DISCUSSION BY KURT A. REICHLE AND JOHN P. YONKUNAS

Once again, Steve Philbrick has taken a concept which makes many actuaries feel uncomfortable and, through lucid writing and clear examples, made it available to all who take the time to read him. Prior to Mr. Philbrick's paper, fitting size of loss distributions has been a tool primarily available only to the "pure actuary." This guide to the Pareto distribution provides all actuaries access to a powerful means of analysis. The strength of this tool is matched only by its simplicity as presented by Mr. Philbrick.

Using data prepared by the Actuarial Committee of the Insurance Services Office (ISO), this review will examine three facets of the single parameter Pareto distribution: the impact of development, the impact of trend, and evidence that the Pareto may overstate the tail of the distribution function. We also will suggest some practical guides for putting the Pareto distribution to use, including an analysis of the sensitivity of the parameter estimate to the number of claims available.

WHY THE PARETO?

Beginning in 1977, the Ad Hoc Increased Limits Subcommittee of ISO (subsequently the Increased Limits Committee) searched for the best-fitting continuous distribution for liability losses. Because no distribution seemed to fit both small and large claims well, the Subcommittee decided instead to look for the best-fitting curve for losses above a lower truncation point. After a good deal of research, the two parameter Pareto was selected as providing the best fit to liability losses. Although many enhancements have been made in the methodology used to derive increased limits factors since 1977, the Pareto curve remains ISO's favored distribution.

Implementation of the two parameter Pareto distribution does require complex formulas, including a set of Newton-Raphson equations used iteratively to solve for the Maximum Likelihood Estimates (MLE) of each parameter. These formulas are not solved easily without the use of a computer, and therefore require extensive programming and computing costs. While this complexity does not pose an insurmountable problem to the "pure actuary," it may hinder the efforts of the "lay actuary" to use models rather than empirical data directly.

The one parameter Pareto distribution is quite simple to use, as demonstrated in Mr. Philbrick's article. Estimates of various moments of the distribution are very simple to calculate, and the formulas are easily remembered. The same is true for many estimates of the parameter, including the MLE. However, to our knowledge, no extensive research has been published on how appropriate the one parameter Pareto is for loss distributions. The two parameter Pareto has a proven track record as an acceptable model for excess losses. A simple mathematical transformation will show that the parameter of the one parameter Pareto is equivalent to a parameter of the two parameter Pareto (see Appendix A). Hence, by using the one parameter version, we obtain much of the power of the two parameter Pareto without its accompanying complexity.

LOSS DEVELOPMENT

The change in the cumulative value of losses for a given accident period has been discussed extensively in the actuarial literature. But very little has been published on how the distribution of individual claims changes as losses mature, and in particular how the parameters underlying that distribution change. A full discussion of loss development and its effect on the Pareto is beyond the scope of this review. We will, however, cite some of our observations from examining data provided by ISO.

An analysis of losses usually begins by segregating the data into various time periods (report year, accident year, policy year, etc.). To put these periods on a comparable basis, two adjustments are commonly utilized: trend and loss development. Although the use of the one parameter Pareto implies that severity trend may be ignored (as discussed in a later section), loss development may not. An adjustment must be made for loss development prior to combining various periods for analysis.

In casualty lines of insurance, loss development generally has a positive impact on losses; i.e., average losses become more severe as the largest losses emerge most slowly. Remember that severity and the Pareto parameter are inversely related. Therefore, an *a priori* expectation is that the Pareto parameter should decrease as an accident period becomes more mature.

As will be seen, the value of the Pareto parameter varies substantially from one valuation to another. Property losses from several accident periods may be combined to derive the parameter with no recognition of the date of loss. Applying the same approach to casualty lines may severely overstate the parameter and understate the excess severity. An excellent example of this is inadvertently included in Mr. Philbrick's paper. In Application 3 in Section VII, Mr. Philbrick combines professional liability claims from four accident years with no adjustment for development, calculating a MLE of the Pareto parameter of 1.408. Deriving the maximum likelihood estimate of each year separately produces Pareto parameters of 1.176, 1.002, 1.570 and 1.746 for 1978 through 1981 respectively. The clear upward trend in these values is to be expected and most often results in an overstatement of the parameter if the claims are simply combined with no adjustment for development.

Additional evidence that the parameter is inversely related to maturity was found when we examined occurrence size distributions (OSD's) provided by ISO. A lower truncation point of $25,000 has been selected. The MLE of the Pareto parameter was calculated by policy year, by evaluation month. A table of parameters for Owners, Landlords, and Tenants (OL&T) Bodily Injury Liability follows.

Policy Year	Evaluation Month						
	27	39	51	63	75	87	99
1975	1.313	1.546	1.412	1.377	1.308	1.283	1.281
1976	1.547	1.467	1.407	1.309	1.258	1.225	
1977	1.539	1.578	1.482	1.389	1.347		
1978	1.644	1.578	1.460	1.364			
1979	1.688	1.518	1.443				
1980	1.700	1.590					
1981	1.717						

As expected, the parameter decreases as the policy year matures. Loss development must be accounted for prior to analysis. One could use a triangulation to adjust immature parameters to their ultimate values.

Note that with the exception of the 27 month evaluation, the parameter is relatively stable across policy years within a given evaluation. We found this to be true for other values of the truncation point and for Products Bodily Injury Liability data.

Why the parameters calculated at 27 months exhibit an upward trend is not clear. It may indicate that data as of 27 months is too immature for analyzing excess losses. It may also indicate a change in industry reserving practices. Such a change would affect the distribution most at the earliest evaluation and least at later maturities.

We suggest that more research be devoted to determining the impact of loss development on the Pareto parameter. We also recommend that the user of the one parameter Pareto not blindly combine data without adjusting for loss development.

TREND

Of all the implications of the Pareto distribution, the most vexing is that trend does not affect excess loss severity, only loss frequency. How can such a distribution be appropriate for casualty-property losses? It is "obvious" that trend changes severity values. The work of ISO in fitting Pareto distributions to excess liability losses provides us with much data to evaluate this property.

As shown in the section on loss development, the Pareto parameter has remained relatively stable across policy periods for a given evaluation, which provides solid evidence that the parameter may be unaffected by trend.

Another empirical test is to examine the value of the average excess claim size over time. We again turn to the OSD's for OL&T Bodily Injury as compiled by ISO. Note that this raw data has not been adjusted for trend or loss development.

It is readily apparent that the average claim sizes have remained stable over time: both across policy years and within policy years. Trend does not appear to affect the average size of loss within a specific excess layer.

A more direct approach is to examine the form of distribution after making a transformation for trend. Assuming uniform trend, the value of the parameter is preserved; that is, q remains unchanged. The mathematical details of this transformation can be found in Appendix B.

How does one explain that the average claim size within a given excess interval remains unaffected after trend (and development)? At first glance it is intuitively unappealing if not totally unacceptable. Is it possible that the Pareto simply is not a realistic model for size of loss distributions?

The explanation is that the forces of trend and development fall upon the frequency side of the equation. As Mr. Philbrick points out, trend and development merely act to shift claims from one layer to another without changing the average in the layer. Instead, the frequency by layer changes as losses develop and occur later in time. So we still are stuck with adjustments for trend and development when the objective is to forecast aggregate loss dollars.

Average Claim Size in Layer $50,000 to $100,000

Policy Year	Evaluation Month						
	27	39	51	63	75	87	99
1975	79,174	77,135	78,306	78,407	80,263	80,462	80,533
1976	77,039	75,303	76,920	78,484	79,264	79,864	
1977	77,742	76,496	76,373	77,540	78,278		
1978	75,247	76,994	78,765	79,026			
1979	73,067	76,232	77,827				
1980	73,789	75,733					
1981	75,011						

Average Claim Size in Layer
$100,000 to $250,000

Policy Year	Evaluation Month						
	27	39	51	63	75	87	99
1975	172,059	165,249	163,967	166,215	167,825	171,715	173,931
1976	170,587	170,295	170,584	173,939	176,422	179,241	
1977	156,528	159,315	159,453	165,980	167,384		
1978	161,748	162,952	167,251	173,905			
1979	156,233	162,447	165,895				
1980	161,820	165,275					
1981	154,519						

In developing increased limits factors or excess loss premium factors, claim frequency drops out of the equation. All that remain are ratios of severities. Therefore, since the parameter is preserved after trend, an adjustment for trend may not be necessary. This could greatly simplify current procedures.

Data we have examined support the conclusion that trend does not affect excess severity. Hence, our preconceptions turned out to be significant stumbling blocks to accepting the Pareto. We hope that other readers will note the strength of the empirical evidence before accepting what appears to be "common sense."

GOODNESS OF FIT

In its initial consideration of the Pareto, the Increased Limits Subcommittee of ISO expressed concern that the Pareto may overstate the tail probabilities. Mr. Philbrick also refers to the fact that "most actual data suggests that the tail of the Pareto is still somewhat too 'thick' at extremely high loss amounts." Empirical evidence for casualty lines demonstrates the greater the truncation point, the larger the parameter estimate. That is, the indicated excess severity declines as one raises the truncation point when fitting the distribution. If excess claims were truly Pareto distributed, then one would obtain the same maximum likelihood estimate of the parameter independent of the truncation point chosen.

To demonstrate this overstatement, we look at Pareto parameters derived from ISO data for liability lines. These data are censored above at $500,000. The Workers' Compensation data are from a single insurer and are unlimited.

PARETO PARAMETERS

Line of Insurance	Lower Truncation Point (000)			
	25	50	100	250
OL&T Bodily Injury	1.281	1.330	1.447	1.508
Products Bodily Injury	.991	1.269	1.714	2.584
Workers' Compensation	1.454	1.715	2.316	2.086

It is clear from these data that, depending on the line of insurance, the Pareto parameter may be influenced greatly by the truncation point chosen. A significant implication of this upward trend is that parameters estimated with a low truncation point will generate conservative estimates of severities in the higher layers. For example, the estimate of the layer $1,000,000 excess of $1,000,000 may be greatly overstated if the truncation point for deriving the parameter is $25,000. The impact of this shortcoming is minimal if the excess layer estimated has a lower bound close to the truncation point. For instance, the estimate of the severity of any layer excess of $25,000 will be close to the actual severity in that layer if the truncation point for deriving the q parameter is close to $25,000. This is true even when the parameter increases rapidly with the truncation point.

ISO data provide evidence to support these conjectures. Displayed in the following table are comparisons of actual and fitted average severities for a selected group of gross layers.

OWNERS, LANDLORDS AND TENANTS BODILY INJURY
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1975 AS OF 99 MONTHS

Losses Limited to	Actual Severity	Difference Between Actual and Fitted Severities for Truncation Point of $25,000	$100,000
$125,000	$119,486	1.8%	1.5%
$150,000	$135,162	2.3%	1.4%
$175,000	$147,818	2.7%	1.1%
$200,000	$158,426	2.9%	0.7%
$250,000	$173,931	3.9%	0.7%
$300,000	$184,779	5.3%	1.1%
$350,000	$190,338	8.0%	2.9%
$400,000	$195,144	10.1%	4.2%
$450,000	$199,153	11.8%	5.2%
$500,000	$202,348	13.4%	6.1%

Two facts are readily apparent from this exhibit. First, the wider the layer being estimated, the greater the potential error. Second, the closer the truncation point is to the lower end of the layer, the smaller the error. For those interested, Appendix C contains similar data for other truncation points and evaluations.

In using the Pareto to derive increased limits factors, the magnitude of these errors is significantly reduced. Losses in excess of the lower truncation point generally represent a small percentage of the total claim count. Since increased limits factors incorporate claims of all sizes, the large percentage of losses below the truncation point reduces the impact of any error in the excess estimate and, therefore, any error in the increased limits factor.

PRACTICAL CONSIDERATIONS

The inability to correctly estimate the Pareto parameter will obviously affect the accuracy of the excess severity. As is commonly true when modelling, the error in the parameters is dependent upon the amount of data available. The Pareto is no exception.

A generally accepted way to express the potential errors in a parameter estimate is a classical credibility approach based on claim counts. Confidence intervals, although complex in their derivation, can be developed and used to indicate the number of claims required to achieve a given level of confidence for a given level of tolerance. For example, it can be shown that 310 claims are necessary to be 90% confident of being within 10% of the true value of the Pareto parameter. Confidence intervals in the following table were generated based on the MLE of the parameter. Formulas for the confidence intervals are developed in Appendix D.

Level of	Level of Confidence				
Tolerance	97.5%	95%	90%	85%	80%
± 5%	2160	1655	1165	890	710
±10%	580	445	310	240	190
±15%	275	210	150	115	90
±25%	115	85	60	45	40
±50%	40	30	20	15	10

This table can give the user an indication of the accuracy of the MLE. Clearly, a large number of excess claims is required for a high degree of accuracy. When sample data lack the credibility required, it is desirable to have available a source of parameters based on a larger volume of data. These parameters can then be used as the complement of credibility to the parameter derived from the data being analyzed. In Appendix E are Pareto parameters from ISO for various sublines of General Liability, Automobile Liability and Professional Liability. When either no data or limited volumes of data are available, these factors can provide reasonable estimates of excess severities.

An important question to answer before determining whether enough claims are available or whether to use ISO factors for credibility weighting is: "How sensitive is the estimate of an average net claim size to errors in the parameter estimate?" The following charts display the error in the estimate of the average net claim size for various layers of loss for a given error in the MLE.

ERROR IN AVERAGE CLAIM COST
PARETO PARAMETER = 1.00

Net Layer	Error in MLE 10%	25%	50%
\$400,000 excess of \$100,000	7.6%	17.7%	31.3%
\$900,000 excess of \$100,000	10.7%	24.0%	40.6%
\$1,900,000 excess of \$100,000	13.6%	29.6%	48.6%

ERROR IN AVERAGE CLAIM COST
PARETO PARAMETER = 1.50

Net Layer	Error in MLE 10%	25%	50%
\$400,000 excess of \$100,000	9.7%	21.9%	37.3%
\$900,000 excess of \$100,000	12.7%	27.6%	44.8%
\$1,900,000 excess of \$100,000	15.1%	31.8%	50.0%

Two generalizations can be drawn from this example. The percentage error varies with both the size of the parameter and the width of the layer being estimated. It is also interesting to note that the error in estimating an average net claim size for a specific layer can easily exceed the error in the MLE.

Because of the special properties of the Pareto, the errors for the layers shown above are dependent only on the relationship of the endpoints to the truncation point. Thus, the error in each of the two layers $400,000 xs $100,000 and $4,000,000 xs $1,000,000, with truncation points of $100,000 and $1,000,000, respectively, is the same, given an identical error in the underlying parameter.

Even though the percent error in a layer varies with the size of the parameter, the absolute dollar error decreases. This may be obvious since severity is inversely proportional to the Pareto parameter. Thus we might be more lenient with a lower degree of tolerance for a larger value of the parameter.

The following table displays absolute dollar errors in various net layers for a 10% error in the MLE.

DOLLAR ERROR IN NET LAYER

Net Layer	$q=1.00$	$q=1.50$
$400,000 excess $100,000	$12,284	$10,756
$900,000 excess $100,000	$24,587	$17,350
$1,900,000 excess $100,000	$40,708	$23,382

CONCLUSION

The empirical data we have examined indicate that the implications underlying the use of the one parameter Pareto are satisfied for casualty lines of insurance. This is not to say that limitations and restrictions on its use do not exist. It would be asking too much of any one parameter distribution to perfectly fit excess losses of all property and casualty lines. But the range of applications of the Pareto are substantial and, therefore, significant to anyone involved in excess pricing. This paper should provide encouragement to those who may have felt intimidated by the complexity of most modelling techniques available to actuaries. At the same time, it provides a powerful tool for those who regularly use more complex models but do not always need ten decimal point accuracy.

Most of the data referenced in this review is the product of the Increased Limits Committee and the staff of the Insurance Services Office. We wish to thank the ISO for allowing us the use of their data and analysis.

APPENDIX A

DERIVATION OF THE ONE PARAMETER PARETO FROM THE TWO PARAMETER PARETO

The one parameter Pareto is a special case of the two parameter Pareto. A common form of the two parameter Pareto and the one currently used by Insurance Services Office is:

$$f(x) = \frac{q \times (b)^q}{(x + b)^{(q+1)}} \qquad \text{for } 0 \le x < \infty \tag{1}$$

In this formula, the value of x represents individual claim sizes. Generally, this form is fit to losses above some lower truncation point.

We wish to derive $g(y)$, where

$$y = (x + b)/b \qquad \text{for } 1 \le y < \infty \tag{2}$$

and

$$dy = dx/b \tag{3}$$

Substituting (2) and (3) into (1) we have,

$$g(y) = f(x) \times (dx/dy)$$

$$g(y) = q \times y^{-(q+1)} \qquad \text{for } 1 \le y < \infty$$

which is the general form of the one parameter Pareto.

APPENDIX B

PARETO AND TREND

This appendix will show that, assuming uniform trend (i.e., all claims sizes trend at the same rate), the value of the parameter q is preserved (remains unchanged). We start by restating the Pareto,

$$f(x) = q \times x^{-(q+1)} \text{ for } 1 \leq x < \infty$$

Under uniform trend we have the following transformation,

$$y = a \times x \text{ for } a \leq y < \infty$$

and,

$$dy = a \times dx$$

Here the multiplicative factor a represents the impact of trend on individual claims.

Making this change of variable and solving for $g(y)$ we have,

$$g(y) = (1/a) \times q \times (y/a)^{-(q+1)} \text{ for } a \leq y < \infty$$

Renormalizing this density function by dividing all values of y by a, we have,

$$z = y/a;\ dz = dy/a \text{ for } 1 \leq z < \infty$$

The transformation then becomes,

$$h(z) = g(y) \times (dy/dz)$$

$$h(z) = q \times z^{-(q+1)}$$

The parameter q in all three density functions is the same and has not been affected by the transformation.

APPENDIX C

COMPARISON OF FITS

Presented in the following exhibits are comparisons of actual severities to data fitted by a one parameter Pareto. All data are from OL&T Bodily Injury as provided by ISO.

Exhibit C-1 displays fits of gross losses excess of \$25,000 using a truncation point of \$25,000. These fits produce an average absolute error of 1.3% and range from 0.0% to 3.8%.

Exhibit C-2 displays fits of gross losses excess of \$100,000 using a truncation point of \$100,000. The absolute errors in these fits average 2.0% and range from 0.0% to 6.1%.

Exhibit C-3 displays fits of gross losses excess of \$100,000 using a truncation point of \$25,000. The absolute errors are much greater in these fits. They average 5.6% and range from 1.9% to 13.4%.

In general, the wider the interval the greater the divergence. But these differences are relatively small when the lower bound of the layer is equal to the truncation point. Exhibit C-3 demonstrates that the error in predicting average severities can be quite large when the lower bound of a layer is much larger than the truncation point.

EXHIBIT C-1

SHEET 1

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1981 AS OF 27 MONTHS
MLE OF THE PARAMETER: 1.7172

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$37,975	$38,655	1.8%
$75,000	$43,649	$44,005	0.8%
$100,000	$47,087	$46,960	−0.3%
$125,000	$48,962	$48,868	−0.2%
$150,000	$50,527	$50,215	−0.6%
$175,000	$51,605	$51,224	−0.7%
$200,000	$52,549	$52,013	−1.0%
$250,000	$53,946	$53,173	−1.4%
$300,000	$54,955	$53,992	−1.8%
$350,000	$55,548	$54,606	−1.7%
$400,000	$56,006	$55,086	−1.6%
$450,000	$56,370	$55,472	−1.6%
$500,000	$56,702	$55,791	−1.6%

EXHIBIT C-1
SHEET 2

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1980 AS OF 39 MONTHS
MLE OF THE PARAMETER: 1.5899

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$38,695	$39,223	1.4%
$75,000	$44,758	$45,213	1.0%
$100,000	$48,499	$48,673	0.4%
$125,000	$50,850	$50,980	0.3%
$150,000	$52,765	$52,653	−0.2%
$175,000	$54,135	$53,933	−0.4%
$200,000	$55,335	$54,951	−0.7%
$250,000	$57,196	$56,484	−1.2%
$300,000	$58,550	$57,595	−1.6%
$350,000	$59,413	$58,446	−1.6%
$400,000	$60,143	$59,123	−1.7%
$450,000	$60,721	$59,677	−1.7%
$500,000	$61,207	$60,141	−1.7%

EXHIBIT C-1
SHEET 3

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1979 AS OF 51 MONTHS
MLE OF THE PARAMETER: 1.4427

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$39,551	$39,922	0.9%
$75,000	$46,500	$46,749	0.5%
$100,000	$51,110	$50,901	−0.4%
$125,000	$53,954	$53,777	−0.3%
$150,000	$56,253	$55,924	−0.6%
$175,000	$58,053	$57,610	−0.8%
$200,000	$59,646	$58,979	−1.1%
$250,000	$61,993	$61,095	−1.4%
$300,000	$63,613	$62,675	−1.5%
$350,000	$64,614	$63,915	−1.1%
$400,000	$65,447	$64,923	−0.8%
$450,000	$66,088	$65,764	−0.5%
$500,000	$66,601	$66,479	−0.2%

EXHIBIT C-1
SHEET 4

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1978 AS OF 63 MONTHS
MLE OF THE PARAMETER: 1.3644

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$39,696	$40,313	1.6%
$75,000	$47,088	$47,633	1.2%
$100,000	$52,090	$52,209	0.2%
$125,000	$55,529	$55,442	−0.2%
$150,000	$58,321	$57,895	−0.7%
$175,000	$60,545	$59,846	−1.2%
$200,000	$62,466	$61,449	−1.6%
$250,000	$65,349	$63,960	−2.1%
$300,000	$67,484	$65,866	−2.4%
$350,000	$68,749	$67,381	−2.0%
$400,000	$69,806	$68,627	−1.7%
$450,000	$70,706	$69,676	−1.5%
$500,000	$71,518	$70,577	−1.3%

EXHIBIT C-1
SHEET 5

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1977 AS OF 75 MONTHS
MLE OF THE PARAMETER: 1.3466

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$39,895	$40,404	1.3%
$75,000	$47,363	$47,841	1.0%
$100,000	$52,203	$52,519	0.6%
$125,000	$55,322	$55,839	0.9%
$150,000	$57,747	$58,367	1.1%
$175,000	$59,602	$60,384	1.3%
$200,000	$61,216	$62,046	1.4%
$250,000	$63,890	$64,657	1.2%
$300,000	$66,044	$66,646	0.9%
$350,000	$67,240	$68,232	1.5%
$400,000	$68,175	$69,539	2.0%
$450,000	$68,979	$70,642	2.4%
$500,000	$69,716	$71,592	2.7%

EXHIBIT C-1
SHEET 6

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1976 AS OF 87 MONTHS
MLE OF THE PARAMETER: 1.2254

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$40,414	$41,043	1.6%
$75,000	$48,555	$49,329	1.6%
$100,000	$53,923	$54,765	1.6%
$125,000	$57,602	$58,746	2.0%
$150,000	$60,653	$61,853	2.0%
$175,000	$63,165	$64,382	1.9%
$200,000	$65,374	$66,503	1.7%
$250,000	$68,936	$69,908	1.4%
$300,000	$71,613	$72,565	1.3%
$350,000	$73,314	$74,728	1.9%
$400,000	$74,685	$76,543	2.5%
$450,000	$75,807	$78,098	3.0%
$500,000	$76,844	$79,455	3.4%

EXHIBIT C-1
SHEET 7

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $25,000
POLICY YEAR 1975 AS OF 99 MONTHS
MLE OF THE PARAMETER: 1.2805

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$50,000	$40,137	$40,748	1.5%
$75,000	$48,060	$48,637	1.2%
$100,000	$53,525	$53,714	0.4%
$125,000	$57,217	$57,379	0.3%
$150,000	$60,188	$60,208	0.0%
$175,000	$62,586	$62,490	−0.2%
$200,000	$64,597	$64,388	−0.3%
$250,000	$67,535	$67,406	−0.2%
$300,000	$69,591	$69,735	0.2%
$350,000	$70,644	$71,614	1.4%
$400,000	$71,555	$73,177	2.3%
$450,000	$72,315	$74,508	3.0%
$500,000	$72,920	$75,661	3.8%

EXHIBIT C-2
SHEET 1

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1981 AS OF 27 MONTHS
MLE OF THE PARAMETER: 2.0623

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$114,905	$119,867	4.3%
$150,000	$127,343	$132,944	4.4%
$175,000	$135,917	$142,187	4.6%
$200,000	$143,421	$149,057	3.9%
$250,000	$154,519	$158,571	2.6%
$300,000	$162,540	$164,833	1.4%
$350,000	$167,257	$169,259	1.2%
$400,000	$170,897	$172,549	1.0%
$450,000	$173,788	$175,088	0.7%
$500,000	$176,430	$177,104	0.4%

EXHIBIT C-2
SHEET 2

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1980 AS OF 39 MONTHS
MLE OF THE PARAMETER: 1.6478

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,647	$120,777	2.7%
$150,000	$132,015	$135,659	2.8%
$175,000	$142,302	$146,940	3.3%
$200,000	$151,305	$155,842	3.0%
$250,000	$165,275	$169,103	2.3%
$300,000	$175,438	$178,602	1.8%
$350,000	$181,915	$185,802	2.1%
$400,000	$187,391	$191,484	2.2%
$450,000	$191,733	$196,104	2.3%
$500,000	$195,380	$199,948	2.3%

EXHIBIT C-2

SHEET 3

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1979 AS OF 51 MONTHS
MLE OF THE PARAMETER: 1.741

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,217	$120,567	2.9%
$150,000	$131,141	$135,022	3.0%
$175,000	$142,041	$145,809	2.7%
$200,000	$151,681	$154,207	1.7%
$250,000	$165,895	$166,513	0.4%
$300,000	$175,709	$175,162	−0.3%
$350,000	$181,765	$181,616	−0.1%
$400,000	$186,812	$186,641	−0.1%
$450,000	$190,693	$190,679	0.0%
$500,000	$193,797	$194,004	0.1%

EXHIBIT C-2
SHEET 4

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1978 AS OF 63 MONTHS
MLE OF THE PARAMETER: 1.5282

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,167	$121,049	1.6%
$150,000	$134,728	$136,499	1.3%
$175,000	$147,124	$148,449	0.9%
$200,000	$157,831	$158,043	0.1%
$250,000	$173,905	$172,639	−0.7%
$300,000	$182,802	$183,351	0.3%
$350,000	$192,851	$191,638	−0.6%
$400,000	$198,746	$198,290	−0.2%
$450,000	$203,759	$203,781	0.0%
$500,000	$208,289	$208,412	0.1%

EXHIBIT C-2
SHEET 5

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1977 AS OF 75 MONTHS
MLE OF THE PARAMETER: 1.5706

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,980	$120,952	2.5%
$150,000	$131,964	$136,198	3.2%
$175,000	$142,660	$147,907	3.7%
$200,000	$151,968	$157,249	3.5%
$250,000	$167,384	$171,357	2.4%
$300,000	$179,804	$181,622	1.0%
$350,000	$186,699	$189,506	1.5%
$400,000	$192,095	$195,797	1.9%
$450,000	$196,731	$200,962	2.2%
$500,000	$200,977	$205,296	2.1%

EXHIBIT C-2
SHEET 6

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1976 AS OF 87 MONTHS
MLE OF THE PARAMETER: 1.2489

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,416	$121,706	1.9%
$150,000	$135,523	$138,568	2.2%
$175,000	$148,781	$152,239	2.3%
$200,000	$160,440	$163,665	2.0%
$250,000	$179,341	$181,931	1.4%
$300,000	$193,369	$196,121	1.4%
$350,000	$202,346	$207,626	2.6%
$400,000	$209,584	$217,242	3.7%
$450,000	$215,508	$225,462	4.6%
$500,000	$220,981	$232,613	5.3%

EXHIBIT C-2
SHEET 7

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1975 AS OF 99 MONTHS
MLE OF THE PARAMETER: 1.4467

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,486	$121,238	1.5%
$150,000	$135,162	$137,087	1.4%
$175,000	$147,818	$149,515	1.1%
$200,000	$158,426	$159,611	0.7%
$250,000	$173,931	$175,194	0.7%
$300,000	$184,779	$186,822	1.1%
$350,000	$190,338	$195,941	2.9%
$400,000	$195,144	$203,348	4.2%
$450,000	$199,153	$209,525	5.2%
$500,000	$202,348	$214,782	6.1%

EXHIBIT C-3
SHEET 1

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1981 AS OF 27 MONTHS
MLE OF THE PARAMETER: 1.7172

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$114,905	$120,620	5.0%
$150,000	$127,343	$135,183	6.2%
$175,000	$135,917	$146,094	7.5%
$200,000	$143,421	$154,618	7.8%
$250,000	$154,519	$167,161	8.2%
$300,000	$162,540	$176,020	8.3%
$350,000	$167,257	$182,657	9.2%
$400,000	$170,897	$187,842	9.9%
$450,000	$173,788	$192,020	10.5%
$500,000	$176,430	$195,471	10.8%

EXHIBIT C-3
SHEET 2

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1980 AS OF 39 MONTHS
MLE OF THE PARAMETER: 1.5899

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,647	$120,908	2.8%
$150,000	$132,015	$136,062	3.1%
$175,000	$142,302	$147,662	3.8%
$200,000	$151,305	$156,893	3.7%
$250,000	$165,275	$170,784	3.3%
$300,000	$175,438	$180,852	3.1%
$350,000	$181,915	$188,559	3.7%
$400,000	$187,391	$194,692	3.9%
$450,000	$191,733	$199,714	4.2%
$500,000	$195,380	$203,921	4.4%

EXHIBIT C-3
SHEET 3

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1979 AS OF 51 MONTHS
MLE OF THE PARAMETER: 1.4427

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,217	$121,248	3.4%
$150,000	$131,141	$137,116	4.6%
$175,000	$142,041	$149,568	5.3%
$200,000	$151,681	$159,689	5.3%
$250,000	$165,895	$175,322	5.7%
$300,000	$175,709	$186,997	6.4%
$350,000	$181,765	$196,159	7.9%
$400,000	$186,812	$203,606	9.0%
$450,000	$190,693	$209,818	10.0%
$500,000	$193,797	$215,108	11.0%

EXHIBIT C-3

SHEET 4

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1978 AS OF 63 MONTHS
MLE OF THE PARAMETER: 1.3644

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,167	$121,431	1.9%
$150,000	$134,728	$137,693	2.2%
$175,000	$147,124	$150,624	2.4%
$200,000	$157,831	$161,254	2.2%
$250,000	$173,905	$177,901	2.3%
$300,000	$182,802	$190,533	4.2%
$350,000	$192,851	$200,578	4.0%
$400,000	$198,746	$208,835	5.1%
$450,000	$203,759	$215,792	5.9%
$500,000	$208,289	$221,767	6.5%

EXHIBIT C-3
SHEET 5

Owners, Landlords, and Tenants
Pareto Goodness of Fit
Gross Losses in Excess of $100,000
Policy Year 1977 as of 75 Months
MLE of the Parameter: 1.3466

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$117,980	$121,473	3.0%
$150,000	$131,964	$137,826	4.4%
$175,000	$142,660	$150,869	5.8%
$200,000	$151,968	$161,617	6.3%
$250,000	$167,384	$178,504	6.6%
$300,000	$179,804	$191,365	6.4%
$350,000	$186,699	$201,622	8.0%
$400,000	$192,095	$210,075	9.4%
$450,000	$196,731	$217,213	10.4%
$500,000	$200,977	$223,356	11.1%

EXHIBIT C-3
SHEET 6

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1976 AS OF 87 MONTHS
MLE OF THE PARAMETER: 1.2254

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,416	$121,762	2.0%
$150,000	$135,523	$138,749	2.4%
$175,000	$148,781	$152,576	2.6%
$200,000	$160,440	$164,171	2.3%
$250,000	$179,241	$182,786	2.0%
$300,000	$193,369	$197,315	2.0%
$350,000	$202,346	$209,143	3.4%
$400,000	$209,584	$219,061	4.5%
$450,000	$215,508	$227,565	5.6%
$500,000	$220,981	$234,983	6.3%

EXHIBIT C-3
SHEET 7

OWNERS, LANDLORDS, AND TENANTS
PARETO GOODNESS OF FIT
GROSS LOSSES IN EXCESS OF $100,000
POLICY YEAR 1975 AS OF 99 MONTHS
MLE OF THE PARAMETER: 1.2805

Losses Limited to	Actual Severity	Fitted Severity	Percent Difference
$125,000	$119,486	$121,630	1.8%
$150,000	$135,162	$138,326	2.3%
$175,000	$147,818	$151,790	2.7%
$200,000	$158,426	$162,993	2.9%
$250,000	$173,931	$180,801	3.9%
$300,000	$184,779	$194,546	5.3%
$350,000	$190,338	$205,632	8.0%
$400,000	$195,144	$214,855	10.1%
$450,000	$199,153	$222,708	11.8%
$500,000	$202,348	$229,517	13.4%

APPENDIX D

CONFIDENCE INTERVALS FOR THE PARETO PARAMETER

This appendix derives a formula that can be used to approximate the number of claims necessary to achieve a given level of confidence for a given level of tolerance in estimating the Pareto parameter. The results of this appendix are based upon the work of Jerry Jurschak in an unpublished paper entitled "The Pareto Distribution and Excess of Loss Reinsurance."

In Mr. Jurschak's paper he shows that the following formula represents a $100(1 - d)\%$ confidence interval for the Pareto parameter,

$$\left\{\frac{b \times q'}{n}, \frac{c \times q'}{n}\right\}$$

where

q' = MLE of the parameter

n = number of claims in the sample

$$b = \frac{1}{4} \times (Z_{d/2} + \sqrt{4n - 1})^2$$

$$c = \frac{1}{4} \times (Z_{(1-d/2)} + \sqrt{4n - 1})^2$$

Z = standard normal values.

Using a classical credibility approach, various values of n can be determined for a given level of confidence and a given level of tolerance (i.e. being within $\pm 10\%$ of the true value of q).

Assume that we wish to be within $100(k - 1)\%$ of the true value of the parameter $100(1 - d)\%$ of the time. The number of claims to comply with these constraints can be determined by solving the following confidence interval for n.

$$\left\{\frac{b \times q \times k}{n}, \frac{c \times q \times k}{n}\right\}$$

Substituting the above formulas into this confidence interval,

$$\frac{1}{4} \times (Z_{d/2} + \sqrt{4n - 1})^2 \times \frac{(q \times k)}{n} \geq q$$

$$q \geq \frac{1}{4} \times (Z_{(1-d/2)} + \sqrt{4n - 1})^2 \times \frac{(q \times k)}{n}$$

Since the absolute values of the standard normal numbers are equal, nothing is lost by dropping the right term. A few algebraic manipulations will produce

$$\frac{\sqrt{4n}}{\sqrt{k}} \geq (Z_{(1-d/2)} + \sqrt{4n-1})$$

and,

$$\sqrt{4n} - (\sqrt{4n-1} \times \sqrt{k}) \geq Z_{(1-d/2)} \times \sqrt{k}$$

For large n, we may question the necessity, bearing in mind the search for a simpler form, of subtracting the 1. In other words,

$$\sqrt{4n} \sim \sqrt{4n-1}$$

Using this simplifying assumption we have

$$\sqrt{4n} - (\sqrt{4n} \times \sqrt{k}) \geq Z_{(1-d/2)} \times \sqrt{k}$$

Solving this equation for n yields

$$n \geq \frac{Z^2_{(1-d/2)} \times k}{4 \times (1 - \sqrt{k})^2}$$

This formula is then used to generate the following table. Note that all figures have been rounded to the nearest multiple of five.

Level of Tolerance	Level of Confidence				
	97.5%	95%	90%	85%	80%
± 5%	2160	1655	1165	890	710
±10%	580	445	310	240	190
±15%	275	210	150	115	90
±25%	115	85	60	45	40
±50%	40	30	20	15	10

APPENDIX E

INDUSTRY VALUES OF THE PARETO PARAMETER q AS PRODUCED BY INSURANCE SERVICES OFFICE

Line of Insurance	Value of q	Truncation Point
GENERAL LIABILITY		
—Products		
—Bodily Injury		
—High Severity	0.938	\$25,000
—Low Severity	0.848	\$25,000
—Property Damage	1.144	\$ 3,000
—Manufacturers and Contractors		
—Bodily Injury		
—All Classes	0.945	\$40,000
—High Severity	0.825	\$35,000
—Low Severity	1.031	\$40,000
—Property Damage	0.987	\$ 4,000
—Owners, Landlords, and Tenants		
—Bodily Injury		
—All Classes	1.245	\$25,000
—High Severity	1.159	\$30,000
—Low Severity	1.600	\$30,000
PROFESSIONAL LIABILITY		
—Physicians	1.141	\$22,000
—Surgeons	1.110	\$22,000
—Hospitals	0.932	\$ 1,000
—Dentists	1.527	\$ 7,000
—Lawyers	2.098	\$ 2,000
COMMERCIAL AUTOMOBILE LIABILITY		
—Zone Rated	0.882	\$ 9,000
—Light/Medium Trucks	1.061	\$ 9,000
—Heavy Trucks	0.941	\$ 9,000
—Extra Heavy Trucks	0.949	\$ 9,000
—Private Passenger, Publics, and Garages	1.080	\$ 9,000