DURATION

RONALD E. FERGUSON

Abstract

This paper begins by highlighting some of the changes in the macroeconomic environment that have affected the way insurers and reinsurers price their products. Attention is next focused on the importance of the time value of money for certain insurance products. The next topic is reinvestment risk and the ways investors try to deal with this problem. Working with fixed income securities, immunization theory and, in particular, duration are discussed. Duration is the word given to the statistic derived by weighting each year (of the bond's life) by the present value of the associated cash flows; all aggregated and divided by the price of the bond. Duration provides a theory or framework that the investor can use to more or less guarantee (as far as the investment or reinvestment risks are concerned) a targeted wealth position.

Today's economic environment has caused all of us to think a lot more about the investment side of the insurance and reinsurance business. While the time value of money has always been at least an implicit factor in insurance and reinsurance pricing, it has now, for better or worse, come out of the closet. Certain insurance products—structured settlements and loss portfolio transfers, for example—put the investment issue on center stage.

The pricing of some products is, in part, dependent upon an accurate determination of the present value of the dollars that will ultimately be paid out. For example, a structured settlement, made in lieu of a lump sum claim payment, guarantees that a specified number of dollars will be paid to the claimant at specified intervals for life or for a determined number of years. A loss portfolio transfer involves the transfer or reinsurance of a defined block or portfolio of known and/or unknown losses from one party to another. A present value is assigned to the portfolio of losses; pricing is largely a function of this estimated present value. Once the ultimate value of the structured settlement or loss portfolio transfer has been determined (estimated), the insurer normally purchases one or more bonds that will fund the claim payments. As we will see, the concept of "duration" is useful in attempting to harmonize the liabilities and the related invested assets.

A factor that makes the time value of money both critical and hard to deal with is that we are living in turbulent financial times. One set of statistics that highlights this is the frequency of prime rate changes. The prime rate changed only twice in the 1940's; 16 times in the 1950's; 17 times in the 1960's; 132 times in the 1970's, and so far, with over two years of the 1980's finished, we've already seen over 70 changes! [1]

Since the time value of money is now an explicit and important part of some of our pricing and the financial environment is unstable or volatile, we will have to pay more attention to the investment side of the business. It is recognized that there are many factors that influence the rate of return of an insurance program. For life insurance, the principal underwriting risks are the mortality, lapse and long term expense assumptions. Property/casualty insurers face critical frequency and severity assumptions, both confounded by price and social inflation.

Once a view is taken on these assumptions, no matter how certain or uncertain they may be, there remain timing (i.e., the actual incidence of payments) and investment risks. This paper does not address underwriting risk (i.e., in the context of this paper, the expected ultimate loss level) or the timing of the related payments. These are important but separate topics. Similarly, this paper does not address the important topic of profit and contingency loadings.

In pricing the financial, as opposed to underwriting, aspect of an insurance or reinsurance arrangement, the first step is usually to calculate a breakeven or internal rate of return. The internal rate of return or yield for an investment is the discount rate that equates the present value of the expected cash outflow with the present value of the expected inflows [2][1]. Next, an attempt is made to design an investment program to produce a return greater than the breakeven or internal rate of return. Obviously there would be little or no incentive to write the business if the internal rate of return cannot be exceeded with an acceptable level of investment risk.

There are, of course, a number of financial risks or reasons why the targeted return might not be achieved. The principal investment related risks are: *timing* (i.e., the timing or incidence of the cash outflows, which might be very predictable for auto physical damage, for example, but quite uncertain for the so-called long tail lines); *credit risk* (i.e., default as to interest and principal) which won't be treated in this paper; and *reinvestment risk* which will be discussed.

Our company recently had a submission that highlighted reinvestment risk in a dramatic way. Stripped of nonessential features and somewhat disguised, the proposal involved a single premium, paid in advance, in return for a commitment to pay \$100,000,000 at the end of 20 years. Assuming the money is invested at 12% per annum and ignoring the credit risk, profit, overhead, taxes, arbitrage opportunities, etc., the price or "pure premium" is $\$100{,}000{,}000/(1.12)^{20}$ or \$10,366,677.

How do we get from \$10,366,677 to \$100,000,000? Let's assume we buy a bond for \$10,366,677 (assume cost = par value = redemption value) with a 12% annual coupon. At the end of 20 years we have:

Our \$10,366,677	(redemption)
and	
\$24,880,023	(20 years of interest, at a 12% coupon)

[1] In mathematical terms, r = the internal rate of return where CF_t is the cash flow for period t such that

$$\sum_{t=0}^{n} \left[\frac{CF_t}{(1+r)^t}\right] = 0$$

(A discussion of the possible shortcomings of the Internal Rate of Return technique can be found in *Financial Theory and Corporate Policy,* by Thomas E. Copeland and J. Fred Weston, Reading, Massachusetts: Addison-Wesley Publishing Company, 1979 edition, pages 28–33).

The rest, a staggering $64,753,300, comes from interest on interest or reinvestment! Put another way, 72% of the return comes from the interest earned on the interest.[2] Clearly, reinvestment can make or break an insurance program. Let's say upon reinvestment we get 10%—not 12%—we'll then be short of the goal by $18,383,000. Of course, rates could go the other way, say to 14%, in which case we would have a windfall profit of $23,602,000.

Volatile interest rates and the recognition of the time value of money in insurance and reinsurance pricing make it appropriate for us to put a new word in our vocabulary—immunization. Unfortunately, the word immunization isn't always used precisely and, perhaps, it means different things to different people. The definition I offer is: the investment risks are immunized if the desired wealth level (of the investment portfolio) has been achieved at the end of the investment horizon (i.e., holding period) regardless of interest rate changes during the holding period. This, of course, implies that all intervening cash flows during the holding period have been met.

Immunization, although much talked about today, is not a new concept. Some fairly sophisticated work was done on immunization theory at least as far back as 1938 by Mr. Frederick R. Macauley [3]. The earliest traces in the actuarial literature date to 1952 when a British life insurance actuary, Mr. Frank Reddington [4], suggested that insurance companies really ought to think about synchronizing their investments and underwriting risks. In this country, the cause has been championed, for the last 10 years or so, by Mr. Irwin T. Vanderhoof, FSA, ACAS [5].

To understand immunization techniques one needs to understand the several ways bonds can be characterized. First, there is the simple, but not very useful, notion of years or term to maturity. This is self-explanatory—a bond maturing in 2002 has a 20 year maturity as of 1982. As demonstrated in the above example, buying a 20 year bond to cover a liability maturing in 2002 does not immunize one from the "disease" of changing interest rates. In a period of volatile interest rates, characterizing a bond as a 20 or 30 or whatever years to maturity really isn't very useful.

Recognition of the fact that the years to maturity isn't a useful way to describe a bond has led to another measure known as the weighted term or years to maturity. Under this approach *all* cash flows occurring over the life of

[2] See Appendix I for some other examples.

the bond (i.e., interest coupons and the redemption value) are used as weights for the year involved. Put on a formula basis:[3]

$$\text{Weighted average term to maturity} = \frac{CF_1 \cdot 1}{\Sigma CF} + \frac{CF_2 \cdot 2}{\Sigma CF} + \frac{CF_3 \cdot 3}{\Sigma CF} \ldots ; \frac{CF_n \cdot \mathrm{n}}{\Sigma CF} \text{ or,}$$

$$\frac{\sum_{t=1}^{n} CF_t(t)}{\sum_{t=1}^{n} CF_t},$$

where

t = year of cash flow (i.e., year 1, year 2, etc.)
CF_t = cash flow in year t
n = number of years to maturity

Thus, a 10 year bond with a 4% coupon would have a weighted average term to maturity of 8.71 years while the same type of bond with an 8% coupon would have a weighted average term to maturity of 8 years (see Table A below). Since this measure recognizes the cash flow differences between the bonds, it is somewhat more useful than the years to maturity in determining a portfolio's overall sensitivity to changing interest rates. Although a better measure than years to maturity, weighted average years to maturity doesn't have (much) operational significance. The problem with this measure is that each dollar has equal weight; that is, the time value is not considered.

The quest for immunization has led to an even more sophisticated and more useful concept known as "duration" of the bond. Duration is a measure of a bond's price volatility. Thus, duration can be derived using differential calculus (see Appendix II).

In simpler terms, duration is a weighted average term to maturity where the years are weighted by the present value of the related cash flow.

[3] In this and all other formulas in this paper, annual end-of-year interest payments have been assumed. It would be relatively easy to modify the formulas to accommodate the more typical mode of semi-annual interest payments.

TABLE A [6]

Bond A—$1,000 face value with a 4% coupon, maturing in 10 years
Bond B—$1,000 face value with a 8% coupon, maturing in 10 years

Weighted Average Term to Maturity (assuming annual interest payments)

Bond A

(1) Year	(2) Cash Flow	(3) Cash Flow/TCF	(4) (1) × (3)
1	$ 40	0.02857	0.02857
2	40	0.02857	0.05714
3	40	0.02857	0.08571
4	40	0.02857	0.11428
5	40	0.02857	0.14285
6	40	0.02857	0.17142
7	40	0.02857	0.19999
8	40	0.02857	0.22856
9	40	0.02857	0.25713
10	1,040	0.74286	7.42860
Sum	$1,400	1.00000	8.71425

Weighted Average Term to Maturity—8.71 Years

Bond B

(1) Year	(2) Cash Flow	(3) Cash Flow/TCF	(4) (1) × (3)
1	$ 80	0.04444	0.04444
2	80	0.04444	0.08888
3	80	0.04444	0.13332
4	80	0.04444	0.17776
5	80	0.04444	0.22220
6	80	0.04444	0.26664
7	80	0.04444	0.31108
8	80	0.04444	0.35552
9	80	0.04444	0.39996
10	1,080	0.60000	6.00000
Sum	$1,800	1.00000	7.99980

Weighted Average Term to Maturity—8.00 Years

$$\text{Duration} = \frac{\displaystyle\sum_{t=1}^{n} \frac{t \cdot CF_t}{(1+y)^t}}{\displaystyle\sum_{t=1}^{n} \frac{CF_t}{(1+y)^t}}$$

CF_t = cash flow in year t
y = yield to maturity[4] (not the coupon rate)
t = year of cash flow
n = number of years to maturity.

Using the above formulas, for example, the 8% bond (maturing in 10 years) discussed above has a duration of 7.25 years compared with a weighted average term to maturity of 8.0 years. (See Table B.)[5]

There are a couple of other ways to compute a bond's duration (see Appendix II for a discussion of volatility and duration), one of which is a crude, but useful shortcut/approximation. If the coupon rate is fairly close to the yield to maturity, say 70% or more, the duration can be very roughly approximated as $1/y + 1$, the formula for a perpetuity (See Appendix II, Section II, Equation 10).

Duration can do some very interesting and wonderful things for the investor seeking to achieve a certain ultimate return or wealth level. Duration allows the interest rate risk (i.e., reinvestment) to be balanced with the price or capital

[4] There are three different yields associated with a bond

1. *Nominal yield* is the ratio of interest to principal (without regard to compounding). This is also called the coupon rate.

2. *Current yield* is the ratio of interest to the amount actually paid for the bond. The current yield overstates the return on premium bonds and understates the return on discount bonds.

3. *Yield to maturity*—sometimes called the net yield to maturity, takes into account all cash flows associated with the bond, i.e., the amount paid, the interest and redemption amounts to be received if the bond is held to maturity.

   $$\text{Price} = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$

   r is the yield to maturity

[5] See Appendix V for a replication of the Table B Duration Values and a simple program to calculate a bond's duration.

TABLE B [7]
DURATION OF A BOND

DURATION (ASSUMING EIGHT PER CENT MARKET YIELD)
BOND A—4% COUPON

(1) Year	(2) Cash Flow	(3) PV at 8%	(4) PV of Flow	(5) PV = % of Price	(6) (1) × (3)
1	$ 40	0.9259	$ 37.04	0.0506	0.0506
2	40	0.8573	34.29	0.0469	0.0936
3	40	0.7936	31.75	0.0434	0.1302
4	40	0.7350	29.40	0.0402	0.1606
5	40	0.6806	27.22	0.0372	0.1860
6	40	0.6302	25.21	0.0345	0.2070
7	40	0.5835	23.34	0.0319	0.2233
8	40	0.5403	21.61	0.0295	0.2360
9	40	0.5002	20.01	0.0274	0.2466
10	1,040	0.4632	481.73	0.6585	6.5850
Sum			$731.58	1.0000	8.1193

Duration—8.12 Years

BOND B—8% COUPON

(1) Year	(2) Cash Flow	(3) PV at 8%	(4) PV of Flow	(5) PV = % of Price	(6) (1) × (3)
1	$ 80	0.9259	$ 74.07	0.0741	0.0741
2	80	0.8573	68.59	0.0686	0.1372
3	80	0.7938	63.50	0.0635	0.1906
4	80	0.7350	55.80	0.0588	0.1906
5	80	0.6806	54.44	0.0544	0.2720
6	80	0.6302	50.42	0.0504	0.3024
7	80	0.5835	46.68	0.0467	0.3269
8	80	0.5403	43.22	0.0432	0.3456
9	80	0.5002	40.02	0.0400	0.3600
10	1,080	0.4632	500.26	0.5003	5.0030
Sum			$1000.00	1.000	7.2470

Duration—7.25 Years

risk. This balancing arises out of the *inverse*[6] relationship between interest rate or reinvestment risk and price risk.

The zero coupon bonds that have recently become fashionable may help to illustrate duration.[7] An obligation to fund a liability of known proportion at the end of 10 years would be totally satisfied by a 10 year zero coupon bond; that is, there is no reinvestment risk. It doesn't matter what happens to interest rates if the only goal is to exactly achieve a certain wealth position at the end of the holding period. Put another way, since a zero coupon bond has no interim cash flow, its *term to maturity* is equal to its *weighted term to maturity* which is also equal to its *duration*.

It turns out that, for several reasons, zero coupon bonds are not a panacea.

- Zero coupon bonds are not as yet widely available.
- Zero coupon bonds may not be available at a credit risk level that suits the investor. (Also note that the *entire* credit risk is "stacked" at the redemption date.)
- Zero coupon bonds, other things being equal, carry a slight premium.
- Tax exempt institutions (such as pension funds) are currently the major investors in these bonds. The tax implication of zero coupon bonds can—to certain investors—be onerous (i.e., the bond owner is subject to tax on income which is accrued—not received).[8]

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[6] Other things being equal, as interest rates rise the price or market value of fixed rate bonds decline and as interest rates decline the price or market value of fixed rate bonds rises.

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[7] According to George L. Shinn's article, "Innovative Approaches to Financing" appearing in the Winter 81/82 issue of *Chief Executive*, J. C. Penney Inc. issued the first public zero coupon bond in April, 1981.

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[8] The obverse of this coin is, of course, a great attraction to the bond issuer, but the IRS wants to spoil the game—a little. See page 43 of the May 5, 1982 edition of the *Wall Street Journal*. Currently the issuer takes a deduction on a pro-rata or equal installment accrual of interest. The proposed IRS change, which will require Congressional approval, will reflect the compounding of accrued interest.

Example: 30 year $1,000 bond, purchased for $50, yielding 10.5%

	Deductible Interest	
	Current tax basis	Proposed
Year 1	$950/30 = $31.67	$50.00 × .105 = $5.25
Year 2	$950/30 = $31.67	$55.25 × .105 = $5.80

It seems clear that such a change in the tax law will, other things being equal, reduce the enthusiasm of would-be zero coupon bond issuers.

An interesting recent development strikes at the first two shortcomings: availability and credit risk. In July of 1982, Merrill Lynch brought to market a cleverly designed new product called TIGR's (Treasury Investment Growth Receipts) [8]. Other investment bankers have since followed suit.

TIGR's are treasury bonds repackaged to look and behave like zero coupon bonds. The new tax act, Tax Equity and Fiscal Responsibility Bill of 1982, effective July 1, 1983, added I.R.C. Section 1232B, which prescribes the tax treatment for bonds that have been stripped (i.e., for which the interest coupons have been separated). An investment bank might, for example, buy a 25-year bond, strip out the 50 interest coupons yielding 51 zero coupon bonds (including the bond itself). The "mini" zero coupon bonds are kept in a custodian bank. The investor gets a receipt as evidence of his claim on the securities.

The last shortcoming, taxes, may yet be solved. Some state and local housing authorities have been issuing zero coupon municipal bonds. Most of the issues so far have call features, thus taking away one of the presumed advantages of zero coupon bonds and Original Issue Discount bonds. (According to Woolridge and Gray, Original Issue Discount, "OID", bonds of which zero coupon is the extreme case are priced to yield as much as 100 basis points less than otherwise comparable full coupon bonds. They offer two reasons: non-callability and immunization. [9])

Working with bond durations, one can achieve nearly the same immunized result offered by zero coupon bonds. In other words, if a company has a 10 year obligation and invests in a bond with a duration of 10 years, which may in fact involve a bond with a term to maturity of 18 years (for example), the return/wealth would be immunized. This happy result comes about because of the counterbalancing of interest rate risk and price risk. In other words, if interest rates go down, the investment return is less than anticipated but there is a counterbalancing capital gain in the market price of the bond. Put another way, by buying a bond with a longer than apparently needed term to maturity—but with the right duration—the investor creates an interest sensitive overhang (i.e., the difference between the duration and the term to maturity) on the bond which is engineered to the right proportions. Consider the following examples in Table C:

TABLE C [10]

Realized Return From a 5-Year 9% Par Bond Over Various Horizon Periods

Reinvestment Rate and Yield-to-Maturity At Horizon		Horizon Period 1 Year	3 Years	4.13 Years	5 Years
	Coupon Income	$90	$270	$372	$450
7%	Capital Gain	$68	$37	$16	$0
	Interest-On-Interest	$2	$25	$51	$78
	Total Dollar Return	$160	$331	$439	$528
	Realized Compound Yield	*15.43%*	*9.77%*	*9.00%*	*8.66%*
9%	Capital Gain	$0	$0	$0	$0
	Interest-On-Interest	$2	$32	$67	$103
	Total Dollar Return	$92	$302	$439	$553
	Realized Compound Yield	*9.00%*	*9.00%*	*9.00%*	*9.00%*
11%	Capital Gain	−$63	−$35	−$16	$0
	Interest-On-Interest	$2	$40	$83	$129
	Total Dollar Return	$29	$275	$439	$579
	Realized Compound Yield	*2.89%*	*8.26%*	*9.00%*	*9.36%*

Table C illustrates a striking compensation effect for investment periods of less than 5 years. For the 3-year period, at the 7% reinvestment rate assumption, the interest-on-interest naturally falls short of the amount required to support a target return of 9%. However, if the bond could be sold at the price corresponding to the assumed 7% yield-to-maturity rate, then a capital gain would be realized which would more than compensate for the lower value of interest-on-interest. Table C illustrates the well-known facts that over the short term, lower interest rates lead to increased returns through price appreciation while, over the longer term, lower interest rates lead to reduced returns through reduced interest-on-interest. For periods lying between the short term and the longer term, it is not surprising to find these two effects providing some compensation for each other.

Duration is particularly useful when pricing a single risk or insurance program. Indeed, it is probably the only way to immunize the investment risks of such individual undertakings. Duration can also be used by the insurance company as it aggregates risks and a corresponding portfolio of invested assets. The required duration of the portfolio could be computed as the weighted average of the various constituent durations or computed on an aggregate basis—by expected payout year (e.g., a certain block of assets with a duration of .5 for payments in the first year, 1.5 for payments in the second year, etc.).

While duration is an elegant and appealing concept, it is not without a few practical problems. First, it's nearly impossible to find bonds with a duration greater than 20 years.[9] Second, a bond's duration changes over time. For example, as one year of a 10 year holding period passes there remain 9 years on the obligation, but the duration has decreased only by perhaps 6/10 of a year. Thus, the liability and corresponding assets are out of synchronization and no longer immunized. Third, when interest rates change the bond duration also changes with the result that the investor is not immunized against *further* interest rate changes. (See Appendix III.) Fourth, sinking funds and call features can make the whole process fairly complicated. These four problems can, however, be overcome by constantly retuning or rebalancing the duration of assets. Fifth, transaction costs and taxes can be a drag on the immunization program.

The duration of the portfolio must be tracked—an ideal computer application—and the portfolio tuned as

1. interest payments become available for investment
2. bonds mature
3. time passes
4. market yields change, and
5. new liabilities are taken on.

The easiest way to envision the retuning is to sell the entire portfolio and reinvest at the new required/computed durations: not the most efficient approach, but easy to understand. Alternatively, retuning can be accomplished by merely shifting funds from longer term bonds to shorter term bonds to shorten the

[9] As noted earlier, duration is a specially weighted average term to maturity—the weights being the present value of the cash flows. A bond with a duration of 20 involves a term to maturity of 45 at 4% yield to maturity with a coupon of 6%, 70 years at 5% YTM. At a 6% YTM, the duration starts to converge on 17.65 at about 120 years. Since bonds are rarely issued with terms succeeding 35 years, it is nearly impossible to achieve a duration of more than 20 years with today's coupon rates and yields to maturity. See also Appendix II, section II and Appendix III Exhibit A2.

duration (the usual requirement as time passes) or vice versa. Simply reinvesting the coupons, as they are paid, in short term bonds may fund and accomplish the needed rebalancing. For a fuller discussion of portfolio rebalancing, see Gushee's article [11].

Another portfolio approach is to strive for a perfect matching of the expected payments and the interest income and redemptions from the investment portfolio. The usual motivation for matching of this type is solvency considerations. The National Association of Insurance Commissioners has, from time to time, expressed an interest in the matter and commissioned Tillinghast, Nelson and Warren, Inc. to study the idea and develop a program or protocol [12].

In theory, an immunized portfolio can be achieved by cash flow matching. This condition will be obtained only if all interest coupons go directly to loss payments so that there is *no* reinvestment exposure. Similarly, all loss payments and maturities have to be precisely matched. All in all, a difficult but not impossible task.

It would, of course, be easy to construct a cash flow matched portfolio using zero coupon bonds. While such a portfolio might be called a cash flow matched portfolio, it is in reality a duration based portfolio.

It is also possible to construct a perfect cash flow matched portfolio using conventional bonds—especially if the liability or payment stream is decreasing over time. If the payment stream is increasing or variable (i.e., up and down from year to year), it may not be possible to achieve perfect matching (i.e., avoid reinvestment risks).

Merrill Lynch and several other investment houses (Salomon Brothers, First Boston and Goldman Sachs, to name a few) have developed cash flow matching models. Under those systems, the customer specifies the "cash flow liability stream". The "system" accesses the firm's bond data base and develops a portfolio consistent with the customer's expected payment profile, the customer's credit risk appetite, and the customer's attitude toward call risk. (The greater the coupon, the greater the call risk. Put the other way around, bonds with lower coupons, other things being equal, will sell at a greater discount and hence afford greater call protection.)

A sample portfolio is set forth in Appendix IV. A study of Appendix IV reveals that all coupon interest goes directly to loss payments leaving no reinvestment risk. There is a very strong similarity or connection between a perfect cash flow matched portfolio and the TIGR's discussed earlier. Indeed, as can be seen from Appendix IV, a perfect cash flow matched portfolio is developed by taking a conventional bond and breaking it into n + 1 (i.e., n coupons plus a redemption value) constituent zero coupon bonds. Thus it can be seen that cash flow matching is actually a subset or type of duration.

In general, a matching program will keep maturities shorter than a duration program. This, of course, will produce a yield penalty in (normal) times of a positive yield curve.[10]

Cash flow matching can be a very useful concept—especially for the regulator. Depending on the objectives and circumstances, matching may or may not be a good investment strategy. Cash flow matching is difficult to achieve at the individual risk or program level (where there's a single payment or where the payments are so small that it would not be feasible to put together a portfolio of bonds.) Duration, on the other hand, can and does work both at the individual risk and portfolio level.

Summary

As Dr. Leibowitz has suggested, "the traditional motivation for bond investment was to secure a fixed cash flow over some appropriate time frame. The typical bond investor was highly risk averse. He was more than willing to sacrifice the excitement of potentially spectacular results in order to achieve a reasonably reliable pattern of return. However, in recent years, the traditional role of bonds as an asset category has been buffeted by a series of dramatic changes in the marketplace. Surging interest rates and an explosion in volatility have characterized recent markets. This environment imposes a harsh dilemma on the bond portfolio manager: how to pursue prudent active strategies and still provide his client with the comfort level that probably served as the primary basis for allocating funds to the fixed-income market in the first place?" [13]

In most insurance and reinsurance pricing, the reinvestment problem is a small, although not unimportant, element in the parcel of risks assumed. For the so-called long tail liability lines and life insurance, and specialty products

[10] A yield curve is a plot of yields (usually on the ordinate) and maturities (usually on the abscissa) for bonds of comparable quality. The bonds differ as to maturities but need to be identical in creditworthiness, e.g., treasury bonds or bonds issued at the same time by the same issuer. The yield curve tells us what a knowledgeable investor requires, other things being equal, to commit for longer times (i.e., maturities). A yield curve is said to be positive or upward sloping if yields increase as maturities lengthen. Yield curves are normally positive reflecting the greater uncertainty and "risk" premium associated with longer maturities. There have been periods, notably in the early 1980's, when the yield curve has been negative (i.e., short duration money commands higher interest rates than long term money).

such as structured settlements and loss portfolio assumptions, the reinvestment risk is very important. Often, this additional risk element is either underestimated or, perhaps more often, simply ignored.

Sometimes an attempt is made to quantify the reinvestment risk, charge an additional premium for the risk, and use conventional investment techniques. In other words the actuary might price the business using yields derived from an immunized investment portfolio—even though the funds might be invested on another basis.

Generally it would be prudent to immunize or harmonize the liabilities and the related invested assets. As discussed in this paper, immunization can be achieved by (exact) cash flow matching or by tuning the investment portfolio to the appropriate duration.

Cash flow matching and duration are very useful concepts and are but the first steps (in a way, building blocks) in more sophisticated contemporary portfolio management. I believe actuaries as well as others in today's insurance company need to be more familiar with the management or harmonization of assets and liabilities. We also need to have some familiarity with the newer active (versus the traditional passive) portfolio management theories and techniques. It is hoped that this paper is a step, even if a modest one, in that direction.

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[17] Macauley, op. cit.

[18] Leibowitz, op. cit., page 5.

[19] Rountree, op. cit., page 4.

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APPENDIX I

MAGNITUDE OF INTEREST-ON-INTEREST TO ACHIEVE 9% REALIZED COMPOUND YIELD FROM 9% PAR BONDS OF VARIOUS MATURITIES [14]

Maturity In Years	Total Dollar Return	Interest-On-Interest At 9% Reinvestment Rate	Interest-On-Interest As Percentage Of Total Return
1	$92	$2	2.2%
2	193	13	6.5
3	302	32	10.7
4	422	62	14.7
5	553	103	18.6
7	852	222	26.1
10	1,412	512	36.2
20	4,816	3,016	62.6
30	13,027	10,327	79.3

APPENDIX II

BOND PRICE VOLATILITY AND DURATION

An idea suggested by Rountree [15] and others is to get a different perspective on duration by developing a measure of bond price volatility. Consider the financial world's rule of thumb that one basis point change in yield for a long term coupon bond drives a ⅛% bond price change. This implies a duration of .125%/.01% or 12.5.

Studying bond price volatility leads to a more rigorous explanation/derivation of duration.

If the price of the bond is P,

$$P = \sum_{t=1}^{N} \frac{CF_t}{(1 + y)^t} \tag{1}$$

we can, using simple differential calculus, measure the change in the price of the bond related to a change in the yield:

$$\lim_{\Delta y \to 0} \frac{\Delta P}{\Delta y} = \frac{dP}{dy} = \sum_{t=1}^{N} \frac{CF_t}{(1 + y)^{t+1}} \cdot (-t) \tag{2}$$

Relating this to the price of the bond,

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^{t+1}}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t}$$

$$= \left(\frac{1}{(1 + y)}\right) \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^t}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t} \tag{3}$$

Thus, it can be seen that duration is a function of bond price volatility

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = \frac{-1}{1 + y} \cdot \text{Duration} \tag{4}$$

Put another way, the relation between the duration of a bond and its price volatility (as set forth by Hopewell and Kaufman [16], is:

$$\lim_{\Delta y \to 0} \frac{\Delta P}{P} = -D^*(\Delta y) \tag{5}$$

where

$\frac{\Delta P}{P}$ = the % change in bond price

D^* = the adjusted duration of the bond in years which is equal to $\left(\frac{1}{1 + y}\right) D$, and

Δy = the change in the market yield.

Rearranging, we get:

$$\lim_{\Delta y \to 0} \frac{\Delta P/\Delta y}{P} = -D^* = \left(\frac{1}{1 + y}\right) \cdot -D = \left(\frac{1}{1 + y}\right) \frac{\Sigma_{t=1}^{N} CF_t(-t)/(1 + y)^t}{\Sigma_{t=1}^{N} CF_t/(1 + y)^t} \tag{6}$$

as in (3).

In practice, the unadjusted duration figure (D) is used when computing the impact of market rate changes. Without the factor $(1/1 + y)$ we have the time-weighted "corporate average-life" formula with each payment period weighted by its present value discount factor as originally proposed by Macauley. [17]

Thus,

$$D^* = \lim_{\Delta y \to 0} \frac{-\Delta P/\Delta y}{P} = \frac{-dP/dy}{P} = \frac{\Sigma_1^N CF_t(t)/(1 + y)^t}{\Sigma_1^N CF_t/(1 + y)^t} \tag{7}$$

Consider a discrete case:

$$D^* = \frac{-\Delta P/\Delta y}{P} = \frac{-\Delta P}{P} \cdot \frac{1}{\Delta y} = \frac{-\text{ price change}}{\text{price}} \cdot \frac{1}{\text{yield change}} \quad (8)$$

(from 5), or

$$D^* = \frac{-(P_2 - P_1)}{(P_1 + P_2)/2} \cdot \frac{1}{y_2 - y_1} = \frac{2(P_1 - P_2)}{P_1 + P_2} \cdot \frac{1}{y_2 - y_1}$$

Example: A 10 year bond with an 8% coupon (at the end of the year)

$P_1 = 99.9665$ $\quad Y_1 = .08005$

$P_2 = 100.0336$ $\quad Y_2 = .07995$

The adjusted duration equals 6.71 From (8)

Duration = 7.247 as in Table A, Bond B. From (4)

APPENDIX II—SECTION II

Consider a perpetuity (of $1)

$$P = \frac{1}{y} \quad (9)$$

$$\frac{dP}{dy} = \frac{-1}{y^2}$$

$$D^* = \frac{-dP/dy}{P} = \frac{1/y^2}{1/y} = \frac{1}{y}$$

$$D = \frac{1}{y} + 1 \quad (10)$$

Thus for very long term bonds, $1/y + 1$ may be a good approximation for duration.

This exercise also sheds some light on a comment in the text that it's not possible with high yields for a bond to have a duration of 20 years or longer. Formula (10) would suggest that durations of 20 or more years can only be achieved when y, the yield, is less than 5.26%.

APPENDIX III

EXHIBIT A1

DURATION OF VARIOUS BONDS ALL PRICED TO YIELD 9% [18]

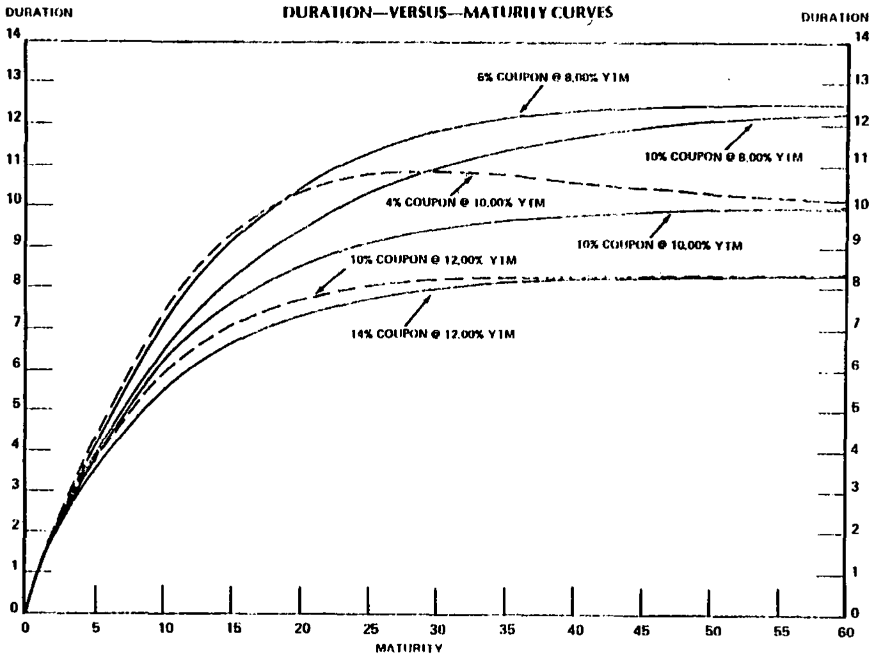
Maturity in Years	Coupon 0%	7.5%	9.0%	10.50%
1	1.00	0.98	0.98	0.98
2	2.00	1.89	1.87	1.86
3	3.00	2.74	2.70	2.66
4	4.00	3.51	3.45	3.38
5	5.00	4.23	4.13	4.05
7	7.00	5.50	5.34	5.20
10	10.00	7.04	6.80	6.59
20	20.00	9.96	9.61	9.35
30	30.00	11.05	10.78	10.59
100	100.00	11.61	11.61	11.61

Ex. A1 shows the *Duration* of various bonds. Returning to the original objective of providing an assured 9% target return over a 5-year period, we can see that one should choose a bond having a *Duration* of 5 years (as opposed to a maturity of 5 years!)

To obtain a *Duration* of 5-years in a 9% par bond, it turns out that one would need a maturity of around 6.3-years.

EXHIBIT A2

DURATION-VERSUS-MATURITY CURVES [19]



APPENDIX IV

DECLINING LIABILITY STREAM [20]

Year	Benefit Payment Required	Cash Flow Matched Portfolio: Par	Name	Coupon	Maturity	Cost	Cash flow
1981	$1,260,000	$505,000	Mt. States Tel.	8.700	9/1/1981	$485,000	$1,260,662.00
1982	1,219,000	528,000	Norway	7.500	6/15/1982	486,615	1,219,927.00
1983	1,175,000	503,000	Australia	8.125	11/15/1983	447,473	1,175,127.00
1984	1,127,000	522,000	Export Dev. Bank	9.850	1/15/1984	475,542	1,127,549.75
1985	1,078,000	521,000	Manuf. Han. Tr.	8.500	6/1/1985	453,270	1,078,699.25
1986	1,026,000	516,000	Sweden	9.500	4/15/1986	448,941	1,027,047.25
1987	974,000	514,000	European Invest. Bk.	9.875	6/1/1987	445,828	975,158.50
1988	919,000	484,000	Ford Mtr. Credit	8.250	11/1/1988	353,823	919,779.75
1989	864,000	493,000	Trailer Train	10.000	5/15/1989	403,023	864,199.75
1990	807,000	461,000	GMAC	7.125	12/1/1990	328,850	807,549.75
1991	748,000	435,000	Ford Mtr. Credit	7.500	11/15/1991	298,815	748,703.50
1992	688,000	424,000	Commercial Credit	7.750	2/15/1992	281,544	688,648.50
1993	628,000	396,000	Heller	7.750	4/1/1993	265,724	628,873.50
1994	567,000	350,000	GMAC	7.750	10/1/1994	238,763	567,528.50
1995	508,000	318,000	Ohio Edison	8.750	9/1/1995	212,427	508,403.50
1996	451,000	289,000	Phil Electric	8.250	8/1/1996	184,870	451,578.50
1997	396,000	269,000	H. F. C.	8.450	1/15/1997	180,351	396,370.75
1998	344,000	239,000	Canada	8.625	4/1/1998	173,784	344,698.63
1999	296,000	201,000	New England Pwr.	8.375	9/1/1999	131,138	296,391.75
2000	252,000	174,000	Carolina P & L	8.750	8/1/2000	118,673	252,558.00
2001	212,000	156,000	Public Svc. Ele.	8.375	5/15/2001	105,030	212,801.00
2002	176,000	126,000	So. Cal. Edison	8.250	7/1/2002	82,618	176,269.00
2003	144,000	105,000	Long Island Light	8.125	12/1/2003	62,174	144,874.00
2004	117,000	90,000	Florida P & L	8.500	1/1/2004	60,050	117,517.50
2005	93,000	73,000	Commonwealth Edison	8.750	3/1/2005	47,451	93,499.00
2006	73,000	56,000	Duke Power	8.375	10/1/2006	36,047	73,305.25
2007	56,000	46,000	Ontario Prov.	8.400	1/15/2007	31,037	56,683.25
2008	42,000	34,000	Central P & L	8.875	9/1/2008	23,514	42,751.25
2009	31,000	27,000	Ches. Pot. Tel.	8.875	6/1/2009	19,412	31,535.63
2010	23,000	21,000	Ohio Bell Tel.	8.750	1/1/2010	14,822	23,418.50
2011	16,000	16,000	Pacific G & E	9.375	2/1/2011	11,408	16,750.00
	$16,307,000					$6,908,017	$16,324,859.51

APPENDIX V

```
10 REM   ************   DURATION CALCULATION   ************
20 REM  THIS PROGRAM WAS WRITTEN IN "BASICA" ON AN IBM PC.
30 REM THIS PROGRAM WILL CALCULATE DURATIONS FOR A SERIES OF BONDS AT FIVE
40 REM YEAR MATURITY INTERVALS UP TO A SPECIFIED MAXIMUM--NN
50 REM WITH THE INPUT SPECIFIED FOLLOWING THE PROGRAM LISTING THE PROGRAM
60 REM WILL REPLICATE THE DURATION VALUES SHOWN IN TABLE B.
70 REM     VARIABLE      DESCRIPTION
80 REM     --------      -----------
90 REM        Y            YIELD TO MATURITY
100 REM      MAT           REDEMPTION VALUE
110 REM       CF           THE COUPON OR INTEREST RECEIVED EACH YEAR
120 REM                    AS IN THE PAPER IT IS ASSUMED THAT THE COUPON
130 REM                    OR INTEREST IS RECEIVED AT YEAR END.
140 REM       NN           NUMBER OF YEARS TO MATURITY
150 REM
160 REM
170 REM
180 REM
190 REM    PROMPT FOR INPUT OF DATA
200 REM
210 PRINT "INPUT YIELD (EG. 8% AS .08)";
220 INPUT Y
230 PRINT "INPUT REDEMPTION VALUE(EG. 1000)";
240 INPUT MAT
250 PRINT "INPUT COUPON (EG. 80)";
260 INPUT CF
270 PRINT "INPUT # OF YEARS TO MATURITY";
280 INPUT NN
290 REM
300 REM    PRINT REPORT HEADINGS
310 REM
320 FOR P=1 TO 5
330 PRINT
340 NEXT P
350 PRINT"     YIELD(%)  REDEMPTION($)  COUPON($)  MATURITY(YRS)  DURATION"
360 PRINT"     --------  -------------  ---------  -------------  --------"
370 REM
380 REM  DURATION CALCULATION
390 REM
400 FOR N=5 TO NN STEP 5
410 NUM=(MAT*N)/(1+Y)^N
420 DEN =MAT/(1+Y)^N
430 FOR T= 1 TO N
440 NUM =NUM +(CF*T)/(1+Y)^T
450 DEN= DEN + CF/(1+Y)^T
460 NEXT T
470 DUR = NUM/DEN
480 PRINT USING"      #.##          #,###       ###       ###          ###.####
";Y*100,MAT,CF,N,DUR
490 NEXT N
500 FOR P=1 TO 5
510 PRINT
520 NEXT P
530 END
```

```
RUN
INPUT YIELD (EG. 8% AS .08)? .08
INPUT REDEMPTION VALUE(EG. 1000)? 1000
INPUT COUPON (EG. 80)? 40
INPUT # OF YEARS TO MATURITY? 25
```

YIELD(%)	REDEMPTION($)	COUPON($)	MATURITY(YRS)	DURATION
8.00	1.000	40	5	4.5907
8.00	1.000	40	10	8.1184
8.00	1.000	40	15	10.6238
8.00	1.000	40	20	12.2635
8.00	1.000	40	25	13.2452

```
OK
```

```
RUN
INPUT YIELD (EG. 8% AS .08)? .08
INPUT REDEMPTION VALUE(EG. 1000)? 1000
INPUT COUPON (EG. 80)? 80
INPUT # OF YEARS TO MATURITY? 25
```

YIELD(%)	REDEMPTION($)	COUPON($)	MATURITY(YRS)	DURATION
8.00	1,000	80	5	4.3121
8.00	1,000	80	10	7.2469
8.00	1,000	80	15	9.2442
8.00	1,000	80	20	10.6036
8.00	1,000	80	25	11.5288

```
OK
```