EXTRAPOLATING, SMOOTHING, AND INTERPOLATING DEVELOPMENT FACTORS

RICHARD E. SHERMAN

Abstract

The purpose of this paper is to provide a practical handbook describing simple yet accurate methods of extrapolating, smoothing, and interpolating development factors. It will focus on the inverse power curve, its properties, and examples of fits obtained to various types of loss experience. It will also illustrate usage of the inverse power curve in addressing a variety of actuarial problems, including the following:

- A lack of mature development experience.
- A lack of credible loss development data.
- Loss data at interim evaluation dates.
- Loss experience at odd, inconsistent evaluation dates.
- A need to break down annual development into quarterly or monthly segments.

The objective of this paper is to enhance the reader's capability in analyzing loss development.

INTRODUCTION

Development factor analysis is fundamental to most actuarial studies for ratemaking and reserving purposes. It is the purpose of this paper to materially enhance the reader's capability in analyzing loss development. A simple, general mathematical function, the inverse power curve, will be presented that usually fits loss experience as well as or better than other functions in common use today. Comparisons of goodness of fit using the inverse power curve and various other functions have been made based on incurred and paid losses, reported and paid claim counts, and primary and excess experience for workers' compensation, medical malpractice, automobile and general liability, automobile physical damage, fidelity, and surety. This is not a theoretical treatise so much as it is a practical guide aimed at presenting simple yet very accurate methods of extrapolating, smoothing, and interpolating development factors. We will focus on effective approaches to dealing with the following common actuarial problems:

- The most mature experience available still indicates the clear potential for further development (either upward or downward) to an ultimate basis.
- Only two or three development factors are available in the loss history, but there is still a need for a full profile of future loss development.
- Development factors for the later stages of development are sparse or fluctuate significantly and the reliability of selecting factors for the most mature stages of development on the basis of one or two historical factors is openly questionable.
- A given body of development data is based on relatively few claims and must be credibility weighted with external data sources while still preserving the *unique* characteristics of that experience.
- All prior development experience is on a year-end basis, but there is a need to incorporate the latest evaluation which is at some point in the middle of the year.
- Available loss experience is at odd, inconsistent evaluation dates.
- There is a need to estimate quarterly or monthly development, but only annual data is available.
- Accident or report quarter development factors are needed, but only annual factors for accident or report years are available.

An approach to dealing with each of these problems will be described in various sections of this paper. Although the examples in this paper are illustrated with the use of one type of mathematical function, many of the techniques can be used with a wide variety of other functions.

SECTION I

EXTRAPOLATION OF INCURRED LOSSES AND PAID LOSSES USING THE INVERSE POWER FUNCTION

The availability of a simple family of curves that closely fit loss development factors of all types for any line of business would be instrumental in advancing the quality of reserve and ratemaking analysis. Research indicates that the family of curves of the form, $1.0 + a(t + c)^{-b}$, which we shall call inverse power curves, comes closer to filling this need than other functions in use today. For example, a comparison of paid loss development factors for workers' compensation (accident year 1969 for the Wausau Insurance Companies) with approximations obtained by fitting the inverse power curve and five other mathematical functions is provided below.

	Development Factors						
Year of Development	Actual	Inverse Power	McClenahan[1]	Geometric[2]	Exponential Decay[3]	Log-Normal[4]	Logarithmic[5]
2:1	1.920	1.889	2.840	1.683	1.309	1.378	1.409
3:2	1.228	1.224	1.329	1.277	1.202	1.190	1.168
4:3	1.098	1.100	1.131	1.147	1.133	1.112	1.103
5:4	1.051	1.056	1.061	1.088	1.087	1.073	1.072
6:5	1.036	1.036	1.031	1.055	1.057	1.051	1.054
7:6	1.025	1.025	1.016	1.035	1.037	1.036	1.044
8:7	1.019	1.018	1.008	1.023	1.025	1.028	1.037
9:8	1.014	1.014	1.004	1.015	1.016	1.022	1.032
10:9	1.011	1.011	1.002	1.010	1.011	1.016	1.026
11:10	1.009	1.009	1.001	1.007	1.007	1.013	1.024
12:11	1.008	1.008	1.001	1.005	1.005	1.011	1.021
Chi-Square Statistic[6]		.001	.307	.039	.289	.216	.191

[1] Charles L. McClenahan, "A Mathematical Model for Loss Reserve Analysis," *PCAS* LXII, 1975, pp. 134–153.

[2] David Skurnick, Discussion of "A Mathematical Model for Loss Reserve Analysis," *PCAS* LXIII, 1976, pp. 125–127.

[3] Obtained by fitting an exponential curve of the form, $y = ae^{bt}$, to the development factors less one.

[4] Derived by fitting a log-normal distribution to the cumulative payments distribution, and then expressing the fitted distribution in terms of development of factors.

[5] Based on fitting a logarithmic curve of the form, $y = a + b \ln t$, to the cumulative payments distribution, and then expressing the fitted distribution in terms of development factors.

[6] Paul H. Hoel, *Introduction to Mathematical Statistics*, 1971, pp. 225–234.

The chi-square statistic for goodness of fit is substantially better for the inverse power curve than for the other functions. Similarly, the size of errors for the inverse power curve is also significantly less, as shown below.

Comparison of Curve Fit Errors

Year of Development	Inverse Power	McClenahan	Geometric	Exponential Decay	Log-Normal	Logarithmic
2:1	−.031	+.920	−.237	−.611	−.542	−.511
3:2	−.004	+.101	+.049	−.026	−.038	−.060
4:3	+.002	+.033	+.049	+.035	+.014	+.005
5:4	+.005	+.010	+.037	+.036	+.022	+.021
6:5	.000	−.005	+.019	+.021	+.015	+.018
7:6	.000	−.009	+.010	+.012	+.011	+.019
8:7	−.001	−.011	+.004	+.006	+.009	+.018
9.8	.000	−.010	+.001	+.002	+.008	+.018
10:9	.000	−.009	−.001	.000	+.005	+.015
11:10	.000	−.008	−.002	−.002	+.004	+.015
12:11	.000	−.007	−.003	−.003	+.003	+.013
Average Absolute Error	.004	.102	.037	.068	.061	.065

Another test of the appropriateness of various functions is the factor to ultimate they indicate. For this purpose we will truncate any development indicated past 80 years (since all permanent disability claimants will presumably have died within this period.) A comparison of development factors from 12 years to 80 years of development is as follows:

Indicated by Case Reserves	1.086
Inverse Power Curve	1.076
McClenahan	1.007
Geometric	1.011
Exponential	1.009
Log-Normal	1.047
Logarithmic	1.537

In the above example, historical patterns have shown that case reserves are adequate to cover IBNR losses as well as changes in reported reserves.

These results are representative of comparisons performed on both paid and incurred losses for most lines of business. This paper will focus on illustrating the usage of the inverse power curve to address a wide range of actuarial problems.

In the following example, incurred losses for an isolated accident year will be extrapolated to an ultimate basis using an inverse power function. The only information we are given is incurred losses for automobile bodily injury liability for accident year 1978 at the following evaluation dates:

Evaluation Date	Incurred Losses	Development Factor
12/31/78	$ 8,479,000	–
12/31/79	13,380,000	1.578
12/31/80	14,678,000	1.097
12/31/81	15,147,000	1.032

We will fit an inverse power curve to the development factors so that the factor at age t will be approximated by $(1 + at^{-b})$.

This fit can be performed in a least squares sense on a computer. For the sake of simplicity we will illustrate another method for fitting this curve which involves the use of only natural logarithms, exponentials, and linear regression. This method is displayed in Exhibit 1. First, we compute the reciprocals of each age of development (t) and we subtract 1.0 from each incurred loss development factor. The natural logarithms of $1/t$ and each development factor minus one are then calculated. A linear regression is then performed with ln $(1/t)$ as the independent variable (x) and ln(factor $-$ 1.0) as the dependent variable (y). In this case, the coefficient of determination (goodness of fit) was .99887. The values of a and b were obtained from the linear least squares trend line ($y = a + bx$) as 2.33259 and 4.19024, respectively. These parameters give us the following equation for the incurred loss development factor at age t:

$$1.0 + 10.30460t^{-4.19024}.$$

The extrapolated estimates in Exhibit 1 were easily obtained by first computing $1/t$ and $\ln(1/t)$ for each future age of development and then using the relationship

$$\ln\,(\text{development factor} - 1.0) = \ln a + b \ln(1/t)$$

from the linear regression to obtain the projections in column (4). These projections were then exponentiated to obtain the projected development factors (less one) in column (2). By adding one to each of these projected factors and taking their product, we obtain a factor to ultimate of 1.0257. This factor, when applied to the latest value of incurred losses for accident year 1978 of $15,147,000, yields an estimated ultimate incurred loss of $15,536,445.

Exhibit 2 provides a comparison of actual and fitted incurred loss development factors for automobile bodily injury liability, general liability, and workers' compensation over 10 to 15 years of development.

The goodness of fit of the inverse power curve can often be improved by adding a third parameter, making the function of the form:

$$1.0 + a\,(t + c)^{-b}.$$

In this case, we define a function, $f(c)$, to be the coefficient of determination (R^2) of the above inverse power curve. The value of $f(c)$ is estimated for a wide range of values of c and a local maximum can be found by numerical analysis techniques. For example, in Exhibit 2, $c = -1$ was used for general liability. This technique is often useful in obtaining a better fit for the earlier periods of development than for later periods. Variations in the c parameter usually have little impact on the projected factors for later periods of development, but have a major effect on varying the shape of the inverse power curve for the earliest periods of development. As an alternative to letting $c = -1$, we may simply redefine the values of t. For example, for the 2:1 development factor, we have defined t as being equal to 2 (its value at the end of the period of development). Alternatively, defining t as its value at the beginning of each development period would result in setting $c = 0$ for the examples in Exhibit 2 and would eliminate this third parameter.

To continue the previous example and to illustrate the versatility of the inverse power function, it will next be used to extrapolate paid losses to an ultimate basis using only the following information:

Evaluation Date	Incurred Paid	Development Factor
12/31/78	$ 3,071,000	–
12/31/79	8,603,000	2.801
12/31/80	11,941,000	1.388
12/31/81	13,541,000	1.134

The method is identical to that used in projecting the incurred factors above and is illustrated in Exhibit 3. A coefficient of determination of .99998 was obtained, indicating an excellent fit. The product of all the extrapolated factors in column 2 is 1.1393, indicating an estimated ultimate loss of $15,427,261 ($13,541,000 × 1.1393). This closely compares with the incurred projection of $15,536,445 developed above.

SECTION II

SOME PROPERTIES OF THE INVERSE POWER FUNCTION

The inverse power curve possesses a characteristic which is essential to obtaining close approximations to actual loss development factors. To show this, let us define some terms. Let d_i represent the development factor for the i^{th} period of development. Let B_i be the "decay" ratio between $(d_i - 1.0)$ and $(d_{i-1} - 1.0)$. We have observed that a common characteristic of loss development data of any type is that B_i tends to increase asymptotically to 1.0 as i increases. This pattern can be verified from Exhibit 2 for general liability incurred losses as follows:

	Decay Ratios (B_i)		
Years of Development	Actual	Smoothed[7]	Inverse Power
3	.333		.300
4	.663	.451	.496
5	.416	.519	.606
6	.506	.563	.675
7	.846	.722	.741
8	.879	.916	.765
9	1.034	.832	.794
10	.633	.785	.814
11	.737	.811	.834
12	1.143	.881	.848
13	.813	.950	.860
14	.923	.794	.870
15	.667		.879

[7] Each smoothed decay ratio is the third root of the product of the corresponding actual factor and the immediately preceding and immediately succeeding factor. For example, $.451 = (.333 \times .663 \times .416)^{1/3}$. This is also equivalent to taking the third root of the decay ratio between a given development factor minus one $(d_i - 1.0)$ and the third subsequent development factor minus one $(d_{i+3} - 1.0)$. For example, $.451 = (.077/.839)^{1/3}$. Both smoothing formulae are based on the assumption that there is a constant decay ratio applicable over a three-year period.

The inverse power curve satisfies this condition since

$$B_i = \frac{a(i)^{-b}}{a(i-1)^{-b}} = \left(\frac{i-1}{i}\right)^b = \left(1 - \frac{1}{i}\right)^b$$

and it is clear that $(1 - (1/i))^b$ increases to 1.0 as i increases.

One simple method of tail analysis assumes that B_i is constant (at least for the later periods of development). It is much more common for the decay ratios to increase than it is for them to remain constant. However, usage of a constant B_i (with a B_i based on more mature experience) can often serve to provide a lower bound for projections of future development.

In loss development experience we have reviewed, the earliest decay ratios are usually very low (.2 to .4) rising to the .7 to .9 range for later periods. It is this property of the inverse power curve which yields generally better fits than other functions. For example, consider the following comparison of decay ratios for the functions compared at the beginning of this paper.

	Decay Ratios						
Year of Develop-ment	Actual	Inverse Power	McClenahan	Geo-metric	Expo-nential Decay	Log-Normal	Loga-rithmic
3	.248	.252	.179	.406	.654	.503	.411
4	.430	.446	.398	.531	.654	.589	.613
5	.520	.560	.466	.599	.654	.652	.699
6	.706	.643	.508	.625	.654	.699	.750
7	.694	.694	.516	.636	.654	.706	.815
8	.760	.720	.500	.657	.654	.778	.841
9	.737	.778	.500	.652	.654	.786	.865
10	.786	.786	.500	.667	.654	.727	.813
11	.818	.818	.500	.700	.654	.813	.923
12	.889	.889	1.000	.714	.654	.846	.875

While many functions can fit loss development factors well over some segment of the history of development, few provide good fits over the entire history. It is the properties of the inverse power curve in terms of decay ratios, as noted above, as well as its flexibility in fitting the very large factors common at early stages of development, that make it a natural candidate for development factor analysis.

Because of the behavior of the decay ratios of the inverse power curve and their correspondence to this type of phenomenon in actual loss development experience, it is usually possible to obtain relatively good approximations of factors for later periods based solely on extrapolations of factors for earlier periods. For example, consider the general liability data in Exhibit 2 and extrapolations based only on the earliest factors:

	Extrapolation Based on			
Years of Development	First 2 Factors	First 3 Factors	First 4 Factors	Actual Factors
2	1.839	1.810	1.874	1.839
3	1.279	1.307	1.283	1.279
4	1.146	1.174	1.146	1.185
5	1.093	1.117	1.092	1.077
6	1.065	1.085	1.064	1.039
7	1.049	1.066	1.048	1.033
8	1.038	1.053	1.037	1.029
9	1.031	1.044	1.030	1.030
10	1.026	1.037	1.025	1.019
11	1.022	1.032	1.021	1.014
12	1.019	1.028	1.018	1.016
13	1.016	1.025	1.015	1.013
14	1.014	1.022	1.014	1.012
15	1.011	1.020	1.012	1.008

Naturally, the reliability of such projected factors is limited by the high degree of variability inherent in the first few factors and the sensitivity of any extrapolation technique to such variability.

While it would be highly desirable to derive a closed-form equation for the product of all extrapolated development factors as an estimate of the age-to-ultimate factor, the author has been unable to solve this problem. A simple program can be written to perform this otherwise cumbersome set of computations.

SECTION III

FITTING THE INVERSE POWER CURVE TO INCURRED LOSSES FROM THE REINSURANCE ASSOCIATION OF AMERICA EXPERIENCE

As an example of the goodness of fit of the inverse power function to excess experience, we have fitted curves to average incurred loss development factors from the 1983 edition of the *Loss Development Study* of the Reinsurance Association of America. In order to reduce fluctuations in this data before performing the curve fits, the mean factor for the latest 10 years was obtained for each year of development.

The curve fits shown in Exhibit 4 indicate that significant upward development is indicated beyond the most mature experience available for medical malpractice and workers' compensation. Upward development of 36.0% is projected for medical malpractice from 14 to 25 years of development. Upward development of 18.5% is estimated for workers' compensation from 25 to 50 years of development, which would no doubt be due to increasing medical costs and benefit changes on permanent disability cases.

SECTION IV

PROJECTING LOSSES IN A DYNAMIC ENVIRONMENT USING THE TWO-DIMENSIONAL INVERSE POWER FUNCTION

The accurate projection of losses in a dynamic environment can best be accomplished if a two-dimensional function can be found which closely approximates recent historical experience and which does not exhibit any detectable bias for any portion of that experience. In this section, the two dimensional inverse power function will be presented and tested and its derivation detailed. In keeping with the guidelines set forth earlier for keeping all analyses simple, we have limited our analytic tools to exponentials, natural logarithms, and linear least squares trend lines. The results are not perceptibly different from those which would be obtained from a computerized two-dimensional least squares fit and the added advantage of being able to perform all computations on a pocket calculator is achieved.

The data used in this test consisted of paid loss development factors for workers' compensation for accident years 1955 to 1980 from the Wausau Insurance Companies. The factors extended out to 12 years of development. The resultant two-dimensional inverse power curve took the following form:

$$PLDF_{AY,t} = 1.0 + (.819663 + .000983AY)t^{(-3.911356\ +\ .027946AY)}$$

Exhibit 5 provides a comparison of the actual and fitted factors using the above function.

In this equation, t represents the year of development of the given paid factor minus 1.0. Thus, for the 2:1 factor, t equals 1.0. This is equivalent to setting c = 1.0 for the three-parameter function. AY represents the accident year, expressed in years since 1900. (Since each set of coefficients is defined in terms of a linear relationship, it does not matter how AY is defined in terms of the initial year.) For example, for accident year 1967, AY = 67. The above two-dimensional function may be viewed as a family of one-dimensional inverse power curves. Sample curves are as follows:

Accident Year	Inverse Power Curve
1957	$1.0 + .876t^{2.318}$
1962	$1.0 + .881t^{-2.179}$
1967	$1.0 + .886t^{-2.039}$
1972	$1.0 + .890t^{1.899}$
1977	$1.0 + .895t^{1.760}$

The above two-dimensional equation was derived by first estimating one-dimensional inverse power curves for the average factors for each of the following groups of accident years:

1955–1959
1960–1964
1965–1969
1970–1974

From these fits, the following inverse power curves were obtained:

Accident Years	Inverse Power Curve	Goodness of Fit
1955–59	$PLDF_t = 1.0 + .877134t^{2.321363}$	.997336
1960–64	$PLDF_t = 1.0 + .880757t^{2.175112}$	.998984
1965–69	$PLDF_t = 1.0 + .880758t^{2.037354}$	.999826
1970–74	$PLDF_t = 1.0 + .893510t^{-1.901515}$	.998100

Linear regression analysis was then applied to the set of coefficients of t, with AY as the independent variable, to obtain the equation:

$$\text{Coefficient of } t \text{ for accident year } AY = .819663 + .000983\ AY.$$

Likewise, a linear trend line was fitted to the exponents of t.

Exhibit 6 provides a test of potential bias which might result from fitting the two-dimensional function to the triangle of factors. There does not appear to be any detectable bias since there are not significant contiguous areas of the triangle for which the signs of the errors are consistently positive or negative.

SECTION V

SELECTING DEVELOPMENT FACTORS FOR THE MOST MATURE PERIODS OF DEVELOPMENT WHEN CREDIBILITY IS LOW

The top portion of Exhibit 7 presents the commonly accepted method for selecting development factors for the most mature periods of development. The arithmetic mean of these factors for each period of development is selected—unless that mean appears too far out of line. We might, for example, want to temper the 6:5 factor because of its unexpected magnitude.

Let us consider the reasonableness of this common practice. Of all of the mean $Y{:}X$ factors, the mean factors for the earlier periods of development are often more reliable indications of future development factors (*unless* some clear trend is present or the magnitude of development is large) than the later mean factors. The earlier mean factors are the average of a greater number of individual factors, each of which is the end result of more claims transactions than those for the later factors. For example, consider the following history of incurred loss development.

Accident Year	Incurred Losses (000's) 1	2	3	4	5	6
1976	1,234	2,340	2,789	2,873	2,841	3,517
1977	1,462	2,506	3,185	3,507	4,071	
1978	1,618	2,657	3,459	3,684		
1979	1,824	2,740	3,378			
1980	1,943	3,087				
1981	2,120					

Ratio of Total Incurred Losses	13,330 / 8,081	12,811 / 10,243	10,064 / 9,433	6,912 / 6,380	3,517 / 2,841
Dollar Weighted Average Development Factor	1.650	1.251	1.067	1.083	1.238

Relative Volume of Losses on which Average Factor is Based:

Numerator	1.000	.961	.755	.519	.264
Denominator	.789	1.000	.921	.623	.277

In the above example, the reliability which should be assigned to each successive factor (after the first) declines sharply. How do we recognize this in the commonly accepted procedures? Not only is it often not recognized, it is usually violated to a successively greater extent as factors are selected for the later periods of development. This process is culminated by placing full reliance on the sole factor available for the oldest period of development. Furthermore, this one factor is heavily impacted by only a few, generally large, claims.

An alternate method of selecting factors is displayed in Exhibit 7. As commonly done, the mean factors are first computed. An inverse power curve is then fitted to the mean factors for the first two periods of development to project the 4:3 factor. (Alternatively, the inverse power curve could be fitted to all the individual factors.) The selected factor (1.110) is then determined as the weighted average of the inverse power curve projection (1.125) and the arithmetic mean of the actual 4:3 factors (1.065). In this simple example, the weights used are the number of actual factors on which each estimate is based. In the case of the arithmetic mean, three factors were used in computing the mean and a weight of three is assigned to 1.065. Nine factors underlie the inverse power curve projection (five 2:1 factors and four 3:2 factors) and its estimate of 1.125 is assigned a weight of nine.

The above process is then repeated, with the next inverse power curve fitted to the first two mean factors and the selected 4:3 factor of 1.110. The projected factor of 1.063 from the curve fit is given a weight of 12, versus a weight of 2 for the mean factor of 1.075. The weighted average of 1.065 then becomes the selected factor. This process can be repeated ad infinitum to select development factors of greater stability and accuracy than can be typically obtained by selecting the mean factors for the most mature periods of development.

Let us further suppose that we have another body of experience for the same line of business. How can this information be properly combined with the more specific, but less credible data we have just analyzed? Of many approaches tried, the following appears to possess the greatest validity. We begin by comparing the residual factors (i.e., the development factor less 1.0) corresponding to the development factors:

Years of Development	Residual Factors Company	Residual Factors "Industry"	Ratio
2:1	.669	.483	1.385
3:2	.250	.167	1.497
4:3	.110	.094	1.170
5:4	.065	.046	1.413
6:5	.054	.033	1.636

The arithmetic mean of the above ratios is 1.420; the median is 1.413; the arithmetic mean of the 3 middle ratios is 1.432. The stability of these ratios suggests that the company's residual factors tend to be about 42% higher than the "industry's." We may then use this assumption to further smooth the selected factors, and, perhaps more importantly, to project the development factors at later, yet to be experienced, stages of development:

Years of Development	"Industry" Factors	Smoothed Company Factors
2:1	1.483	1.686
3:2	1.167	1.237
4:3	1.094	1.133
5:4	1.046	1.065
6:5	1.033	1.047
7:6	1.028	1.040
8:7	1.019	1.027
9:8	1.012	1.017
10:9	1.009	1.013

SECTION VI

ESTIMATING QUARTERLY DEVELOPMENT FACTORS FROM ANNUAL FACTORS FOR A GIVEN ACCIDENT (REPORT) YEAR

In this section, a method will be presented for estimating quarterly development factors for a given accident (or report) year based only on annual development factors. The inverse power function is again used extensively. Applications for this technique appear in subsequent sections and include:

1) How to incorporate loss development information at odd evaluation dates. An example of this would be the inclusion of loss data as of June 30, 1983 in an analysis of annual development factors which are all at year end.
2) How to analyze loss development when all evaluation dates are odd. As an example, we will perform an analysis on accident years 1979–1982 incurred losses where the only data available is at the following evaluation dates: July 31, 1980, November 30, 1981 and April 30, 1982.
3) Performing more precise discount calculations by translating annual development factors into quarterly or monthly factors.

For simplicity in our current example, we will assume that the only information we have on accident year 1980 loss payments for workers' compensation is:

Evaluation Date	Cumulative Paid Losses	Paid Loss Development Factor
December 31, 1980	\$11,300,000	—
December 31, 1981	25,817,000	2.285
December 31, 1982	35,040,000	1.357

In actuality, we have used data which includes quarterly evaluation dates and development factors, but we shall pretend that we do not have this and attempt to approximate it from the above information. The process is started by deriving two initial approximations of quarterly factors—one for each annual interval. Consider first calendar year 1981. There are four quarterly development factors we want to estimate, with t (in quarters as of the end of each period) equal to 5, 6, 7, and 8. The average t value for these factors is 6.5. We know that the product of these four quarterly factors is the annual factor of 2.285. A first approximation for the average of these four factors is the fourth root of 2.285, or 1.229. We assign this to the average t-value of these factors (6.5). Similarly, an average factor of 1.079 is estimated for 1982 and assigned to an average t-value of 10.5. With this, we have two points with which to determine a two-parameter inverse power curve ($1.0 + 14.583516\, t^{-2.219212}$), which forms the basis for our first approximation of the quarterly factors:

t	Factor	t	Factor
5	1.410	9	1.111
6	1.275	10	1.088
7	1.195	11	1.071
8	1.144	12	1.059

We note that in both cases, the product of these factors exceeds the annual factor, indicating the need for an improved approximation.

$$2.458 = 1.410 \times 1.275 \times 1.195 \times 1.144$$
$$1.371 = 1.111 \times 1.088 \times 1.071 \times 1.059$$

In the first case, the actual annual factor of 2.285 is .9296 of the above product of 2.458. The fourth root of .9296 (.9819) gives us an "average" correction factor to apply to our first set of approximations for calendar year 1981. Instead of applying this adjustment, it would be more accurate to distribute the total adjustment in proportion to the development factors less 1.0.

$$.410 + .275 + .195 + .144 = 1.024$$
$$1.0 - .9296 = .0704$$

$$(.410/1.024) \times .0704 = .0282$$
$$(.275/1.024) \times .0704 = .0189$$
$$(.195/1.024) \times .0704 = .0134$$
$$(.144/1.024) \times .0704 = .0099$$

$$(1.0 - .0282) \times 1.410 = 1.370$$
$$(1.0 - .0189) \times 1.275 = 1.251$$
$$(1.0 - .0134) \times 1.195 = 1.179$$
$$(1.0 - .0099) \times 1.144 = 1.133$$

After analogous adjustments to the quarterly factors for calendar year 1982, we have a full set of second approximations. We then fit an inverse power curve to this second set of approximations to smooth the factors and produce our third and final set of estimates.

	Approximations			Actual	
t	First	Second	Third	Factors	Error
2			3.500	3.531	−.031
3			2.067	1.971	+.096
4			1.585	1.657	−.072
5	1.410	1.370	1.366	1.382	−.016
6	1.275	1.251	1.251	1.245	+.006
7	1.195	1.179	1.181	1.160	+.021
8	1.144	1.133	1.137	1.145	−.008
9	1.111	1.110	1.107	1.112	−.005
10	1.088	1.087	1.086	1.079	+.007
11	1.071	1.070	1.070	1.063	+.007
12	1.059	1.058	1.058	1.064	−.006

The final set of approximations differs from the actual data to such a small degree that such differences may be attributable only to random fluctuations in the actual loss experience. If these approximations are used, we may, for example, refine present value calculations.

Payments During Calendar year	Present Value at 8% as of January 1, 1980: Based on Annual Payments	Based on Quarterly Payments
1980	96.23%	95.36%
1981	89.10	89.32
1982	82.50	82.71
	89.66%	89.51%

SECTION VII

INCORPORATING LOSS DEVELOPMENT DATA FROM ODD EVALUATION DATES

This section provides an application of the techniques of the last section to a very common problem. For illustration, let us assume that we have incurred losses for accident years 1980–82 as of each year end and have just received the latest evaluation (June 30, 1983). How do we incorporate this information which doesn't fit in our standard triangle? Without a systematic approach, this is typically a frustrating situation.

Accident Year	Incurred Losses (000's) as of *X* Months of Development: 12	24	36	48
1980	$24,132	$40,746	$55,109	$62,328*
1981	27,782	45,929	55,712*	
1982	26,368	36,704*		
1983	15,961*			

*as of June 30, 1983

Accident Year	Incurred Loss Development Factors: 24:12	36:24	48:36
1980	1.689	1.352	1.131*
1981	1.653	1.213*	
1982	1.392*		

*6-months factors

In the above situation, usage of the June 30, 1983 data seems particularly important since it provides half of the known development factors. The first step is to determine what time interval serves as the least common denominator for the time lags between any two successive evaluations. In this case, t is six months, so we define it in terms of six-month intervals. We use the same techniques as described in the last section to break down the annual data into semiannual factors. It may then be compared with the actual semiannual factors from the first half of 1983.

	Incurred Loss Development Factors (Y:X Months)					
Source of Factors	12:6	18:12	24:18	30:24	36:30	42:36
Breakdown of Annual Experience		1.352	1.243	1.182	1.144	
First Half of 1983		1.392		1.213		1.129
Inverse Power Curve Fitted to All of the Above Factors	1.618	1.368	1.255	1.192	1.152	1.125

The inverse power curve factors can then be used to project each year's losses as of June 30, 1983 to 42 months of development as well as to extrapolate losses to ultimate. In the above approach, we have effectively used all of the loss history available to make projections.

SECTION VIII

ANALYZING LOSS DEVELOPMENT WHEN ALL EVALUATION DATES ARE ODD

In the following example, we will deal with the analysis of loss development when the evaluation dates are completely inconsistent. For accident years 1979–82, the only evaluation dates available are July 31, 1980, November 30, 1981, and April 30, 1982. Since the dates are 16 months and 5 months apart, the least common denominator is one month and we must break down the data into monthly factors. We will denote each data point as a two-dimensional vector, with the first coordinate being the age of the accident year at the given evaluation date, and the second being incurred losses.

Accident Year	(Months of Development, Incurred Losses (000's))			
1979		(19,2413)	(35,3952)	(40,4245)
1980	(7,450)	(23,3120)	(28,3660)	
1981	(11,1201)	(16,2134)		
1982	(4,123)			

Accident Year	(Months of Development, Development Factor)		
1979		(35:19, 1.643)	(40:35, 1.073)
1980	(23:7, 6.933)	(28:23, 1.172)	
1981	(16:11, 1.777)		

For each development period, we derive a first approximation of a monthly incurred loss development factor for a month in the middle of the period by taking the n^{th} root of the development factor, where n is the length of the interval in months.

Accident Year	(Months of Development, Development Factor)		
1979		(27.5:26.5, 1.032)	(38:37, 1.014)
1980	(15.5:14.5, 1.129)	(26:25, 1.032)	
1981	(14:13, 1.122)		

An inverse power curve is then fitted to all of the above points to estimate monthly development factors up to 40 months. The factors from this curve are then accumulated to produce approximations of the actual factors.

Accident Year	(Months of Development, Development Factor)		
1979		(35:19, 1.690)	(40:35, 1.073)
1980	(23:7, 10.753)	(28:23, 1.184)	
1981	(16:11, 1.965)		

In this first iteration, our approximations are all significantly too high and we adjust our estimated monthly factors by correction factors equal to the n^{th} root of the quotient of the actual factor to the approximated factor. For example, the approximation of (23:7, 6.936) is (23:7, 10.753), so the correction factor is the 16^{th} root of (6.936/10.753), or .973. Thus, the new monthly development factor is revised from (15.5:14.5, 1.129) to (15.5:14.5, 1.129 × .973). With all of these new monthly factors, we fit another inverse power curve and estimate an entire new set of monthly factors, which are then used to approximate the known factors. This iteration process is repeated until there is no further improvement in minimizing the sum of the squares of the differences between the approximated factors and the known factors. In this case, the final curve is $(1.0 + 31.010659\ t^{-2.109624})$ and the sum of the squares of the differences is less than .001. With a full set of monthly factors, losses as of 4, 16, 29 and 40 months can be projected to ultimate.

SECTION IX

ESTIMATING QUARTERLY ACCIDENT QUARTER DEVELOPMENT FACTORS FROM ANNUAL ACCIDENT YEAR FACTORS

It is sometimes desirable to estimate quarterly development factors for individual accident quarters, but the only data available is that of annual development factors for separate accident years. In this section we will illustrate a procedure for deriving such a refinement in loss development history.

If quarterly factors are not available for each accident year, then they must first be estimated as in Section VI. We shall use the third approximation factors from that section as the starting point for our analysis. For simplicity, we will assume that the incurred (or paid) losses as of one quarter of development are the same for all four accident quarters. If d_i represents the i^{th} development factor and q represents losses as of one quarter of development, then incurred losses by accident quarter and quarter of development are as follows:

Accident	Quarters of Development				
Quarter	1	2	3	4	5
1	q	qd_1	qd_1d_2	$qd_1d_2d_3$	$qd_1d_2d_3d_4$
2	q	qd_1	qd_1d_2	$qd_1d_2d_3$	$qd_1d_2d_3d_4$
3	q	qd_1	qd_1d_2	$qd_1d_2d_3$	$qd_1d_2d_3d_4$
4	q	qd_1	qd_1d_2	$qd_1d_2d_3$	$qd_1d_2d_3d_4$

From the above, we can derive equations for each of the quarterly factors for the accident year:

$$(q + qd_1)/q = 3.500$$
$$(q + qd_1 + qd_1d_2)/(q + qd_1) = 2.067$$
$$(q + qd_1 + qd_1d_2 + qd_1d_2d_3)/(q + qd_1 + qd_1d_2) = 1.585$$
$$\frac{(qd_1 + qd_1d_2 + qd_1d_2d_3 + qd_1d_2d_3d_4)}{(q + qd_1 + qd_1d_2 + qd_1d_2d_3)} = 1.366$$

These equations can be solved successively to produce a first set of approximations of the quarterly accident quarter factors:

$$d_1 = 2.500 \qquad d_4 = 1.228$$
$$d_2 = 1.494 \qquad d_5 = 1.237$$
$$d_3 = 1.133 \qquad d_6 = 1.132$$

While these first approximations do not progress downward in a smooth fashion, an inverse power curve may be fitted to these approximations to add consistency. This second set of factors should be tested in relation to how closely they can reproduce the original accident year factors.

SECTION X

A SIMPLE, ALTERNATIVE METHOD FOR ESTIMATING DEVELOPMENT BEYOND THE MOST MATURE EXPERIENCE AVAILABLE

Because of the nature of the inverse power curve, it cannot be fitted to development factors less than 1.0, since this would involve taking the natural logarithm of a negative number. If development is generally upward, but there is an occasional factor less than 1.0, such factors can be removed by smoothing techniques (such as replacing d_i by $(d_{i-1}d_id_{i+1})^{1/3}$ or $(d_{i-2}d_{i-1}d_id_{i+1}d_{i+2})^{1/5}$). If incurred losses generally develop downward in some segment of the loss triangle, then an alternative method of extrapolation of losses is needed. Such a method is presented in this section. It is based on noting relationships between paid losses during a given development period (for a given accident or report period) and the change in outstanding losses during that same period.

It will be helpful to first present some mathematical notation. Loss payments during the i^{th} period of development will be denoted by P_i, and outstanding losses at the end of the i^{th} period of development by O_i. Incurred losses at the end of the t^{th} period of development are then equal to $O_t + \Sigma_{i=1}^{t} P_i$.

At the end of the t^{th} period of development, the ultimate value of unpaid losses is $\Sigma_{i=t+1}^{\infty} P_i$. We wish to find some equivalent expression for this in terms of O_t. Suppose that, after some stage of development, there is a constant relationship between P_i and $(O_{i-1}-O_i)$. That is, $P_i = \alpha(O_{i-1}-O_i)$. Then

$$\sum_{i=t+1}^{\infty} P_i = \sum_{i=t+1}^{\infty} \alpha(O_{i-1} - O_i) = \alpha \sum_{i=t+1}^{\infty} (O_{i-1} - O_i) = \alpha O_t$$

since O_t decreases to zero as t increases. If we can determine the value of α, the runoff ratio, then we have a quick estimate of the ultimate value of unpaid losses (αO_t), where O_t is the latest evaluation of outstanding losses. Estimating α is easy since we can obtain estimates of it for every development period and accident or report period:

$$\alpha = P_i/(O_{i-1} - O_i)$$

Suppose that we find that for the more mature periods of development that paid losses are generally 80% of the decline in outstanding losses. Then, assuming that the runoff ratio (α) is constant for all future periods of development, the ultimate value of unpaid losses is simply 80% of the latest value of outstanding losses.

Exhibits 8 through 10 present this application of the method to automobile liability data. With the consistent pattern of downward development of incurred losses shown in Exhibit 8, there is a need to anticipate further favorable development for accident year 1975. Exhibit 9 displays the calculation of runoff ratios for accident year 1975 while Exhibit 10 displays all available runoff ratios. A runoff ratio of 60% was selected on the basis of Exhibit 10, and application of this ratio to the latest outstanding losses for Accident Year 1975 produced an estimate ($3,919,000) of the ultimate value of outstanding losses. This estimate is equivalent to an incurred loss development factor to ultimate of .975, which has been applied in Exhibit 8.

Exhibit 11 displays runoff ratios for a company with severely deficient reserves. It should be noted that the runoff ratios never stabilize and continue to increase with age. In this case, application of some of the higher runoff ratios may only provide a lower bound for an estimate of ultimate losses.

Once the runoff ratios stabilize for all development periods beyond a certain point, the ultimate value of outstanding losses may be estimated by αO_t for each of the accident or report years which have reached that stage of maturity.

CONCLUDING REMARKS

It is hoped that the research and practical applications presented in this paper can serve as a foundation from which others can make further advancements in the field of loss development analysis.

EXHIBIT 1

EXTRAPOLATION OF INCURRED LOSS DEVELOPMENT FACTORS USING AN INVERSE POWER FUNCTION AUTOMOBILE BODILY INJURY LIABLITY—ACCIDENT YEAR 1978

Age (t)	(1) $1/t$	(2) Incurred Loss Development Factor − 1.0	(3) ln $(1/t)$	(4) ln (Development Factor − 1.0)
2	.500	0.578	−0.693	−0.548
3	.333	0.097	−1.100	−2.333
4	.250	0.032	−1.386	−3.442
		Extrapolated Estimates		
5	.200	0.0122	−1.609	−4.410
6	.167	0.0057	−1.792	−5.176
7	.143	0.0030	−1.946	−5.822
8	.125	0.0017	−2.079	−6.379
9	.111	0.0010	−2.197	−6.873
10	.100	0.0007	−2.303	−7.318
11	.091	0.0004	−2.398	−7.715
12	.083	0.0003	−2.485	−8.080
13	.077	0.0002	−2.565	−8.415
14	.071	0.0002	−2.639	−8.726
15	.067	0.0001	−2.708	−9.015

Notes

(1) The least squares regression was performed on the data for ages 2, 3, and 4, as shown above, which has been rounded to three places.

(2) The extrapolated estimates were derived from the least squares trend line ($y = a + bx$), with $a = 2.33259$ and $b = 4.19024$.

EXHIBIT 2

COMPARISON OF ACTUAL AND FITTED INCURRED LOSS DEVELOPMENT FACTORS USING AN INVERSE POWER FUNCTION

Years of Development	Auto Bodily Injury Liability		General Liability		Workers' Compensation	
	Actual	Fitted	Actual	Fitted	Actual	Fitted
2	1.634	1.680	1.839	1.886	1.493	1.490
3	1.094	1.077	1.279	1.266	1.167	1.159
4	1.025	1.022	1.185	1.132	1.094	1.082
5	1.008	1.009	1.077	1.080	1.046	1.052
6	1.003	1.004	1.039	1.054	1.033	1.036
7	1.003	1.002	1.033	1.040	1.028	1.027
8	1.001	1.002	1.029	1.030	1.019	1.021
9	1.000	1.001	1.030	1.024	1.012	1.017
10	1.001	1.001	1.019	1.020	1.010	1.014
11	—	—	1.014	1.016	1.011	1.012
12	—	—	1.016	1.014	1.010	1.010
13	—	—	1.013	1.012	1.009	1.009
14	—	—	1.012	1.010	1.008	1.008
15	—	—	1.008	1.009	1.007	1.007
Goodness of Fit (R^2)	.98462		.98278		.98551	
Parameters						
$a =$	.68047		.88614		.48984	
$b =$	3.14215		1.73380		1.62362	
$c =$	−1.00000		−1.00000		−1.00000	

Notes

(1) The actual factors above represent composite experience from five major carriers for each line of business.

(2) The goodness of fit is measured by the coefficient of determination (R^2).

EXHIBIT 3

Extrapolation of Paid Loss Development Factors Using an Inverse Power Function Automobile Bodily Injury Liability—Accident Year 1978

Age (t)	(1) $1/t$	(2) Paid Loss Development Factor − 1.0	(3) ln ($1/t$)	(4) ln (Development Factor − 1.0)
2	.500	1.801	−0.693	+0.588
3	.333	0.388	−1.100	−0.947
4	.250	0.134	−1.386	−2.010
		Extrapolated Estimates		
5	.200	0.0578	−1.609	−2.850
6	.167	0.0291	−1.792	−3.536
7	.143	0.0163	−1.946	−4.114
8	.125	0.0099	−2.079	−4.613
9	.111	0.0064	−2.197	−5.055
10	.100	0.0043	−2.303	−5.453
11	.091	0.0030	−2.398	−5.809
12	.083	0.0022	−2.485	−6.135
13	.077	0.0016	−2.565	−6.435
14	.071	0.0012	−2.639	−6.713
15	.067	0.0009	−2.708	−6.972

Note

The extrapolated estimates were derived from the least squares trend line ($y = a + bx$), with $a = 3.18478$ and $b = 3.75038$.

EXHIBIT 4

COMPARISON OF ACTUAL AND FITTED INCURRED LOSS DEVELOPMENT FACTORS
REINSURANCE ASSOCIATION OF AMERICA EXPERIENCE

Years of Development	Automobile Liability		General Liability		Medical Malpractice		Workers' Compensation	
	Actual*	Fitted	Actual*	Fitted	Actual*	Fitted	Actual*	Fitted
2:1	1.760	1.619	2.300	2.290	7.876	6.104	1.634	1.630
3:2	1.227	1.264	1.541	1.536	2.172	2.480	1.285	1.287
4:3	1.100	1.123	1.295	1.287	1.654	1.717	1.169	1.172
5:4	1.061	1.062	1.171	1.177	1.334	1.429	1.134	1.118
6:5	1.031	1.033	1.109	1.119	1.150	1.288	1.092	1.088
7:6	1.015	1.018	1.093	1.085	1.156	1.208	1.053	1.068
8:7	1.015	1.011	1.060	1.064	1.163	1.158	1.055	1.055
9:8	1.008	1.007	1.046	1.050	1.120	1.124	1.048	1.046
10:9	1.006	1.004	1.045	1.039	1.133	1.101	1.039	1.039
11:10	1.000	1.003	1.039	1.032	1.023	1.084	1.036	1.034
12:11	1.001	1.002	1.022	1.027	1.058	1.070	1.014	1.029
13:12	1.001	1.001	1.024	1.022	1.090	1.060	1.017	1.026
14:13	1.001	1.001	1.004	1.019	1.063	1.052	1.030	1.023
15:14	1.000	1.001	1.019	1.016	1.089	1.046	1.023	1.021
16:15	1.000	1.000	1.008	1.014		1.040	1.016	1.019
17:16	1.001	1.000	1.010	1.012		1.036	1.032	1.017
18:17	.999	1.000	1.008	1.011		1.032	1.005	1.016
19:18	1.000	1.000	1.018	1.010		1.029	1.021	1.015
20:19	1.000	1.000	1.004	1.009		1.027	1.015	1.014
21:20	.999	1.000	1.005	1.008		1.024	1.037	1.013
22:21	1.000	1.000	1.017	1.007		1.022	.996	1.012
23:22	1.000	1.000	1.000	1.006		1.020	1.038	1.011
24:23	1.000	1.000	.997	1.006		1.019	1.026	1.010
25:24	1.000	1.000	1.000	1.005		1.017	1.018	1.010

* These factors are the average of the latest 10 accident years for each given year of development from the 1983 edition of the RAA's *Loss Development Study*.

EXHIBIT 5

Comparison of Actual and Fitted Paid Loss Development Factors
Two-Dimensional Inverse Power Functions (Workers' Compensation)

Paid Loss Development Factors

Accident Year	2:1 Actual	2:1 Fitted	3:2 Actual	3:2 Fitted	4:3 Actual	4:3 Fitted	5:4 Actual	5:4 Fitted	6:5 Actual	6:5 Fitted	7:6 Actual	7:6 Fitted
1955	1.832	1.874	1.160	1.169	1.065	1.064	1.032	1.033	1.017	1.019	1.013	1.012
1956	1.807	1.875	1.167	1.172	1.064	1.066	1.042	1.034	1.024	1.020	1.017	1.013
1957	1.869	1.876	1.161	1.176	1.067	1.069	1.033	1.035	1.025	1.021	1.017	1.014
1958	1.863	1.877	1.182	1.179	1.079	1.071	1.039	1.037	1.023	1.022	1.016	1.014
1959	1.852	1.878	1.178	1.183	1.075	1.073	1.035	1.038	1.023	1.023	1.015	1.015
1960	1.897	1.879	1.181	1.187	1.073	1.075	1.037	1.040	1.024	1.024	1.018	1.016
1961	1.884	1.880	1.189	1.191	1.079	1.078	1.047	1.041	1.024	1.025	1.016	1.017
1962	1.871	1.881	1.201	1.195	1.073	1.080	1.045	1.043	1.029	1.026	1.022	1.018
1963	1.934	1.882	1.206	1.199	1.088	1.083	1.042	1.045	1.028	1.028	1.022	1.019
1964	1.827	1.883	1.198	1.203	1.074	1.086	1.045	1.047	1.028	1.029	1.019	1.020
1965	1.856	1.884	1.212	1.207	1.086	1.088	1.044	1.048	1.023	1.030	1.016	1.021
1966	1.893	1.885	1.213	1.211	1.090	1.091	1.050	1.050	1.032	1.032	1.023	1.022
1967	1.858	1.886	1.215	1.216	1.097	1.094	1.050	1.052	1.034	1.033	1.024	1.023
1968	1.879	1.887	1.229	1.220	1.100	1.097	1.060	1.055	1.035	1.035	1.027	1.024
1969	1.920	1.887	1.228	1.224	1.098	1.100	1.051	1.057	1.036	1.036	1.025	1.025
1970	1.870	1.888	1.219	1.229	1.091	1.104	1.055	1.059	1.036	1.038	1.029	1.027
1971	1.813	1.889	1.221	1.234	1.093	1.107	1.056	1.061	1.040	1.040	1.028	1.028
1972	1.906	1.890	1.240	1.239	1.110	1.110	1.062	1.064	1.042	1.042	1.035	1.030
1973	1.967	1.891	1.249	1.244	1.123	1.114	1.071	1.067	1.047	1.044	1.033	1.031
1974	1.926	1.892	1.253	1.249	1.117	1.118	1.073	1.069	1.048	1.046	1.034	1.033
1975	2.027	1.893	1.269	1.254	1.130	1.122	1.076	1.072	1.058	1.048		
1976	1.923	1.894	1.260	1.259	1.125	1.126	1.071	1.075				
1977	1.892	1.895	1.242	1.264	1.124	1.129						
1978	1.892	1.896	1.248	1.270								
1979	1.903	1.897										

EXHIBIT 5
(Continued)

Comparison of Actual and Fitted Paid Loss Development Factors Two-Dimensional Inverse Power Function (Workers' Compensation)

Accident Year	Paid Loss Development Factors									
	8:7		9:8		10:9		11:10		12:11	
	Actual	Fitted	Actual	Fitted	Actual	Fitted	Actual	Fitted	Actual	Fitted
1955	1.010	1.009	1.005	1.006	1.004	1.005	1.003	1.004	1.002	1.003
1956	1.009	1.009	1.008	1.007	1.005	1.005	1.006	1.004	1.003	1.003
1957	1.011	1.010	1.006	1.007	1.004	1.005	1.005	1.004	1.003	1.003
1958	1.010	1.010	1.006	1.007	1.005	1.006	1.006	1.004	1.004	1.004
1959	1.010	1.011	1.007	1.008	1.006	1.006	1.004	1.005	1.004	1.004
1960	1.012	1.011	1.009	1.008	1.007	1.006	1.006	1.005	1.004	1.004
1961	1.015	1.012	1.008	1.009	1.006	1.007	1.006	1.005	1.005	1.004
1962	1.014	1.013	1.009	1.009	1.008	1.007	1.005	1.006	1.006	1.005
1963	1.013	1.013	1.010	1.010	1.009	1.008	1.008	1.006	1.007	1.005
1964	1.015	1.014	1.008	1.011	1.004	1.008	1.005	1.007	1.004	1.005
1965	1.013	1.015	1.010	1.011	1.008	1.009	1.005	1.007	1.006	1.006
1966	1.015	1.016	1.011	1.012	1.010	1.009	1.010	1.008	1.007	1.006
1967	1.016	1.017	1.015	1.013	1.010	1.010	1.009	1.008	1.008	1.007
1968	1.020	1.018	1.013	1.014	1.013	1.011	1.012	1.009	1.009	1.007
1969	1.019	1.019	1.014	1.014	1.011	1.011	1.009	1.009	1.008	1.008
1970	1.018	1.020	1.015	1.015	1.011	1.012	1.009	1.010		
1971	1.025	1.021	1.019	1.016	1.013	1.013				
1972	1.022	1.022	1.019	1.017						
1973	1.025	1.023								

EXHIBIT 6

Test of Bias: Signs of Errors
Fit of Two-Dimensional Inverse Power Function to Workers' Compensation Paid Loss Development Factors

	Years of Development										
	2:1	3:2	4:3	5:4	6:5	7:6	8:7	9:8	10:9	11:10	12:11
1955	+	+	−	+	+	−	−	+	+	+	+
1956	+	+	+	−	−	−	0	−	0	−	0
1957	+	+	+	+	−	−	−	+	+	−	0
1958	+	−	−	−	−	−	0	+	+	−	0
1959	+	+	−	+	0	0	+	+	0	+	0
1960	−	+	+	+	0	−	−	−	−	−	0
1961	−	+	−	−	+	+	−	+	+	−	−
1962	+	−	+	−	−	−	−	0	−	+	−
1963	−	−	−	+	0	−	0	0	−	−	−
1964	+	+	+	+	+	+	−	+	+	+	+
1965	+	−	+	+	+	+	+	+	+	+	0
1966	−	−	+	0	0	−	+	+	−	−	−
1967	+	+	−	+	−	−	+	−	0	−	−
1968	+	−	−	−	0	−	−	+	−	−	−
1969	−	−	+	+	0	0	0	0	0	0	0
1970	+	+	+	+	+	−	+	0	+	+	
1971	+	+	+	+	0	0	−	−	0		
1972	−	−	0	+	0	−	0	−			
1973	−	−	−	−	−	−	−				
1974	−	−	+	−	−	−					
1975	−	−	−	−	−						
1976	−	−	+	+							
1977	+	+	+								
1978	+	+									
1979	−										

EXHIBIT 7

Estimation of Selected Development Factors Using the Inverse Power Curve

Workers' Compensation

Accident Year	Incurred Loss Development Factors 2:1	3:2	4:3	5:4	6:5
1976	1.896	1.192	1.030	.989	1.238
1977	1.714	1.271	1.101	1.161	
1978	1.642	1.302	1.065		
1979	1.502	1.233			
1980	1.589				
Average Factor	1.669	1.250	1.065	1.075	1.238
Fitted Curve—First 2 Factors ($ILDF = 1.0 + 3.584t^{-2.442}$)	1.669	1.250	1.125		
Weight for Average Factor			3/12		
Weighted Factor ($3/12 \times 1.065 + 9/12 \times 1.125$)			1.110		
Fitted Curve—First 2 Average Factors and Weighted 4:3 Factor ($ILDF = 1.0 + 4.117t^{-2.582}$)	1.683	1.238	1.113	1.063	
Weight for Average Factor				2/14	
Weighted Factor ($2/14 \times 1.075 + 12/14 \times 1.063$)				1.065	
Fitted Curve—First 2 Average Factors and Weighted 4:3 and 5:4 Factors ($ILDF = 1.0 + 4.040t^{-2.572}$)	1.680	1.239	1.114	1.064	1.041
Weight for Average Factor					1/15
Weighted Factor ($1/15 \times 1.238 + 14/15 \times 1.041$)					1.054
Selected Factors	1.669	1.250	1.110	1.065	1.054

EXHIBIT 8

Development of Accident Year Incurred Losses

Automobile Liability

Accident Year	Incurred Losses (000's) As of X Years of Development: 1	2	3	4	5	6	Factor To Ultimate	Projected Ultimate Incurred Losses
1975	121,943	116,946	113,249	110,057	106,055	103,343	.975	100,759
1976	129,645	125,138	121,514	115,652	111,277		.950	105,713
1977	146,500	139,283	131,289	124,856			.915	114,243
1978	157,940	148,253	140,551				.876	123,123
1979	158,590	153,068					.839	128,424
1980	168,432						.802	135,082

Accident Year	Incurred Loss Development Factors: 2:1	3:2	4:3	5:4	6:5	
1975	.959	.968	.972	.964	.974	
1976	.965	.971	.952	.962		
1977	.951	.943	.951			
1978	.939	.948				
1979	.965					
Average Factor	.956	.958	.958	.963	.974	.975

EXHIBIT 9

ESTIMATION OF RUNOFF RATIOS

Automobile Liability
Accident Year 1975

Evaluation As of December 31,	(1) Incurred Losses	(2) Cumulative Paid Losses	(3) Unpaid Losses (1)−(2)	(4) Change in Paid Losses	(5) Change in Unpaid Losses	(6) Runoff Ratio (4)/(5)
1975	$121,943	$36,710	$85,233			
1976	116,946	60,839	56,107	+$24,129	−$29,126	82.8%
1977	113,249	74,393	38,856	+ 13,554	− 17,251	78.6
1978	110,057	85,877	24,180	+ 11,484	− 14,676	78.3
1979	106,055	92,707	13,348	+ 6,830	− 10,832	63.1
1980	103,343	96,840	6,503	+ 4,133	− 6,845	60.4

Note
Amounts in columns 1 through 5 are in thousands of dollars.

EXHIBIT 10

RUNOFF RATIOS

Automobile Liability

Accident Year	Runoff Ratio During X Year of Development: 2	3	4	5	6
1975	82.8%	78.6%	78.3%	63.1%	60.4%
1976	85.0	80.9	67.1	64.2	
1977	80.0	69.1	67.8		
1978	77.0	72.3			
1979	86.2				

EXHIBIT 11

ESTIMATION OF RUNOFF RATIOS

Example Company Nearing Receivership
Accident Year 1973

Evaluation As of December 31,	(1) Incurred Losses	(2) Cumulative Paid Losses	(3) Unpaid Losses (1)—(2)	(4) Change in Paid Losses	(5) Change in Unpaid Losses	(6) Runoff Ratio (4)/(5)
1973	$10,458	$ 2,987	$7,471			
1974	14,294	8,896	5,398	+$5,909	−$2,073	285.0%
1975	15,857	13,329	2,529	+ 4,433	− 2,870	154.5
1976	17,160	15,672	1,488	+ 2,343	− 1,040	225.3
1977	18,287	17,630	657	+ 1,958	− 831	235.6
1978	19,675	19,202	473	+ 1,572	− 184	854.3

Note
Amounts in columns 1 through 5 are in thousands of dollars.