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A NOTE ON LOSS DISTRIBUTIONS

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This review will be divided into four sections. First, there are general comments about the paper; next, there are more specific comments and suggestions regarding standardized notation; third, there is a discussion of the Bickerstaff formula; and finally, the notation is extended to other actuarial concepts.

GENERAL COMMENTS

Over the years, many papers have been written on actuarial topics which relate to loss distributions of one form or another. Each author has been free to select the notation used to represent the various concepts, and this freedom has been exercised vigorously. Although this may have resulted in compact notation for a particular paper, the overall result is a plethora of "standards" which are often inconsistent.

Mr. LaRose has attempted to create some order out of this confusion and has succeeded admirably. He has developed a notation (based on the notation originally used by Finger [1]) and applied it to a wide variety of actuarial concepts.

The author actually accomplishes two important goals. First, and most obviously, the author succeeds in defining a reasonably concise notation which can be used to clearly represent many of the important actuarial concepts related to loss distributions. One measure of success is the compactness of the notation. In most cases, the resulting formula is quite compact. In the few exceptions, such as in the case of a disappearing deductible, the resulting formula is no more obscure than that using the original notation.

Second, the use of this standardized notation clearly points out the equivalence of certain actuarial concepts. Although the author makes this point in his conclusion, I think it deserves additional emphasis. The student who encounters Part 9 for the first time should find the going much easier when it is realized that excess ratios, table M charges, excess loss ratios, ELPF's, burning ratios, and stop loss factors are all related concepts.

STANDARDIZED NOTATION

The only concern I have is that this notation might become a de facto standard, without consideration of whether any improvement could be made. The review by Mr. Hewitt included some suggestions for alternative notation; I would like to add to this discussion.

The area defined by $X1(r)$ is referred to in statistics texts as the truncated distribution (with truncation point r) [2]. Similarly, the area defined by $X2(r)$ is referred to as the censored distribution (with censorship point r). Thus, the substitution of XT and XC for $X1$ and $X2$ would provide a useful mnemonic reference. The choice for $X3$ is not as obvious, but I suggest that XS would work.

As the use of risk theory becomes more widespread, we should extend our notation beyond concepts related to means and include variance concepts. One possibility would be to introduce the variables YT, YC and YS defined as follows:

$$YT(x) = \frac{1}{\beta}\int_0^x t^2 dF(t)$$

$$YC(x) = YT(x) + \frac{t^2}{\beta}\int_x^\infty dF(t)$$

$$YS(x) = 1 - YC(x) \quad \text{where } \beta = \textstyle\int_0^\infty t^2 dF(t)$$

Another possibility would be to define these variables using $(t - \alpha)^2$ instead of t^2, so that the variables represent percentages of the total variance, rather than percentages of the total sum of squares. More research needs to be done to determine which, if either, of these two possibilities would be preferable.

BICKERSTAFF

Mr. LaRose shows how the formula for net loss cost in Mr. Bickerstaff's paper [3] can be rewritten in his notation. Unfortunately, he has perpetuated the error in the original formula.

In the original paper, a formula is developed for the net loss cost of auto physical damage coverage. The original formula is reproduced here:

$$\begin{aligned} \textit{Net Loss Cost} = AC_n[&\alpha(1 + r)^{n-1} - DG(D) \\ &- \alpha(1 + r)^{n-1}H(D) - \alpha(1 + r)^{n-1}J(Ld^{n-1}) \\ &+ Ld^{n-1}\, G(Ld^{n-1})] \end{aligned}$$

The functions G, H, and J are related to the loss cost distribution and the first moment distribution. These distributions are based upon loss costs in year 0. To develop the correct loss costs in year n, two types of adjustments are needed.

1. The mean loss cost and list price must be adjusted for inflation and depreciation, respectively. These adjustments are well documented in the original paper.
2. The deductible and list price used as input to the functions must also be adjusted for inflation. This adjustment is not as well documented.

Because the distributions themselves are not changed when used to calculate results for year n, the input values must be stated in terms of year 0. (The impact of a \$100 deductible will be different in year n than in year 0.) The correct adjustment is to divide D and Ld^{n-1} by $(1 + r)^{n-1}$.

If the tables at the end of Bickerstaff's paper are examined, it will be clear that $D/(1 + r)^{n-1}$ <u>is</u> used, rather than D, even though the formula does not include the adjustment.

However, it does not appear that this adjustment was made to the list price. It may be that the factor d^{n-1} is intended to include this adjustment, although that does not appear likely from the text. The correct formula, reflecting these adjustments, is as follows:

$$\begin{aligned} \textit{Net Loss Cost} = AC_n[&\alpha(1 + r)^{n-1} - DG(D/(1 + r)^{n-1}) \\ &- \alpha(1 + r)^{n-1}\, H(D/(1 + r)^{n-1}) \\ &- \alpha(1 + r)^{n-1}J(Ld^{n-1}/(1 + r)^{n-1}) \\ &+ Ld^{n-1}G(Ld^{n-1}/(1 + r)^{n-1})] \end{aligned}$$

or, expressed in Mr. LaRose's notation:

$$\begin{aligned} \textit{Net Loss Cost} = AC_n[&\alpha(1 + r)^{n-1} - D[1 - F(D/(1 + r)^{n-1})] \\ &- \alpha(1 + r)^{n-1}X1(D/(1 + r)^{n-1}) \\ &- \alpha(1 + r)^{n-1}[1 - X1(Ld^{n-1}/(1 + r)^{n-1})] \\ &+ Ld^{n-1}[1 - F(Ld^{n-1}/(1 + r)^{n-1})]] \end{aligned}$$

which can be simplified to:

$$\begin{aligned} &\textit{Net Loss Cost} = AC_n[\alpha(1 + r)^{n-1} - D[1 - F(D/(1 + r)^{n-1}) \\ &- \alpha(1 + r)^{n-1}X1(D/(1 + r)^{n-1}) \\ &- \alpha(1 + r)^{n-1}X3(Ld^{n-1}/(1 + r)^{n-1})]] \end{aligned}$$

OTHER ACTUARIAL CONCEPTS

1. Workers' Compensation Experience Rating

Mr. LaRose indicates that the D-ratios in workers' compensation cannot be written in his notation. Although it is slightly awkward, the D-ratio can be written at least partly in his notation.

Recall that the formula for the primary portion of each loss is as follows [4]:

$$Ap = A \quad \text{when } A \leq I$$

$$Ap = \frac{A}{A + C}(I + C) \quad \text{when } A > I$$

The D-ratio, which is the ratio of the average primary losses to average total losses, can then be written as follows:

$$\text{D-ratio} = \frac{\int_0^I xdF(x) + I\int_I^\infty dF(x) + (I + C)\int_I^\infty (x/x + C)\, dF(x)}{\int_0^\infty xdF(x)}$$

The first two terms are $X2(I)$, so we can rewrite the formula as:

$$\text{D-ratio} = X2(I) + (I + C)\frac{\int_I^\infty (x/(x + C))\, dF(x)}{\int_0^\infty xdF(x)}$$

2. *Fratello Formula*

Subsequent to the completion of his paper, Mr. LaRose also used his notation to express the formula in Fratello's paper [5]. The results are shown below. It should be noted that, while the notation was originally used to study *loss* distributions, it can also be used to study other types of distributions as well (e.g., wage distributions as in Fratello).

Let α = average weekly wage
p = nominal % of compensation
A = minimum weekly benefit/p
B = maximum weekly benefit/p
$a = A/\alpha$
$b = B/\alpha$
t = weekly wage of a worker
$F(t)$ = c.d.f. of t

then, the limit factor is

$$X2(b) - X1(a) + a$$

3. *Table L*

The formulae used in Table L can be considered an extension of those used in Table M with the added consideration of individual loss limitations. However, a minor change to the LaRose notation is needed to express these formulae. If we write the expression for $X1(r)$ with the denominator written out, we have

$$X1(r) = \frac{\int_0^r xdF(x)}{\int_0^\infty xdF(x)}$$

Note in particular that the distribution, used in the numerator and denominator are identical.

If we examine the formulae used by Skurnick [6], we find that the denominator has been omitted (as it is equal to 1).

$$\psi^*(r) = \int_0^r (r - s)dF^*(S)$$

However, the omitted denominator is <u>not</u> $\int_0^\infty sdF^*(s)$ but $\int_0^\infty sdF(s)$. Here, the distributions in the numerator and denominator are different. We can overcome this by defining a new set of distributions as follows:

$$XI^*(x) = \frac{\int_0^x tdF^*(t)}{\int_0^\infty tdF(t)}$$

$$X2^*(x) = XI^*(x) + \frac{x(1 - F^*(x))}{\int_0^\infty tdF(t)}$$

$$X3^*(x) = \frac{\int_x^\infty (t - x)dF^*(t)}{\int_0^\infty tdF(t)}$$

In the specific case of Table L, the denominators are identically 1, so they may be omitted.

Now we can restate the Skurnick formulae in terms of this notation:

$$\begin{aligned}\phi^*(r) &= \int_r^\infty (s - r)dF^*(s) + k \\ &= X3^*(r) + k\end{aligned}$$

$$\begin{aligned}\psi^*(r) &= \int_0^\infty (r - s)dF^*(s) \\ &= rF^*(r) - XI^*(r) \\ &= r - X2^*(r)\end{aligned}$$

The relationship between the charge and the savings can also be derived. However, note that the relationship between $X3$ and $X2$ is slightly changed when we work with $X3^*$ and $X2^*$

$$\begin{aligned}X3^*(r) &= \int_r^\infty (t - r)dF^*(t) \\ &= \int_r^\infty tdF^*(t) - r\int_r^\infty dF^*(t) \\ &= 1 - k - XI^*(r) - r(1 - F^*(r)) \\ &= 1 - k - X2^*(r)\end{aligned}$$

Thus,

$$\begin{aligned}\phi^*(r) &= X3^*(r) + k \\ &= 1 - X2^*(r) \\ &= 1 - r + \psi^*(r)\end{aligned}$$

To be consistent with the notation I proposed earlier, I would suggest using XTL, XCL, and XSL instead of XI^*, $X2^*$, and $X3^*$ respectively, where L could be a mnemonic for either loss limitation or Table L.

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4. R. H. Snader, "Fundamentals of Individual Risk Rating and Related Topics," CAS Study Note, Part III.
5. B. Fratello, "The Workmen's Compensation Injury Table and Standard Wage Distribution Table—Their Development and Use in Workmen's Compensation Ratemaking," *PCAS* XLII, 1955, p. 110.
6. D. Skurnick, "The California Table L," *PCAS* LXI, 1974, p. 117.