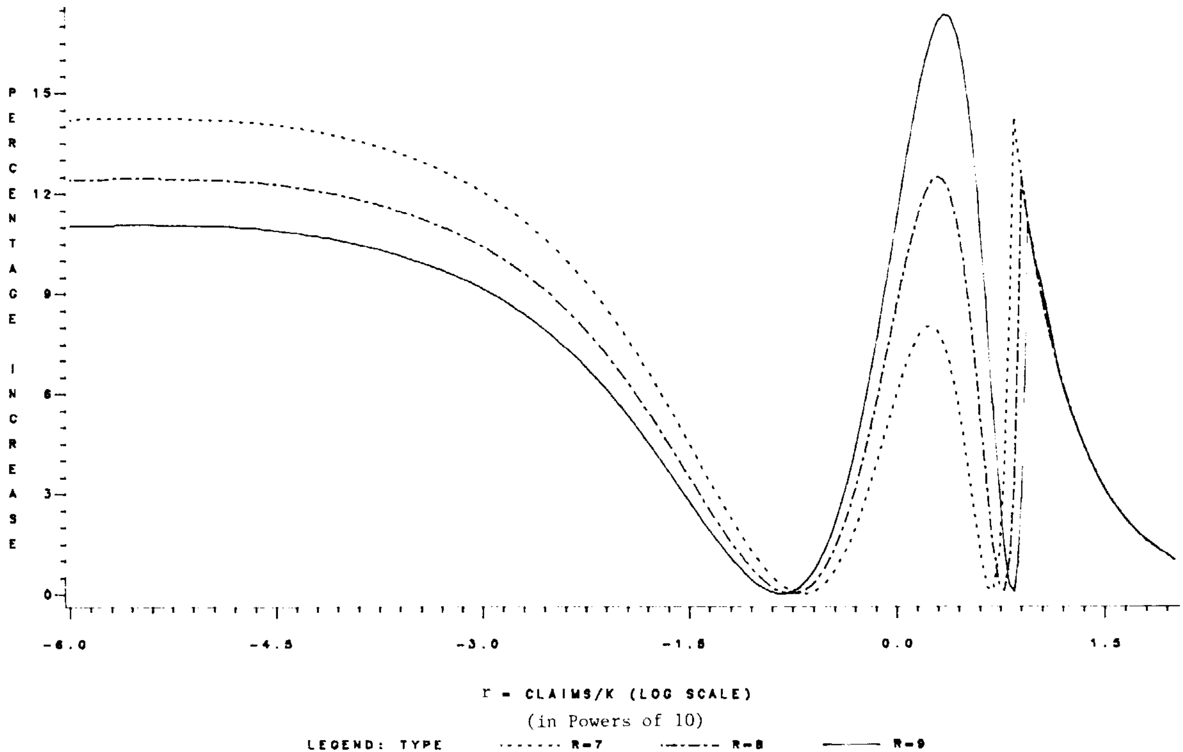


EXHIBIT 4
PART 1

BAYESIAN CREDIBILITY

DIFFERENCE IN CREDIBILITY DUE TO MISESTIMATING k
ESTIMATED CREDIBILITY MINUS CORRECT CREDIBILITY

r	$T = 1/3$	$T = 1/2$	$T = 2/3$	$T = .8$	$T = 1.25$	$T = 1.5$	$T = 2$	$T = 3$
0	0%	0%	0%	0%	0%	0%	0%	0%
.01	2	1	0	0	0	0	0	−1
.02	4	2	1	0	0	−1	−1	−1
.05	8	4	2	1	−1	−2	−2	−3
.10	14	8	4	2	−2	−3	−4	−6
.25	23	13	7	4	−3	−6	−9	−12
.50	27	17	10	5	−5	−8	−13	−19
.75	26	17	10	6	−5	−10	−16	−23
1.00	25	17	10	6	−6	−10	−17	−25
1.50	22	15	9	5	−5	−10	−17	−27
2.00	19	13	8	5	−5	−10	−17	−27
5.00	10	8	5	3	−3	−6	−12	−21
10.00	6	4	3	2	−2	−4	−8	−14
20.00	3	2	2	1	−1	−2	−4	−8
50.00	1	1	1	0	0	−1	−2	−4
100.00	1	0	0	0	0	0	−1	−2

Note: r = Exposures $\div$ k $\quad T = \dfrac{\tilde{k}}{k} = \dfrac{\text{Estimated Bayesian Credibility Parameter}}{\text{Correct Bayesian Credibility Paramer}}$

$$\Delta Z = \frac{r(1 - T)}{(1 + r)(T + r)}$$ See Appendix III.

EXHIBIT 4
PART 2

DIFFERENCE IN CREDIBILITY DUE TO MISESTIMATING K

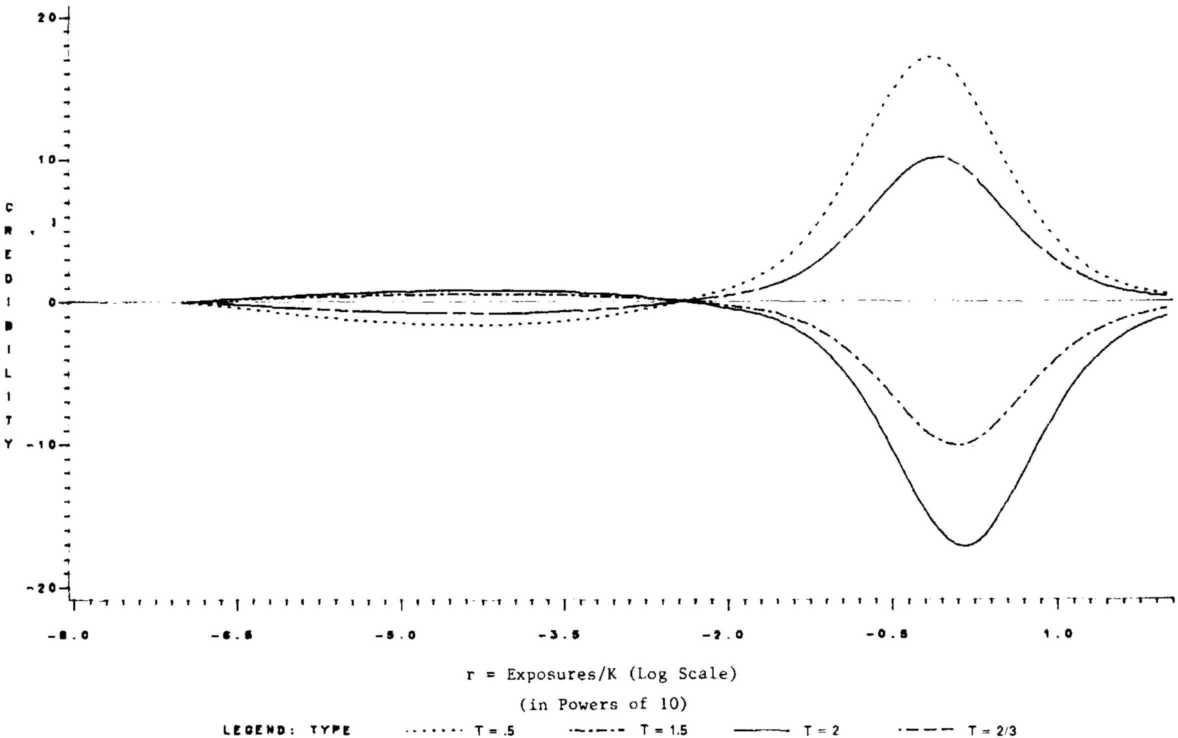


EXHIBIT 5
PART 1

BAYESIAN CREDIBILITY
INCREASE IN VARIANCE DUE TO MISESTIMATING k

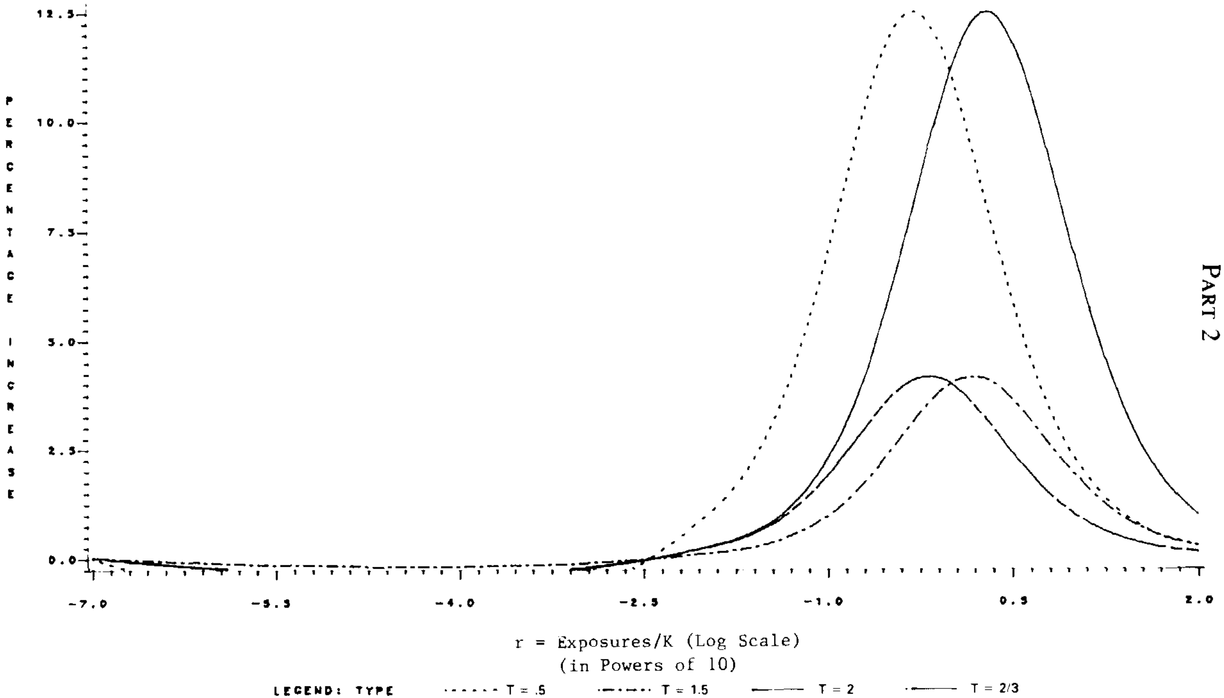
r	$T = 1/3$	$T = 1/2$	$T = .6$	$T = 2/3$	$T = .8$	$T = 1.25$	$T = 1.5$	$T = 1.75$	$T = 2$	$T = 3$
0	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
.01	4	1	0	0	0	0	0	0	0	0
.02	7	2	1	0	0	0	0	0	0	1
.05	15	4	2	1	0	0	1	1	1	2
.10	24	7	3	2	0	0	1	2	2	4
.25	33	11	6	3	1	1	2	4	5	9
.50	32	13	7	4	1	1	3	6	8	16
.75	28	12	7	4	1	1	4	7	10	21
1.00	25	11	6	4	1	1	4	7	11	25
1.50	20	9	5	4	1	1	4	8	12	30
2.00	16	8	5	3	1	1	4	8	13	32
5.00	8	4	3	2	1	1	3	6	10	31
10.00	4	2	1	1	0	0	2	4	7	24
20.00	2	1	1	1	0	0	1	2	4	15
50.00	1	0	0	0	0	0	0	1	2	7
100.00	0	0	0	0	0	0	0	1	1	4

Note: r = Exposures $\div$ k $\quad T = \dfrac{\tilde{k}}{k} = \dfrac{\text{Estimated Bayesian Credibility Parameter}}{\text{Correct Bayesian Credibility Parameter}}$

$$\frac{\Delta V}{V} = \frac{r(T-1)^2}{(T+r)^2}$$ See Appendix IV.

EXHIBIT 5
PART 2

INCREASE IN VARIANCE
DUE TO MISESTIMATING K



APPENDIX I

This appendix derives an expression for the relative increase in variance that occurs when one uses a value for the credibility other than that indicated by Bayesian credibility. It is shown that the variance is given by a parabola.[11] The bottom of the parabola, i.e. minimum variance, occurs when the value for credibility indicated by Bayesian credibility is used. For different values near this, the increase in variance is relatively small.

Let X be a random variable whose distribution depends on a parameter θ. Let the mean of X for the value of the parameter θ be given by $\mu(\theta) = E[X/\theta]$.

Let F be an estimate of μ that gives weight a to the observed value X and weight $1 - a$ to the overall mean M.

$$F = aX + (1 - a)M$$

where $M = E(X) = E_\theta[E[X/\theta]]$.

F is a function of the parameter a.

We wish to determine the variance of the estimate F around the mean μ, averaged over all possible values of the parameter θ.

$$\text{Let } V(a) = E_\theta[E[(F - \mu)^2/\theta]].$$

$$\text{Let } \tau^2 = VAR_\theta[\mu(\theta)] = E_\theta[(\mu(\theta) - M)^2] = \text{“between variance”}$$

$$\delta^2 = E_\theta[VAR[X/\theta]] = \text{“within variance.”}$$

$$F - \mu = a(X - \mu) + (1 - a)(M - \mu)$$

$$\begin{aligned}(F - \mu)^2 &= a^2(X - \mu)^2 + (1 - a)^2(M - \mu)^2 \\ &\quad + 2a(1 - a)(X - \mu)(M - \mu)\end{aligned}$$

$$E[(F - \mu)^2/\theta] = a^2 VAR[X/\theta] + (1 - a)^2(M - \mu(\theta))^2$$

$$V(a) = E_\theta[E[(F - \mu)^2/\theta]] = a^2\delta^2 + (1 - a)^2\tau^2$$

Thus, this variance is given by a parabola in a. $V(a) = a^2\delta^2 + (1 - a)^2\tau^2$. It has a minimum when the derivative is zero.

$$0 = 2a\delta^2 - 2(1 - a)\tau^2$$

$$a = \frac{\tau^2}{\tau^2 + \delta^2}$$

[11] This well known result is given for example in Appendix B of Meyers [7].

$$V\left(\frac{\tau^2}{\tau^2 + \delta^2}\right) = \frac{\delta^2\tau^2}{\delta^2 + \tau^2}$$

Thus, combining the observed value with the overall mean reduces the variance. It is interesting to note in passing that

$$\frac{1}{\text{Minimum Variance}} = \frac{\tau^2 + \delta^2}{\delta^2\tau^2} = \frac{1}{\delta^2} + \frac{1}{\tau^2}$$

$$= \frac{1}{\text{the variance if you use observation}}$$

$$+ \frac{1}{\text{the variance if you use overall mean}} .$$

It is useful to think in terms of the reciprocals of the variance. We want to maximize the reciprocal variance by combining our two estimates. The maximum reciprocal variance is just the sum of the two individual reciprocal variances. Thus, the best that can be done is to double the reciprocal variance (when the two individual variances happen to be equal) and thus halve the variance.[12]

The usual expression for the Bayesian credibility is the value for the parameter a that gives the minimum variance, $Z_B = \tau^2/(\tau^2 + \delta^2)$.

The variance is larger than the minimum for $a = Z_B + \Delta Z$. In this case,

$$\Delta V = V(Z_B + \Delta Z) - V(Z_B) = V'(Z_B)\Delta Z + V''(Z_B)\,\frac{(\Delta Z)^2}{2}$$

where V' and V'' are the first and second derivatives, respectively. (Higher derivatives are zero since V is given by a parabola.) Then

$$\Delta V = (\Delta Z)^2(\delta^2 + \tau^2)$$

$$\frac{\Delta V}{V} = \frac{(\Delta Z)^2(\delta^2 + \tau^2)^2}{\delta^2\tau^2}$$

$$\frac{\Delta V}{V} = \frac{(\Delta Z)^2}{Z_B(1 - Z_B)} .$$

This is the desired expression for the relative increase in variance that occurs when the value used for the credibility is other than that indicated by Bayesian credibility.

[12] A related result is given in Appendix C of Chapter 2 of the ISO Credibility White Paper [3]. The optimal weights to assign to the individual estimates are inversely proportional to the variances.

APPENDIX II

This appendix explores the behavior of the expression derived in Appendix I for the relative change in variance. It is shown that $\Delta V/V$ has the smallest maximum for $R = 8$.

$$\text{Let } R = \frac{F}{k} = \frac{\text{Standard for Full Credibility}}{\text{Bayesian Credibility Parameter}}$$

$$r = \frac{n}{k} = \frac{\text{Number of Claims}}{\text{Bayesian Credibility Parameter}}.$$

$$\text{Let } g(r,R) = \frac{(Z_B - Z_C)^2}{(Z_B)(1 - Z_B)}.$$

This is the expression derived in Appendix I for $\Delta V/V$. However,

$$\frac{1}{Z_B} = \frac{n+k}{n} = 1 + \frac{1}{r} = \frac{1+r}{r}$$

$$\frac{1}{1 - Z_B} = \frac{n+k}{k} = 1 + r$$

$$Z_B - Z_C = \begin{cases} \dfrac{n}{n+k} - \left(\dfrac{n}{F}\right)^{.5} & 0 \leqq n \leqq F \\[2ex] \dfrac{n}{n+k} - 1 & F \geqq \mathrm{n} \end{cases}$$

$$Z_B - Z_C = \begin{cases} \dfrac{\mathrm{r}}{1+r} - \left(\dfrac{\mathrm{r}}{R}\right)^{.5} & r \leqq R \\[2ex] -\dfrac{1}{1+r} & r \geqq R \end{cases}$$

Therefore, if $r \geqq R$,

$$g(r,R) = \left(-\frac{1}{1+r}\right)^2 \left(\frac{1+r}{r}\right)(1+r) = \frac{1}{r},$$

and, if $r \leqq R$,

$$g(r,R) = \left(\frac{r}{1+r} - \left(\frac{r}{R}\right)^{.5}\right)^2 \left(\frac{1+r}{r}\right)(1+r) = \left(\sqrt{r} - \frac{(1+r)}{\sqrt{R}}\right)^2.$$

For any given R, the local maximums on the interval $0 \leqq r \leqq R$ occur at $r = 0$, $r = R/4$, $r = R$.[13]

$$g(0,R) = g(R,R) = 1/R$$
$$g(\tfrac{1}{4}R,R) = 1/R + \tfrac{1}{2}(R/8 - 1)$$

Thus, $\text{MAXIMUM}_r\, g(r,R) = \begin{cases} 1/R & R \leqq 8 \\ 1/R + R/16 - \tfrac{1}{2} & R \geqq 8 \end{cases}$

Thus, $\text{MINIMUM}_R\, \text{MAXIMUM}_r\, g(r,R) = 1/8$, which occurs when $R = 8$.

[13] The first and last are endpoints. The second has $\partial g/\partial r = 0$. The other points where the partial derivative is zero are the minimums where $g = 0$. For $r \geqq R$, $g(r,R) = 1/r$ and is decreasing.

APPENDIX III

The appendix details the derivation of an expression for the maximum difference in Bayesian credibilities that occurs when an estimated value for the Bayesian credibility parameter k is used, rather than the correct value of the parameter.

$$\text{Let } T = \frac{\tilde{k}}{k} = \frac{\text{Estimate of Bayesian Credibility Parameter}}{\text{Correct Bayesian Credibility Parameter}}$$

$$r = \frac{n}{k} = \frac{\text{Exposures}}{\text{Correct Bayesian Credibility Parameter}} .$$

Then the difference in credibilities is

$$\Delta Z = \frac{n}{n + \tilde{k}} - \frac{n}{n + k} = \frac{r}{r + T} - \frac{r}{r + 1}$$

$$\Delta Z = \frac{r(1 - T)}{(1 + r)(T + r)} .$$

As expected, when k is overestimated, $(T > 1)$, the estimated credibility is too low, $(\Delta Z < 0)$.

Taking the partial derivative of ΔZ with respect to r indicates that ΔZ has a maximum when $r = T^{.5}$. The maximum value of $|\Delta Z|$ is

$$|\Delta Z|_{\text{Max}} = \frac{|T - 1|}{(T^{.5} + 1)^2} .$$

As expected, this quantity has a minimum value of zero at $T = 1$, i.e., when the Bayesian credibility parameter is correctly estimated. This expression has the same value for T and $1/T$. In other words, when k is misestimated by a given factor, the magnitude of the maximum difference in the credibility is the same whether k is overestimated or underestimated.

APPENDIX IV

This appendix derives an expression for the relative increase in variance that occurs when an estimated value for the Bayesian credibility parameter k is used, rather than the correct parameter value. An expression for the maximum relative increase in variance is also derived.

$$\text{Let } T = \frac{\bar{k}}{k} = \frac{\text{Estimate of Bayesian Credibility Parameter}}{\text{Correct Bayesian Credibility Parameter}}$$

$$r = \frac{n}{k} = \frac{\text{Exposures}}{\text{Correct Bayesian Credibility Parameter}} .$$

Then, from equation (5),

$$\frac{\Delta V}{V} = \frac{(\Delta Z)^2}{Z_B(1 - Z_B)} ,$$

but, as is shown in Appendix III,

$$\Delta Z = \frac{r(T - 1)}{(1 + r)(T + r)} .$$

Also note that

$$\frac{1}{Z_B} = \frac{n + k}{n} = \frac{1 + r}{r}$$

$$\frac{1}{1 - Z_B} = \frac{n + k}{k} = 1 + r$$

Substituting in equation (5) gives

$$\frac{\Delta V}{V} = \frac{r(T - 1)^2}{(T + r)^2}$$

Taking the partial derivative with respect to r indicates a maximum when $r = T$. Therefore, the maximum value of $\Delta V/V$ is

$$\left(\frac{\Delta V}{V}\right)_{\text{Max}} = \frac{(T - 1)^2}{4T} .$$

As expected, this quantity has a minimum value of zero at $T = 1$, i.e., when the Bayesian credibility parameter has been correctly estimated. This expression has the same value for T and $1/T$. In other words, the maximum relative increase in variance is the same whether k has been overestimated or underestimated by a given factor.

In Appendix III, the same behavior was noted for the maximum difference in credibility. The factor by which k is misestimated, rather than $|k - \bar{k}|$, the difference between the estimated and correct values, is the important quantity.[14]

[14] Therefore, we would expect that confidence intervals for k would not be symmetric around our best estimate. Rather, they should be larger on the high end and smaller on the low end. This behavior was noted in Section 7 of Meyers [8].