TRANSFORMED BETA AND GAMMA DISTRIBUTIONS AND AGGREGATE LOSSES

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DISCUSSION BY ORIN M. LINDEN AND FRED KLINKER

One of the most important problems in collective risk theory has been the computation of the distribution of aggregate losses given individual frequency and severity distributions. Various approaches have been tried since the subject was first introduced by Filip Lundberg more than seventy-five years ago (Cramér [1]). These include approximation, simulation, and actual computation using numerical techniques. (A stochastic approach is also possible and the reviewers hope to discuss this in a later paper.) Approximations have been used with mixed success over the years. An appeal to the central limit theorem "justifies" a normal approximation if the number of claims is large (Beard, Pentikäinen, Pesonen [2]). This has not been satisfactory. Other approximations, such as normal power, Esscher, Gamma, Pareto, and just about any other distribution, have been used based on various theoretical (we can "prove" it) or empirical (it works) arguments. The use of these approximations has not been entirely satisfactory. The reviewers offer a reason for this later.

Another approach, the so called Monte Carlo simulation method, gives much better results. (For an elementary discussion of simulation see Gordon [3].) Simulation gives much better results but has three major drawbacks. First, it can be extraordinarily expensive in computer time, especially with large frequencies. Second, it's subject to the "whims" of the random number generators used. Third, it offers little insight into why a distribution behaves as it does. It has, however, been used very successfully and, up until very recently, it was the best alternative available in most cases.

In the last year or so two very good techniques have been introduced. The first, using a discrete density for the severities, uses a recursive formula and computes the aggregate loss density directly (Panjer [4]). The second, using a piece-wise linear severity, inverts the characteristic function of the distribution (Meyers and Heckman [5]). Both of these methods use numerical techniques. While the reviewers have not used these methods, we do feel that they are very good and that the problems associated with them are decidedly minor.

Despite Panjer's, and Meyers's and Heckman's results, there are very good reasons to have a good approximation formula. It's simple, quick, easy to use, and requires little mathematical knowledge to understand. In addition, for some applications, it's just as good as other techniques. Thus, a pricing formula may often be programmed into a hand calculator. In his paper Gary Venter proposes such an approximation using what he calls the Transformed Gamma Distribution (TGD). By adding a third parameter, α, to the ordinary gamma distribution the author can match up to three moments of the actual distribution. He writes down all the necessary formulas to compute the distribution and, as an example, applies it to the computation of excess ratios used to price aggregate stop loss insurance. The author then introduces the Transformed Beta Distribution (TBD) and explains that the combination of a TGD with a gamma, done in a certain way, produces a TBD. (This is similar to the combining of a Poisson frequency with a gamma to produce a negative binomial frequency.) This property is used to model one form of parameter uncertainty (that of λ). Going back to his prior example the author shows how incorporating such risk into his model almost doubles the expected excess loss over $1,000,000 in this case. Finally, the author compares the TGD to the more exact computations provided by Meyers and Heckman. The TGD itself, while not fitting badly, doesn't fit extremely well either. However, the excess ratios computed from the fitted TGD are extremely close to the exact methods. We will comment on these two statements shortly and show how a much closer fit to the distribution may be obtained by using a sum of TGD's.

The paper provides a large amount of useful information. APL programs are presented to do most of the necessary computations including the solving of two simultaneous equations. The reviewers used these programs and had no trouble reproducing any of the work in the paper. The incomplete gamma program is especially nice to have. A discussion of Gaussian quadrature, for numerical integration, appears in Appendix F. These features make the paper a useful reference document.

Before getting to the heart of our review we will make a few remarks.

The author comments that to use the TGD the skewness must be greater than the coefficient of variation. We did not investigate this. If the author has a reason for this we'd like to see it. In any event this doesn't seem to be a large limitation. All the distributions we've used recently have had this property.

The part of the paper we find least convincing is the section dealing with parameter risk. The author seems very impressed with the transforming of a TGD into a TBD. So much, in fact, that he makes the assumption that λ is transformed gamma distributed. He is content to ignore uncertainty in α and r. This seems to be a somewhat artificial assumption. (It does, however, simplify the computations.) The expected value of the TGD is given by $E(X) = \Gamma(r + (1/\alpha))/\lambda\Gamma(r)$. Thus, a smaller λ implies a larger expected loss. Since most insurers don't go broke and most risks don't produce extraordinarily large losses, we would expect most λ's to be near or larger than the expected value of λ. That is we expect $P(\lambda > E(\lambda)) > .5$. Using the parameters in the example we compute $P(\lambda > 1.144\text{E-}6) = 1 - G(1.144\text{E-}6, 2.597, 1.47, 1{,}288{,}500) = .65$. This result is expected and calms the mind somewhat but we would have expected a larger percentage intuitively. We also believe uncertainty in α and r should be considered. Of course to do so would greatly complicate the calculations.

Earlier on we commented on the fit of a TGD to the actual distribution. Looking at the cumulative distribution offers no insight into the nature of the errors. We argue that, in general, the TGD, TBD, or any other mono-modal density can't fit the aggregate density function very well due to the presence of multiple modes on the density. (By this we do not mean the possibility of having zero loss with positive probability. This spike at the origin is properly accounted for by the author's model.) Exhibit I plots the actual density, from Exhibit 3 of the paper, against the transformed gamma approximation. The differences, due to the modes, are obvious. Exhibit II gives an even more severe case. Both of these distributions resemble those we've used.

We also show in the exhibits a modified TGD we've invented which retains much of the simplicity of the original model yet does a much better job in explaining the modes of the distribution. The actual model we used is

$$(1)\quad \tilde{F}(x) = \sum_{n=0}^{[x/m]} Q(n)\,[P(0|n)\pi(x - nm) + (1 - P(0|n))G(x - nm;\, r_n,\, \alpha_n,\, \lambda_n)]$$

Notation is as follows:

m = maximum possible loss per occurrence

$Q(n)$ = probability of n occurrences of size m (total losses) in a time period

$P(0|n)$ = probability of no occurrences of size less than m (partial losses) given n total losses

$$\pi(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$G(x;\, r,\, \alpha,\, \lambda)$ = the TGD

Appendix II describes the method of fitting the $(r_n, \alpha_n, \lambda_n)$'s. Appendix I gives formulas for $Q(n)$ and $P(0|n)$ for Poisson and negative binomial frequencies. Note that the above sum requires a maximum of $1 + [L/m]$ terms where L is the excess loss limit. In general no more than five terms are needed. All terms in the sum are readily calculable with just a little more programming than is necessary to compute $G(x; r, \alpha, \lambda)$ alone. In many cases $P(0|n)$ is small and thus the π-terms can be ignored. However, the required programming is so simple it's not necessary to do so.

The reviewers applied the above model to the cases shown in Exhibits I and II. A glance at these exhibits clearly indicates a substantial increase in accuracy. In particular, this approximation is able to pick up the multi-modal behavior of the aggregate density function. This is something that both the TGD and the TBD could not do.

We note that parameter risk can be considered in a way similar to that used in the paper. As will be seen later, $\tilde{F}$ has a very simple form in the case of a Poisson frequency. Thus, it is particularly simple to incorporate parameter risk. However, due to time constraints, we did not investigate this.

For completeness we consider the computation of excess ratios. Exhibits IV and V show comparisons of actual excess ratios to those computed from the TGD approximation and our modified TGD approximation. (Formulas to do the calculations for the modified TGD appear in Appendix III.) A look at these exhibits indicates that there is not much difference in using any of the three methods.

This result puzzled us at first, so we tried a fit to two other curves, a Pareto and a normal (see Exhibit IV). Excess ratios computed from the normal were also very close to the actual ratios. However, the more highly skewed Pareto provided ratios that were generally much higher. We speculate that the integral involved in the definition of the excess ratio smooths things out significantly, so that as long as the approximating curve isn't too highly skewed the formula for excess ratios is very robust. The performance of the Pareto supports this.

The form of our modified TGD is indicated by understanding the causes of multi-modality in the aggregate density. To do this we define additional notation as follows:

$S(X)$ = probability of an occurrence $< x$
P = probability of having an occurrence of at least size $M = 1 - S(M)$
$S_1(X) = \begin{cases} S(x)/(1-p) & x < m \\ 1 & x \geq m \end{cases}$
n^* as a superscript represents nth convolution
$P(n|n_0)$ = probability of n partial losses given n_0 total losses.

With the above notation the aggregate loss distribution is given by

$$F(X) = \sum_{n=0}^{\infty} P(n)S^{n^*}(x)$$

(Note that $S^{0^*}(x) = \pi(x)$)

In the following we consider separately the effects of partial losses and total losses. Clearly the conditional distribution of aggregate losses, given n partial losses and n_0 total losses, is given by $S_1^{n^*}(x - n_0m)$. Thus $F(X)$ can be written

$$F(X) = \sum_{n_0=0}^{\infty} Q(n_0)\left[\sum_{n=0}^{\infty} P(n|n_0)S_1^{n^*}(x - n_0m)\right]$$

$$= \sum_{n_0=0}^{\infty} Q(n_0)\left[P(0|n_0)\pi(x - n_0m) + \sum_{n=1}^{\infty} P(n|n_0)S_1^{n^*}(x - n_0m)\right]$$

Define $G(x|n_0) \equiv \dfrac{\sum_{n=1}^{\infty} P(n|n_0)S_1^{n^*}(x)}{1 - P(0|n_0)}$

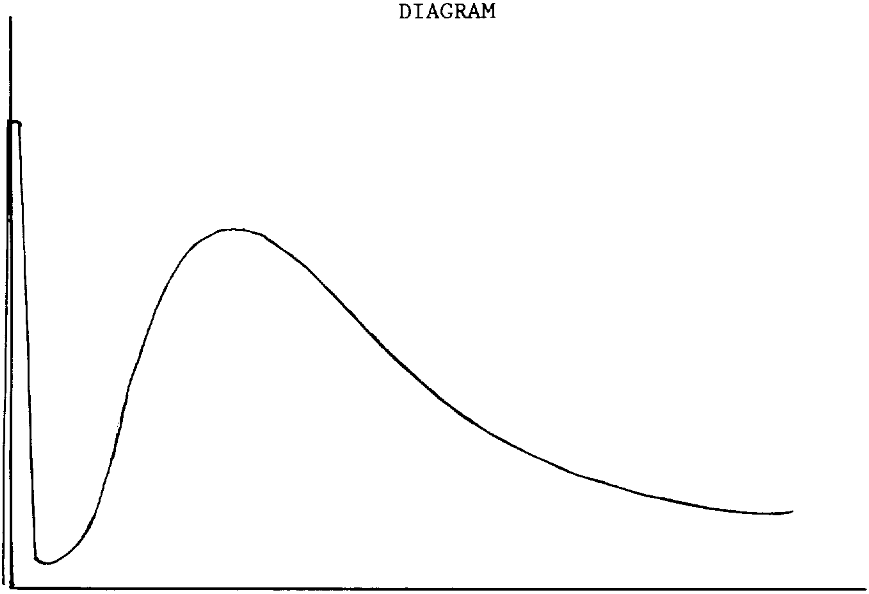
Then

$$(2)\ F(X) = \sum_{n_0=0}^{\infty} Q(n_0)[P(0|n_0)\pi(x - n_0m) + (1 - P(0|n_0))G(x - n_0m|n_0)]$$

The major problem arising in considering the modes of the density of $F(X)$ is in examining the fine structure of $G'(X)$. We believe that for any reasonable frequency and severity distributions (or combinations thereof) $G'(X)$ will have a primary mode that tends to dominate all of its secondary modes. (Consider, for example, a Poisson frequency and a gamma severity.) That is, we can think of $G'(X)$ as being essentially mono-modal. However, we should recognize that these secondary modes probably exist in most cases. They seem to give rise to much less important modes on the density of $F(X)$. Our simulation investigations tend to support this view.

With this in mind, we see that $F(X)$ is essentially a sum, weighted by the $Q(n_0)$'s, of distributions whose densities consist of a δ-function followed by a mono-modal distribution (see diagram).

DIAGRAM



Since the δ-functions have mass of only $P(0|n_0)$ they tend to have little effect on the shape of the density of $F(X)$. Thus, from (2), $F(X)$ will tend to have modes appearing at approximately the points where the $G'(x - n_0m|n_0)$ peak.

The above argument for the existence of modes hinges on the existence of a maximum loss. As a check Exhibit III shows the density of a distribution function with unlimited severity. The appearance of only one mode supports the argument.

The author recognizes a spike in the density of $F(X)$ at the origin and fits the rest of the distribution to a transformed gamma. What we do in (1) is recognize all spikes and fit each $G(x|n_0)$ to a TGD. Thus if $G(x;\, r_n,\, \alpha_n,\, \lambda_n)$ is fitted to replace $G(X|n_0)$ then (2) is transformed into (1) yielding our model.

In the case of a Poisson frequency,

$$P(n|n_0) = e^{-\omega(1-p)}(\omega(1 - p))^n/n!$$

independent of n_0. Hence $G(x|n_0) = G(x|0)$ is also independent of n_0. Thus, (2) becomes

$$(3)\ \tilde{F}(X) = \sum_{n=0}^{\infty} e^{-\omega p}(\omega p)^n/n![e^{-\omega(1-p)}\pi(x - nm) + (1 - e^{-\omega(1-p)})G(x - nm; r, \alpha, \lambda)]$$

where $G(x; r, \alpha, \lambda)$ fits to $G(x|0)$. This is the approximation used in Exhibits I and II.

REFERENCES

[1] H. Cramér, "Collective Risk Theory," *The Jubilee Volume of Försäkringsaktiebolaget Skandia,* 1955.

[2] R. E. Beard, T. Pentikäinen, and E. Pesonen, *Risk Theory,* 2nd Edition, Chapman and Hall, 1977.

[3] G. Gordon, *System Simulation,* 2nd Edition, Prentice Hall, 1978.

[4] H. H. Panjer, "Recursive Evaluation of a Family of Compound Distributions," *ASTIN Bulletin,* Vol. 12, No. 1, page 22.

[5] P. E. Heckman, and G. G. Meyers, "The Calculation of Aggregate Loss Distributions from Claim Severity and Claim Count Distributions," *PCAS* LXX (1983).

EXHIBIT IA

A COMPARISON OF AVERAGE DENSITIES

Aggregate Loss Interval (× 1000)	Average Densities[1] Characteristic Function Method[2] ($\times 10^{-6}$)	Transformed Gamma[3] ($\times 10^{-6}$)	Modified Transformed Gamma[4] ($\times 10^{-6}$)
0–25	2.032	2.484	2.264
25–50	3.132	2.556	2.724
50–75	2.872	2.540	2.784
75–100	2.668	2.500	2.696
100–125	2.452	2.436	2.536
125–150	2.216	2.352	2.328
150–175	1.992	2.252	2.100
175–200	1.788	2.148	1.860
200–225	1.604	2.028	1.628
225–250	1.436	1.908	1.400
250–275	1.944	1.776	1.944
275–300	2.088	1.652	1.912
300–325	1.808	1.524	1.760
325–350	1.588	1.396	1.584
350–375	1.376	1.276	1.408
375–400	1.192	1.152	1.228
400–425	1.024	1.040	1.064
425–450	.884	.932	.908
450–475	.760	.832	.768
475–500	.656	.740	.644
500–525	.668	.648	.660
525–550	.624	.572	.592
550–575	.524	.496	.512
575–600	.440	.432	.440
600–625	.368	.372	.376
625–650	.308	.324	.316
650–675	.256	.272	.260
675–700	.212	.232	.220
700–725	.180	.200	.180
725–750	.148	.164	.148

(1) Average Density = (difference of the values of the cumulative distribution at the endpoints of the interval)/25,000.

(2) From Venter, Exhibit 3, Page 1, Column 2.

(3) From Venter, Exhibit 3, Page 1, Column 6.

(4) See Exhibit IV, Note (2) for parameters.

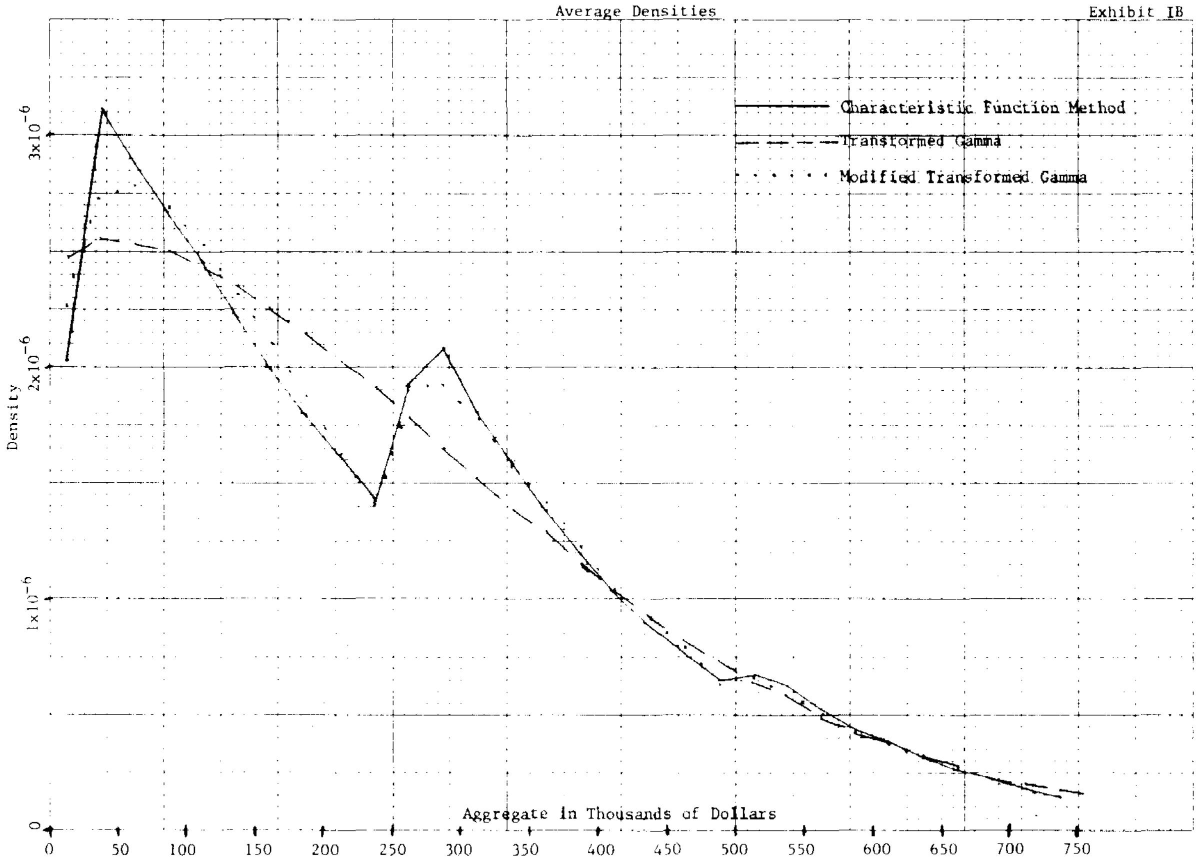

Average Densities
Exhibit IB
Characteristic Function Method
Transformed Gamma
Modified Transformed Gamma
Density
3x10^-6
2x10^-6
1x10^-6
0
Aggregate in Thousands of Dollars
0
50
100
150
200
250
300
350
400
450
500
550
600
650
700
750

EXHIBIT IIA

A COMPARISON OF AVERAGE DENSITIES—ANOTHER DISTRIBUTION

	Average Density		
Aggregate Loss Interval (× 24,076)	Simulation Method(1) (× 10^{-6})	Transformed Gamma(2) (× 10^{-6})	Modified Transformed Gamma(3) (× 10^{-6})
0–1	2.949	6.152	5.552
1–2	3.697	2.251	2.512
2–3	2.886	1.675	1.939
3–4	2.201	1.387	1.626
4–5	1.744	1.206	1.415
5–6	1.578	1.078	1.258
6–7	1.288	.9828	1.133
7–8	1.080	.9074	1.030
8–9	.9968	.8461	.9429
9–10	.7892	.7950	.8670
10–11	.7061	.7515	.8001
11–12	.6646	.7137	.7404
12–13	.6230	.6807	.6865
13–14	.5400	.6513	.6377
14–15	.5400	.6250	.5931
15–16	.4984	.6011	.5521
16–17	.4153	.5795	.5144
17–18	.3738	.5596	.4796
18–19	.3738	.5412	.4472
19–20	.3323	.5241	.4173
20–21	.2907	.5082	.3893
21–22	.3323	.4933	.3633
22–23	.2907	.4793	.3391
23–24	.2492	.4661	.3165
24–25	.2907	.4535	.2953
25–26	.2077	.4416	.2756
26–27	.2492	.4302	.2571
27–28	.2077	.4194	.2398

Notes appear on continuation of exhibit.

EXHIBIT IIA (continued)

A Comparison of Average Densities—Another Distribution

Aggregate Loss Interval (× 24,076)	Average Density: Simulation Method(1) (× 10^{-6})	Transformed Gamma(2) (× 10^{-6})	Modified Transformed Gamma(3) (× 10^{-6})
28–29	.2077	.4089	.2237
29–30	.1661	.3989	.2085
30–31	.2077	.3893	.1944
31–32	.1661	.3800	.1811
32–33	.1661	.3710	.1688
33–34	.1661	.3622	.1572
34–35	.1246	.3538	.1463
35–36	.1661	.3456	.1362
36–37	.1661	.3376	.1267
37–38	.1246	.3298	.1179
38–39	.1246	.3221	.1096
39–40	.1661	.3147	.1018
40–41	.1246	.3075	.0946
41–42	.4153	.3003	1.311
42–43	1.371	.2934	1.136
43–44	1.163	.2866	.8074
44–45	.9968	.2799	.6629
45–46	.7476	.2733	.5722
46–47	.6646	.2668	.5066
47–48	.4984	.2605	.4554
48–49	.4569	.2543	.4134
49–50	.4153	.2481	.3778
50–51	.3323	.2421	.3470
51–52	.2907	.2362	.3198
52–53	.2907	.2303	.2955
53–54	.2492	.2246	.2737
54–55	.2077	.2189	.2539
55–56	.1661	.2133	.2358

Notes appear on continuation of exhibit.

EXHIBIT IIA (continued)

A COMPARISON OF AVERAGE DENSITIES—ANOTHER DISTRIBUTION

Aggregate Loss Interval (× 24,076)	Average Density: Simulation Method[1] (× 10^{-6})	Transformed Gamma[2] (× 10^{-6})	Modified Transformed Gamma[3] (× 10^{-6})
56–57	.2077	.2078	.2192
57–58	.1661	.2024	.2039
58–59	.1661	.1971	.1898
59–60	.1246	.1919	.1767
60–61	.1246	.1867	.1646
61–62	.0830	.1816	.1533
62–63	.1246	.1766	.1429
63–64	.1246	.1717	.1331
64–65	.0830	.1669	.1240
65–66	.1246	.1621	.1156
66–67	.0830	.1574	.1077
67–68	.0830	.1528	.1003
68–69	.0830	.1483	.09340
69–70	.08307	.1438	.08697
70–71	.08307	.1394	.08097
71–72	.08307	.1351	.07536
72–73	.04153	.1309	.07012
73–74	.08307	.1268	.06523
74–75	.04153	.1227	.06067
75–76	.08307	.1187	.05640
76–77	.08307	.1148	.05242
77–78	.04153	.1110	.04870
78–79	.04153	.1072	.04524
79–80	.04153	.1036	.04200
80–81	.04153	.09999	.03899
81–82	.04153	.09648	.03618
82–83	.04153	.09304	.03356
83–84	.2077	.08969	.3286

Notes appear on continuation of exhibit.

EXHIBIT IIA (continued)

A Comparison of Average Densities—Another Distribution

Aggregate Loss Interval (× 24,076)	Average Density: Simulation Method(1) (× 10^{-6})	Transformed Gamma(2) (× 10^{-6})	Modified Transformed Gamma(3) (× 10^{-6})
84–85	.2492	.08641	.1722
85–86	.2077	.08322	.1364
86–87	.1661	.08010	.1163
87–88	.1246	.07706	.1025
88–89	.08307	.07409	.09184
89–90	.08307	.07120	.08321
90–91	.08307	.06839	.07593
91–92	.08307	.06566	.06964
92–93	.08307	.06300	.06410
93–94	.08307	.06042	.05916
94–95	.04153	.05791	.05471
95–96	.04153	.05547	.05067
96–97	.04153	.05311	.04699
97–98	.04153	.05082	.04361
98–99	.04153	.04860	.04051
99–100	0	.04645	.03765

(1) This distribution is based on a Poisson frequency with mean 13.7376 and a Pareto severity
$F(X) = 1 - (B/(B + x))^{\delta}$ with $B = 264.7$ and $\delta = .45128063$
censored at 1,000,000.
The small scale fluctuations are due to our simulation routine which only calculates distributions to .001. Note that .001/24076 = .04153 E-6.

(2) See Exhibit V, Note (2) for parameters.

(3) See Exhibit V, Note (3) for parameters.

Exhibit IIB

Average Densities

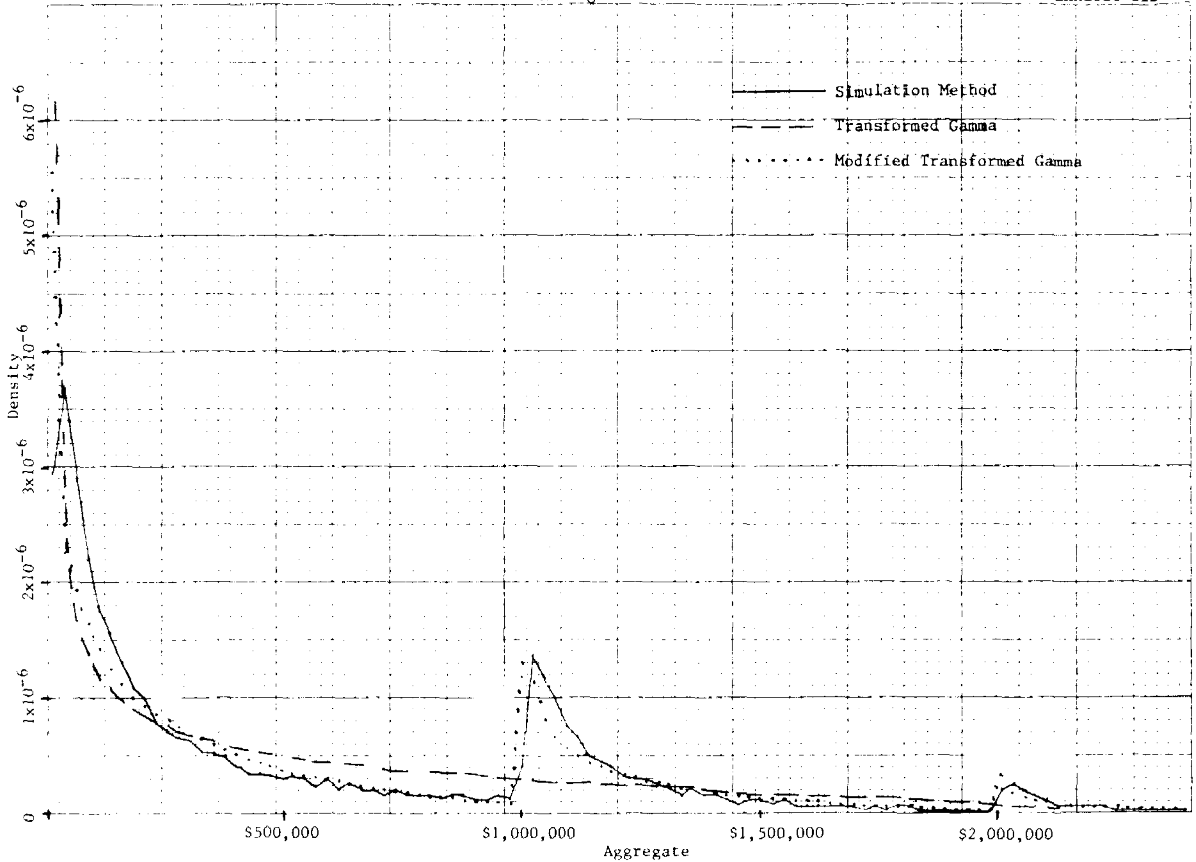


EXHIBIT IIIA

AVERAGE DENSITY
AGGREGATE DISTRIBUTION FOR A SEVERITY WITHOUT A CENSOR[1]

Aggregate Loss Interval (× 120,380)	Simulation Method[2] ($\times 10^{-7}$)
0–1	.7476
1–2	5.732
2–3	12.63
3–4	16.20
4–5	14.12
5–6	11.05
6–7	7.809
7–8	5.150
8–9	3.323
9–10	1.994
10–11	1.412
11–12	.8307
12–13	.5815
13–14	.3323
14–15	.3323
15–16	.1661
16–17	.1661
17–18	.08307
18–19	.08307
19–20	0
20–21	.08307
21–22	0
22–23	.08307
23–24	0
24–25	0
25–26	0
26–27	.08307
27–28	0
28–29	0
29–30	0

(1) Poisson frequency with $\omega = 13.7376$ and a Pareto severity with $B = 65{,}721$ and $\delta = 2.5$ censored at 10^{12}.

(2) The small scale fluctuations are due to our simulation routine which only calculates distributions to .001. Note that .08307 E-7 = .001/120380.

Exhibit IIIB

Aggregate Densities

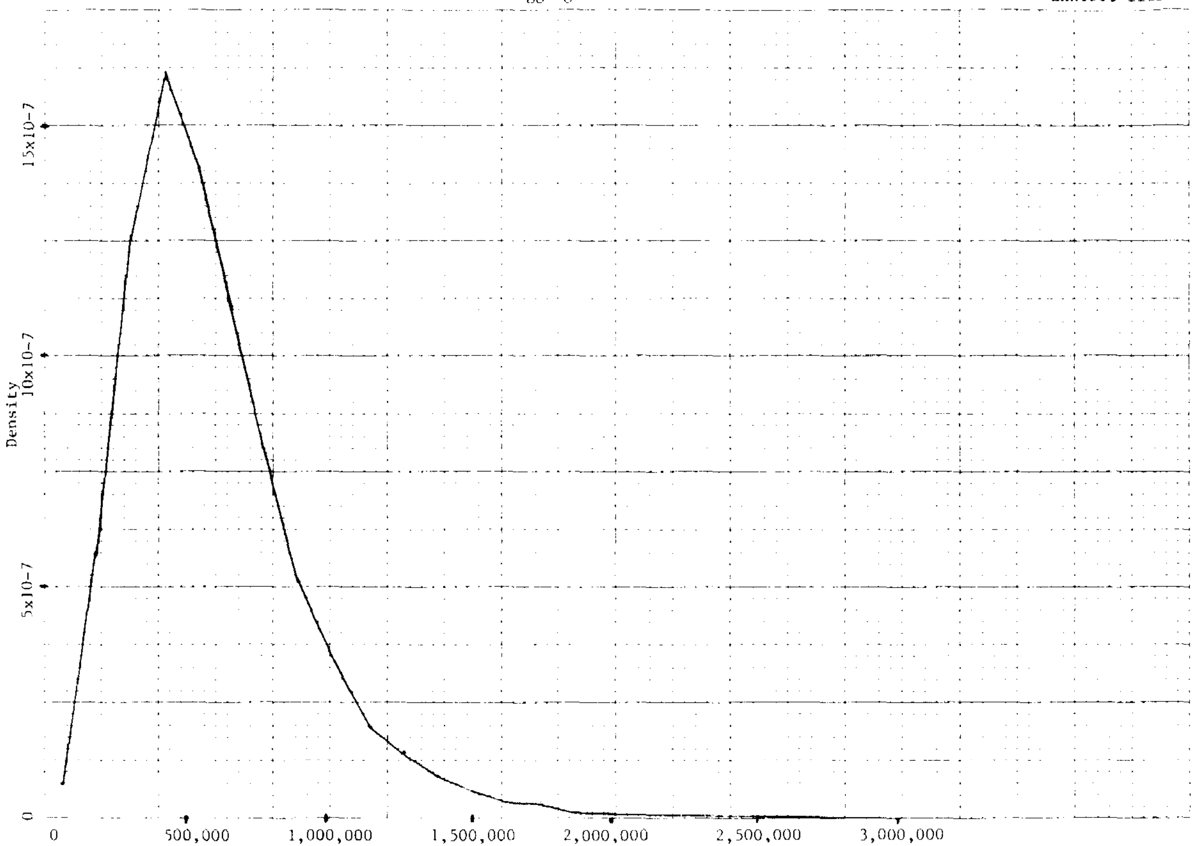


EXHIBIT IV

Comparison of Excess Ratios from Distributions in Venter's Exhibit 3

Aggregate Loss Amount (× 1000)	Characteristic Function Method[1]	TGD[1]	Modified TGD[2]	Normal[3]	Pareto[4]
25	.9016	.9031	.9026	.9033	.9062
50	.8107	.8125	.8116	.8131	.8236
75	.7273	.7283	.7276	.7292	.7506
100	.6507	.6503	.6504	.6517	.6859
125	.5806	.5786	.5798	.5801	.6282
150	.5163	.5129	.5152	.5145	.5768
175	.4573	.4529	.4562	.4546	.5307
200	.4030	.3984	.4022	.4001	.4893
225	.3529	.3491	.3525	.3507	.4521
250	.3066	.3047	.3066	.3062	.4185
275	.2642	.2650	.2648	.2662	.3881
300	.2273	.2295	.2279	.2305	.3605
325	.1951	.1981	.1955	.1987	.3354
350	.1672	.1702	.1674	.1706	.3126
375	.1431	.1457	.1430	.1458	.2917
400	.1221	.1243	.1219	.1241	.2727
425	.1039	.1055	.1036	.1051	.2552
450	.0880	.0893	.0878	.0887	.2392
475	.0742	.0752	.0741	.0745	.2244
500	.0622	.0631	.0622	.0622	.2109
525	.0518	.0528	.0519	.0518	.1984
550	.0430	.0439	.0432	.0429	.1868
575	.0357	.0364	.0358	.0353	.1761
600	.0296	.0301	.0296	.0290	.1662
625	.0245	.0247	.0245	.0237	.1570
650	.0202	.0203	.0202	.0192	.1485
675	.0167	.0165	.0166	.0155	.1406
700	.0137	.0134	.0136	.0125	.1332
725	.0112	.0109	.0112	.0100	.1263
750	.0091	.0088	.0091	.0080	.1199
775	.0074	.0070	.0074	.0063	.1139
800	.0060	.0056	.0059	.0050	.1082
825	.0048	.0045	.0048	.0039	.1030
850	.0039	.0035	.0039	.0030	.0980

Notes appear on following page.

EXHIBIT IV (continued)

COMPARISON OF EXCESS RATIOS FROM DISTRIBUTIONS IN VENTER'S EXHIBIT 3

(1) From Venter, Exhibit 3.

(2) Fit by method of Appendix II.
$\omega = 13.7376$
$m = 250{,}000$
$p = 0.0241$
$r = 0.7568$
$\alpha = 1.55601$
$\lambda = 4.3616\text{E-}6$

(3) Fit to match first two moments.

$$\text{Distribution Function} = \frac{1}{\sqrt{2\pi}\,\sigma(1 - \Phi(-\mu/\sigma))} \int_0^x \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right] dt$$

$\Phi(x)$ = Standard Normal Distribution
$\mu = -31{,}828.4$
$\sigma = 327{,}408.6$

(4) Fit to match first two moments.
$F(X) = 1 - (B/(B + X))^{\delta}$
$B = 807{,}039$
$\delta = 4.22815586$

EXHIBIT V

COMPARISON OF EXCESS RATIOS FROM DISTRIBUTION IN EXHIBIT II

Aggregate Loss Amount ($\times 10^5$)	Simulation(1)	TGD(2)	Modified Transformed Gamma(3)
1	.8599	.8660	.8649
2	.7542	.7555	.7567
3	.6663	.6595	.6647
4	.5877	.5750	.5843
5	.5164	.5000	.5124
6	.4511	.4335	.4470
7	.3888	.3744	.3863
8	.3304	.3220	.3292
9	.2753	.2757	.2748
10	.2226	.2349	.2224
11	.1792	.1990	.1818
12	.1500	.1677	.1513
13	.1269	.1405	.1269
14	.1079	.1170	.1070
15	.0913	.0968	.0904
16	.0767	.0795	.0762
17	.0641	.0649	.0638
18	.0525	.0525	.0527
19	.0420	.0422	.0426
20	.0324	.0336	.0333
21	.0250	.0266	.0262
22	.0200	.0209	.0210
23	.0160	.0162	.0170
24	.0131	.0125	.0139
25	.0110	.0095	.0114

(1) ω = 13.7376 Poisson Frequency
m = 1,000,000 B = 264.7
$S(X) = 1 - (B/(X + B))^{\delta}$ Pareto Severity δ = .45128063

(2) ω = 13.7376 α = 2.56852
r = 0.174667 λ = 4.94882E-7

(3) ω = 13.7376 r = 0.383347
m = 1,000,000 α = 1.42077
p = 0.0243 λ = 1.54E-6

APPENDIX I

$P(n)$, $Q(n)$, and $P(n|n_0)$

$P(n)$ is the probability of n losses in a time period; p is the probability of a total loss (of size m) given that a loss has occurred.

$Q(n)$ is the probability of n total losses. Then

$$Q(n) = \sum_{j=0}^{\infty} P(n + j) \binom{n + j}{n} p^n(1 - p)^j$$

$P(n|n_0)$ is the probability of n partial losses given that n_0 total losses have occurred. Then

$$P(n|n_0) = \binom{n + n_0}{n} p^{n_0}(1 - p)^n/Q(n_0)$$

If $P(n)$ is Poisson, then so are $Q(n)$ and $P(n|n_0)$. Likewise, $P(n)$ negative binomial implies that $Q(n)$ and $P(n|n_0)$ are also negative binomial. The form of the functions remains the same; only the parameters change.

	Poisson Parameter*	Negative Binomial Parameters**	
	α	α_1	α_2
$P(n)$	ω	x	q
$Q(n)$	ωp	x	$q/(p + q - pq)$
$P(n\|n_0)$	$\omega(1 - p)$	$x + n_0$	$p + q - pq$

Note the following interesting fact about the negative binomial case.

$$E(n|n_0) \equiv \sum_{n=0}^{\infty} nP(n|n_0) = (x + n_0) \left(\frac{1 - p - q + pq}{p + q - pq}\right)$$

As the number of total losses increases, so does the expected number of partial losses. This lends support to the usual interpretation of the negative binomial distribution as being associated with situations of positive contagion. (See for example Meyers and Heckman [5].)

* The form of the Poisson is Poisson $(n) = e^{-\alpha}\alpha^n/n!$

** Negative Binomial $(n) = \binom{n + \alpha_1 - 1}{n} \alpha_2^{\alpha_1}(1 - \alpha_2)^n$

APPENDIX II

Moments of $G(x|n_0)$

Recall $G(x|n_0) = \frac{1}{1 - P(0|n_0)} \sum_{n=1}^{\infty} P(n|n_0)\mathcal{S}_1^{n*}(x)$

where $\mathcal{S}_1^{n*}(x)$ is the n^{th} convolution of the cumulative distribution function of the partial losses. Setting $p = 1 - S(m) =$ the probability of a total loss,

$$\mathcal{S}_1(x) = \begin{cases} 0 & x < 0 \\ S(x)/(1 - p) & 0 \le x < m \\ 1 & m \le x \end{cases}$$

The program is as follows:

1) One already knows $E(n^j|n_0) = \sum_{n=0}^{\infty} n^j P(n|n_0)$

and $E(x^j) = (1 - p) \int_0^m x^j d\mathcal{S}_1(x) + pm^j$ for $j = 1$ to 3.

(If $P(n|n_0)$ is Poisson or negative binomial, then the $E(n^j|n_0)$ are tabulated, and presumably one has already calculated the $E(x^j)$.)

Calculate $E^*(n^j|n_0) = \frac{\sum_{n=1}^{\infty} n^j P(n|n_0)}{1 - P(0|n_0)} = \frac{E(n^j|n_0)}{1 - P(0|n_0)}$

and $E^*(x^j) = \int_0^m x^j d\mathcal{S}_1(x) = \frac{E(x^j) - pm^j}{1 - p}$ for $j = 1$ to 3.

2) $\mu_N = E(n|n_0)$
$\sigma_N^2 = E^*(n^2|n_0) - E^{*2}(n|n_0)$
$\gamma_N\sigma_N^3 = E^*(n^3|n_0) - 3E^*(n^2|n_0)E^*(n|n_0) + 2E^{*3}(n|n_0)$
$\mu_x = E^*(x)$
$\sigma_x^2 = E^*(x^2) - E^{*2}(x)$
$\gamma_x\sigma_x^3 = E^*(x^3) - 3E^*(x^2)E^*(x) + 2E^{*3}(x)$

3) Calculate for each n_0 needed, μ_L, σ_L, and γ_L of $G(x|n_0)$ function using the first three formulas of Venter's Appendix C.

4) Calculate the transformed Gamma parameters α_{n_0}, λ_{n_0}, and r_{n_0} by matching the three moments in (3).

Note that if $P(n)$, hence $P(n|n_0)$, is Poisson, then $P(N|n_0)$ and $G(x|n_0)$ are actually independent of n_0 and you need only calculate one triplet α, λ, r for all the G's.

APPENDIX III

Computation of Excess Ratios

$$\text{Define } \tilde{F}(X) = \sum_{n=0}^{[x/m]} Q(n)[P(0|n)\pi(x - nm) + (1 - P(0|n))G(x - nm; r_n, \alpha_n, \lambda_n)]$$

$$\text{Then } E(x) = [mp + (1 - p)E_{\mathcal{S}_1}(x)]E(n)$$

$$\text{where } E_{\mathcal{S}_1}(x) = \int_0^\infty x d\mathcal{S}_1(x) \text{ and } E(n) = \sum_{n=0}^{\infty} nP(n).$$

(Note: The above must be proved and anyone wishing to see a proof can contact the reviewers.)

$$\text{Then } R(a) \equiv \int_a^\infty (x - a)d\tilde{F}(x)/E(x) = 1 - \frac{1}{E(x)} \sum_{n=0}^{[a/m]} Q(n) \Bigg[P(0|n)nm + (1 - P(0|n)) \left\{G\left(a - nm; r_n + \frac{1}{\alpha_n}, \alpha_n, \lambda_n\right) \frac{\Gamma(r_n + (1/\alpha_n))}{\lambda\Gamma(r_n)} + G(a - nm; r_n, \alpha_n, \lambda_n)nm\right\}\Bigg] - \frac{a}{E(x)}(1 - \tilde{F}(a))$$

Although this appears complicated it is really quite simple to compute since usually not many terms are needed.

In the case of a Poisson (with $E(n) = \omega$),

$$P(n|n_0) = e^{-\omega(1-p)}(\omega(1 - p))^n/n!$$

independent of n_0. Therefore λ_{n_0}, α_{n_0} and r_{n_0} are also independent of n_0.

$$\text{Then } \tilde{F}(X) = \sum_{n=0}^{[x/m]} e^{-\omega p} \frac{(\omega p)^n}{n!} [e^{-\omega(1-p)}\pi(x - nm) + (1 - e^{-\omega(1-p)})G(x - nm; r, \alpha, \lambda)]$$

$$E(X) = p\omega m + (1 - p)\omega E_{\mathcal{G}_1}(x)$$

$$= p\omega m + (1 - e^{-\omega(1-p)}) \frac{\Gamma(r + (1/\alpha))}{\lambda\Gamma(r)} \quad \dagger$$

$$R(a) = 1 - \frac{1}{E(X)} \sum_{n=0}^{[a/m]} e^{-\omega p} \frac{(\omega p)^n}{n!} \left[e^{-\omega(1-p)} nm \right.$$

$$+ (1 - e^{-\omega(1-p)}) \left(G \left(a - nm; r + \frac{1}{\alpha}, \alpha, \lambda \right) \frac{\Gamma(r + (1/\alpha))}{\lambda\Gamma(r)} \right.$$

$$\left.\left. + G(a - nm; r, \alpha, \lambda)nm \right) \right] - \frac{a}{E(X)} (1 - \bar{F}(a))$$

†Note: $(1 - e^{-\omega(1-p)}) \dfrac{\Gamma(r + (1/\alpha))}{\lambda\Gamma(r)}$

$$= (1 - e^{-\omega(1-p)}) \int_0^\infty x \, d\left(\frac{\sum_{n=1}^{\infty} P(n|n_0)\mathcal{G}_1^{n*}(x)}{1 - P(0|n_0)} \right)$$

$$= (1 - e^{-\omega(1-p)}) \sum_{n=1}^{\infty} e^{-\omega(1-p)} \frac{(\omega(1 - p))^n}{n!} \frac{nE_{\mathcal{G}_1}(x)}{1 - e^{-\omega(1-p)}}$$

$$= (1 - p)\omega E_{\mathcal{G}_1}(x)$$