DISCUSSION OF PAPER PUBLISHED IN VOLUME LXX

REINSURING THE CAPTIVE/SPECIALTY COMPANY

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VOLUME LXX

DISCUSSION BY OAKLEY E. VAN SLYKE

Mr. Steeneck has presented the basic principles of applying utility theory in reinsurance pricing in an admirable fashion. His article is straightforward and comprehensive. The footnotes provide an excellent bibliography of the current literature on the subject.

The interested reader is particularly directed to the monograph by Leonard Freifelder (Freifelder (1976)). These two works complement each other well.

Utility theory has been useful to this reviewer as a means of achieving a fresh viewpoint on a problem, rather than as a simplistic solution to the problem of finding numerical results (e.g., rates) that adequately reflect one's risk aversion. If the user avoids the pursuit of simple answers through abstract formulas, he can find much of practical value in the methods discussed by Mr. Steeneck. This is especially true for exponential utility functions.

Several of Mr. Steeneck's points merit discussion. This review also provides an opportunity to show two minor results of the reviewer's investigations into utility theory.

PRACTICALITY

Utility theory is practical. We are all familiar with the inadequacy of the simple calculation of expected value. Using utility theory only requires that we shift our mental framework from calculating the $E(X)$ and $E(X-u)^2$ to include the calculation of $E(U(X))$.

This mental shift will be clear if a train of thought developed by Steeneck on page 257 is followed, together with a change in terminology. Let the utility function be

$$RAC(X,c) = c(1 - \exp(x/c)).$$

$RAC(X,c)$ is the "risk adjusted cost" of outflow X with utility scale c.

Then for an aggregate loss distribution

$$F(X) = \Pr(x < X)$$
$$RAC_F(c) = c \ln E_F (\exp (x/c))$$

In other words, the risk adjusted cost of an aggregate loss distribution is a scale adjustment in c of the calculation of the expected value of the aggregate loss distribution. The adjustment in c scales down each possible loss to its multiple of c, inflates it using the exponential function, takes an average, and then backs out the scale adjustments by taking the log and multiplying by c.

In the example given by Steeneck on page 259, c was \$4,000,000.

The use of c instead of $1/r$ makes sense. It puts the constant in real units, dollars, instead of imaginary ones, $(\text{dollars})^{-1}$.

$RAC(c)$ is a simple concept which includes a great deal of information about $F(X)$. (As we shall see, it includes all of the information about $F(X)$.) We expect to see the day when $RAC(c)$ will be computed as routinely as Var (X) is today, particularly when it can be expressed in closed form.

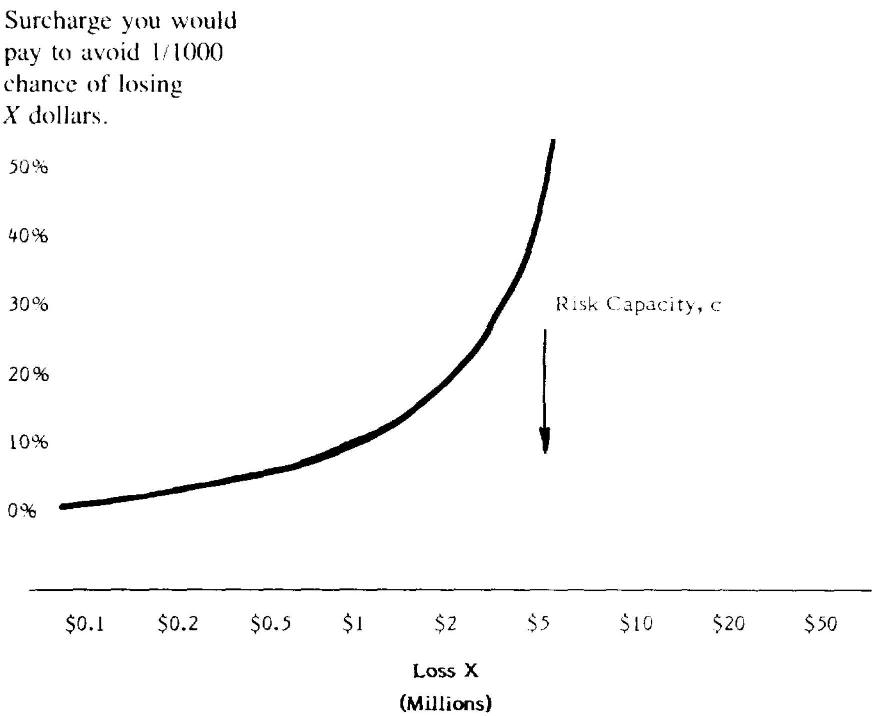
ESTIMATING RISK CAPACITY

It is easy to estimate the risk capacity, c, well enough for practical applications. Reinsurance exists because all insurance companies have a limited capacity to bear risk. In some cases, risk aversion is so high that the firm will do whatever possible to ensure that a catastrophic loss will not bankrupt the firm. In practical applications, however, the level of loss at which management begins to get really concerned is quite a bit less than the level of loss that would bankrupt the firm. In insurance jargon, we call this level of loss the firm's "risk capacity."

Exhibit 1 shows an example of the risk capacity of a particular firm. In this example, the height of the line shows the surcharge the reinsured would be willing to pay to avoid a 0.1% chance of losing a sum of money. The reinsured would be willing to pay only about 0.1% of the sum of money if this sum, X, were not very great. It would not pay a significant surcharge to avoid a 0.1% chance of losing \$10,000. If the amount were much greater than the reinsured's risk capacity, however, then the firm would be willing to pay much more than 0.1% of the possible loss.

EXHIBIT 1

SURCHARGE FOR RISK CURVE



Move the bottom scale left or right until it is in the right place for your decision. Your risk capacity, c, will be below the vertical arrow.

Because of the reinsured's limited capacity to bear risk, management is willing to pay a surcharge (risk charge) to avoid financial fluctuations. To avoid a 0.1% chance of paying out $1,000,000, for example, management is willing to pay something in excess of 0.1% of $1,000,000. The additional amount is called a "risk charge." The total amount management is willing to pay, perhaps $1,100 in this example, is called the "risk adjusted cost" (*RAC*) of the risk's probability distribution.

As Cozzolino (1979) has pointed out, the selection of risk capacity c is not even necessary to make a decision. All that it is generally necessary is that one's risk capacity is known to be in a certain range.

The technique suggested by Cozzolino is to show the risk adjusted cost for one's own aggregate loss distribution with and without the inclusion of the reinsurance contract being evaluated. Each net aggregate loss distribution leads to a unique risk adjusted cost profile. Exhibit 2 shows the profile for a reinsurance decision about a possible cession that involves a considerable amount of risk. In this example, if the reinsurer's risk capacity is less than about $2,000,000, he will not accept the retrocession.

The success of this technique hinges on the fact that more risky alternatives will always have curves that slope downward more steeply than less risky alternatives. As a result, different options will produce risk profile curves that intersect one another if there are significant differences in the uncertainty of results for the options. Obviously, if the risk profile curve for one option is lower than the risk profile curve for another option regardless of one's risk capacity, it is the more attractive alternative.

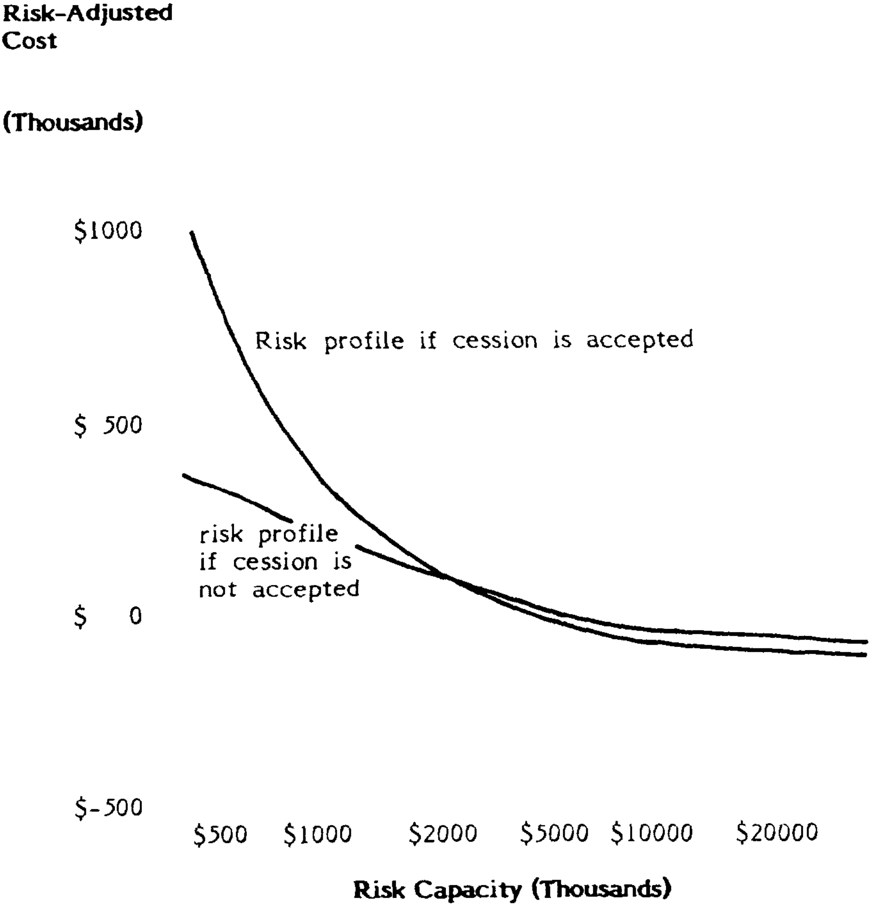
REINSURANCE NEGOTIATIONS

Reinsurance makes sense even when the reinsurer is more risk averse than the reinsured. Steeneck's statement, "If the reinsurer has the same utility function or is less risk averse, a deal can be struck" is unnecessarily restrictive. This is seen in practice as small reinsurers take small pieces of treaties reinsuring large primary companies.

The reason is simple: The reinsured losses are not correlated with the reinsurer's losses; they are correlated with the reinsured's losses.

EXHIBIT 2

RISK PROFILE CURVES FOR THE REINSURANCE DECISION



A risk profile is a display of $RAC(X,c)$ for a range of c. A risk profile is a unique mapping of an aggregate loss distribution $F(X)$.

To the excess writer of, say, \$1,000,000 xs \$1,000,000, the risk looks like

$$RAC_e = c_e \ln E\left[\exp\left(\frac{\text{Max}(0, X^{**} - \$1 \text{ million})}{c_e}\right)\right]$$

where X^{**} is limited to \$2,000,000, and the excess writer's risk capacity is c_e.

To the primary writer, the cession is worth

$$RAC_p = c_p \ln E\left[\exp\left(\frac{X^{**}}{c_p}\right)\right] - c_p \ln E\left[\exp\left(\frac{X^{*}}{c_p}\right)\right]$$

$$= c_p \ln \frac{E\left[\exp\left(\frac{X^{**}}{c_p}\right)\right]}{E\left[\exp\left(\frac{X^{*}}{c_p}\right)\right]}$$

where X^* is limited to \$1,000,000, and the primary writer's risk capacity is c_p.

Reinsurance makes sense when

$$RAC_p > RAC_e$$

This leads to two thoughts:

— One's own risk profile and estimates of the risk profiles of the potential players in a reinsurance deal can help one create a negotiating strategy. Changes in the terms of the reinsurance arrangement can be reflected in changes in the risk profiles. This will identify ways to change the deal to improve it for all parties.

— This analysis makes it clear why new entries always appear in the reinsurance market. Reinsurers have portfolios of losses that are correlated with the potential cession. In workers' compensation, for example, losses in various contracts may be correlated through inflation, benefit level changes, or loss of statutory immunities or defenses. The new entries have risk capacity arising from their own cash flow, but do not have existing portfolios of losses that are correlated with the new cession. (Of course, presumably, they do not have the underwriting expertise of the experienced writer, either.)

EXPONENTIAL UTILITY VS. THE VARIANCE PRINCIPLE

Two advantages of using exponential utility instead of the popular variance principle are:

— Exponential utility provides the correct asymptotic behavior as the loss being considered gets large and its probability gets small. This is illustrated in Exhibit 3.

In contrast, the variance principle leads to premiums greater than the loss itself.

— Exponential utility leads to a more distinct concept of risk capacity. Exhibit 4 shows that the disutility associated with a loss in excess of one's capacity (as defined above) reflects a marked aversion to losses greater than one's risk capacity. This agrees with our intuitive understanding of how we accept and cede risks. The variance principle, in contrast, does not show such a distinct "flinch point."

ESTIMATION

New methods of estimating aggregate loss distributions make practical application much easier. Monte Carlo simulation is readily available, although somewhat costly in terms of computer time. Monte Carlo simulation handles virtually all practical problems including multiline contracts. Monte Carlo simulation also gives the flexibility to break apart workers' compensation losses by type of injury, distinguish various sublines of liability coverage, and so on.

Aggregate distributions are receiving more attention recently. Heckman and Meyers (1983) describe a method of calculating aggregate loss distributions by a method of characteristic functions. Venter (1983a) shows an application of a method of numerical estimation developed by Panjer. Jewell (1983) extends Panjer's work to a dynamic risk portfolio. Each of these authors shows how to calculate the expected value of an excess premium as well as first dollar losses. We can expand Venter's conclusion (from page 69) to read:

> "By approximating the severity distribution with discrete probabilities, the aggregate distribution and excess premium functions and the risk adjusted cost can thus be estimated recursively."

Venter (1983c) has dicussed the advantages of modeling aggregate loss distributions with transformed Gamma distributions. Distributional models may lead directly to general formulas for the risk adjusted cost.

EXHIBIT 3

Asymptotic Behavior

Surcharge to avoid 1/1000 chance of losing x units of risk capacity, expressed as a multiple of X.

$$S = (\ln(.999 + .001 \exp(x/c)) \div .001\, x/c) - 1$$

(exponential utility principle)

or

$$S = .999X/c$$

(variance principle)

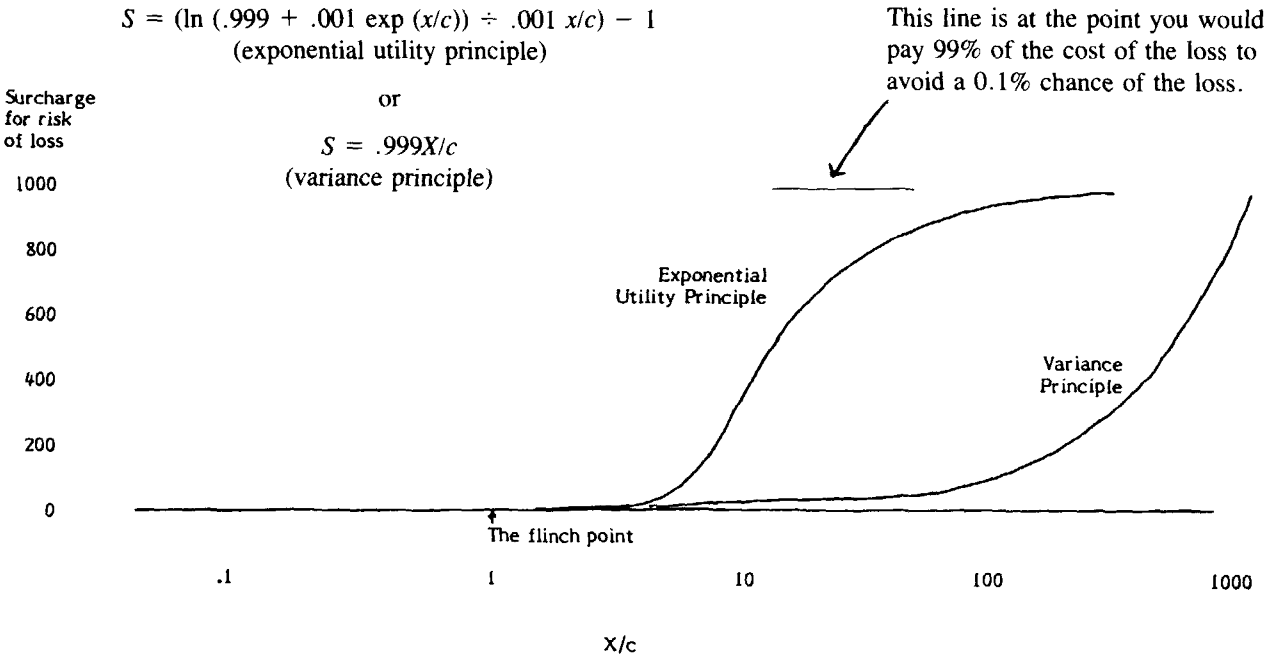
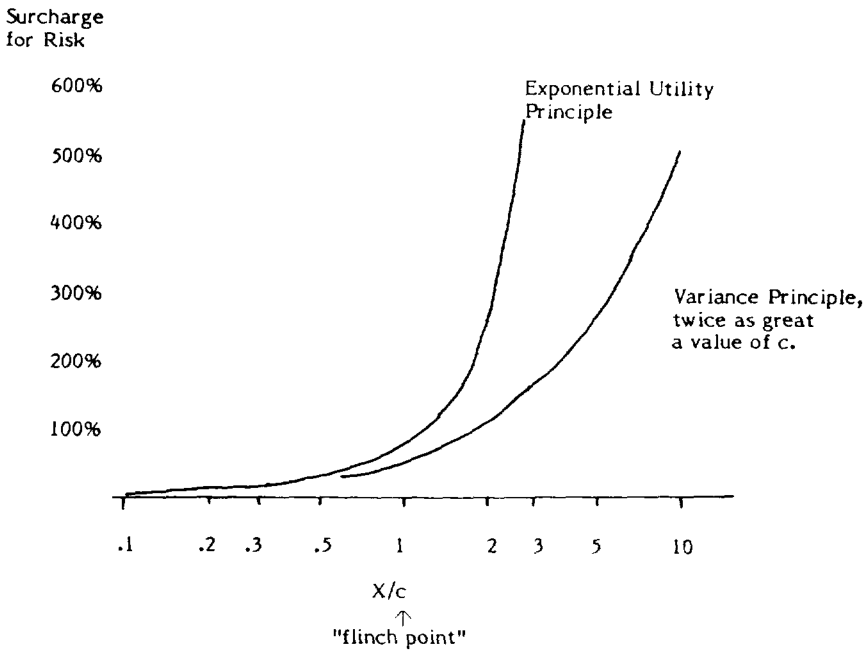


EXHIBIT 4

THE FLINCH POINT



DECREASING AVERSION TO RISK

Venter (1983b) has pointed out the theoretical advantages of

$$RAC\ (X,c,p) = c\left(1 - \exp\left(-\frac{X^p}{c}\right)\right)$$

or some other utility function with decreasing aversion to risk.

This may be a valuable point, but in practice a reinsurer is not likely to vary its risk capacity significantly in response to a loss under a single treaty. It is more realistic to expect a reinsurer to become more or less aggressive in response to a series of losses, a change in the competitive marketplace, or some other factor affecting many treaties. In short, the refinement will not matter in most practical applications.

Indeed, as we have seen, it is easy to explain the search for one's risk aversion if risk aversion is taken to be constant. It is difficult to develop such a procedure if one's risk aversion is supposed to be expressed as a function of the surplus left *after* the loss.

Most importantly, using an exponential utility function does not necessarily result in a misstatement of our utility function. We can be correct if we can correctly see the utility of $(a - X)$ from our vantage point at a. We can be as averse to $(a - X)$ as we wish.

DISTRIBUTIONAL STATISTICS

Characteristic functions and moment generating functions (m.g.f.'s) can be used in tandem to derive simple results for frequently used models. As Heckman and Meyers (1983) showed,

$$\phi_F\ (t) = E\ [\exp(itx)] = \int_0^\infty \exp\ (itx)\ dF(x).$$

where ϕ_F is the characteristic function of $F(x)$.

This leads directly to

$$RAC_F(c) = c \ln E_F\ [\exp\ (x/c)] = c \ln \int_0^\infty \exp\ (x/c)\ dF(x)$$

$$= c \ln \phi_F\left(\frac{1}{ic}\right)$$

where the subscript F refers to the aggregate loss distribution.

They also showed that

$$\phi_{F*G}(t) = \phi_F(t)\,\phi_G(t)$$

and that the characteristic function for an aggregate loss distribution F (with claim severity distribution S) is:

$$\phi_F(t) = \sum_{n=0}^{\infty} p(n)(\phi_S(t))^n.$$

This leads directly to

$$RAC_F(c) = c \ln \sum_{n=0}^{\infty} p(n) \exp\,[n \cdot (RAC_s(c)/c)]$$

where RAC_s is the risk adjusted cost of a single claim.

The $RAC_s\,(c)$ is also closely related to the moment generating function of the severity distribution

$$RAC_s(c) = c \ln M_s\left(\frac{1}{c}\right).$$

We mentioned earlier that the $RAC_F(c)$ contained all the information in $F(X)$. This is now clear because the m.g.f. of a probability distribution is unique (Hogg and Klugman (1984), page 19). Hogg and Klugman have shown (page 50) that if the moment generating function of the severity distribution, $M_s(t)$, is known, and the claim frequency distribution is Poisson, the moment generating function of the aggregate loss distribution is

$$M_F(t) = \exp[\lambda(M_s(t) - 1)]$$

The risk adjusted cost is therefore

$$\begin{aligned} RAC_F(c) &= c \ln\,[M_F(1/c)] \\ &= c \ln\,[M_s(1/c) - 1]. \end{aligned}$$

For example, if the claim size distribution is exponential

$$p(x) = \frac{1}{\sigma} \exp\left(-\frac{x-\theta}{\sigma}\right)$$

then

$$M_s(t) = \frac{\exp\,(t\theta)}{1 - \sigma t}$$

$$M_s\left(\frac{1}{c}\right) = \frac{\exp(\theta/c)}{1 - (\sigma/c)}$$

The roles of σ, θ and c as scale adjustments are clear. This leads to the following risk adjusted costs:

$$\begin{aligned} RAC_F(c) &= c\lambda\left[\frac{\exp(\theta/c)}{1 - \sigma/c} - 1\right] \\ &= \frac{c\lambda}{1 - \sigma/c}\left[\frac{\sigma}{c} + \exp(\theta/c) - 1\right]. \end{aligned}$$

If $\theta = 0$,

$$RAC_F(c) = c\lambda \frac{\sigma}{c - \sigma}.$$

If θ/c is close to 0,

$$RAC_F(c) \doteqdot c\lambda \frac{\sigma + \theta}{c - \sigma}.$$

This development suggests several obvious extensions to be pursued:

—To determine the risk adjusted cost if the claim frequency distribution is negative binomial.

—To determine the risk adjusted costs for other severity distributions for which the m.g.f. is known in closed form.

—To determine the risk adjusted costs for truncated versions of distributions for which the m.g.f. is infinite.

—Numerical approximations based on m.g.f.'s, characteristic functions, or recursive methods.

PROBABILITY, UTILITY, AND PRESENT VALUE

The time value of money is important in many practical problems. In these problems a present value factor $v(i)$ can be associated with each event that produces a loss $X(i)$. The functions v and X may be continuous or discrete.

Interest should be handled in such a way that the distributive property applies to the function RAC. That is, the risk adjusted cost of a possible set of events should be independent of how fine a description one makes of the set of possible events.

The function

$$RAC(c) = c \ln \Sigma\, p(i)\, v(i) \exp (X_i/c)$$

meets this criterion. So does its continuous counterpart

$$RAC(c) = c \ln \int_0^{\infty} v(x) \exp (X/c)\, dF(X).$$

With this definition, the total risk adjusted cost RAC of a set of possible events with risk adjusted costs $RAC(i)$ is the risk adjusted cost of all possible events, with each taken at its present value:

$$RAC_0 = c \ln \sum_i p(i)\, v(i) \exp (RAC(i)/c)$$

$$= c \ln \sum_i p(i)\, v(i) \left[\sum_{\substack{within \\ i}} p(x)\, v(x) \exp (x/c) \right]$$

$$= c \ln \sum_i \sum_{\substack{within \\ i}} (p(i)p(x))(v(i)v(x)) \exp (x/c).$$

In practice, then, probability and present value are almost interchangeable concepts. Present value and utility are not interchangeable concepts. This surprising result follows from the distributive property.[1]

CONCLUSION

The reader is encouraged to try the utility-user's viewpoint in practical problems. Starting perhaps with a discrete decision (such as whether or not to underwrite a particular risk or block of risks), decide on your risk capacity using Exhibit 1 or Exhibit 4. Sketch the risk profile curves for the decision by calculating a few points on each. Think about the interplay between your risk capacity and the decision you prefer (yes or no). Are you being consistent? Have you learned anything about the decision you didn't know before? With use, this additional viewpoint may begin to feel as natural as considering both probability and time in the decision.

[1] It would be reasonable to postulate that the multiplicative associativity of $p(i)$ with $p(x)$ and $v(i)$ with $v(x)$ follows directly from a distributive property on RAC. I have not been able to prove this, nor find an exception. A friend of mine says he proved it on a popcorn box at an Oilers game, but lost the proof. I would like a demonstration of whether or not "Oilers postulate" is true.

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