REINSURING THE CAPTIVE/SPECIALTY COMPANY

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Abstract

This paper primarily discusses one quantitative excess of loss reinsurance pricing technique. European actuarial literature of the 1960's explores mathematical utility theory in the context of insurance. Recently, Freifelder and Cozzolino have written about exponential utility's value in pricing. This paper explores the relationship between wealth, reinsurance dollars and retention/cession. It is hoped that actuaries can supplement management judgment on cost effective reinsurance programs with analyses such as described here.

Introduction

Much has been written about reinsurance lately. The topic has scored highly in topics of current interest to actuaries, regulators, and others. The scope of this reinsurance paper is limited to selecting and pricing an excess of loss reinsurance coverage for a captive or specialty company. Many of these are single line insurers, so applying theory is simplified.

I intend to introduce risk theory but concentrate on utility theory concepts and applications. I believe utility theory presents an entire framework for risk-reward evaluation. A contract of reinsurance can be consummated only when an offer and acceptance has occurred. Since both parties to the contract have different and distinct expectations, each must be realistic in evaluating cost versus benefit. Utility theory allows for a mathematical treatment of the problem.

Reinsurance Programs

Virtually every insurance company must concern itself with the various forms of reinsurance that are available and the functions they perform. The establishment of a good reinsurance program is essential (a) to contain to a manageable level claim variance and (b) to reduce adverse effects on company growth and solvency caused by claim variance.

> It has been said that the object of reinsurance is, in the first place, to protect the direct writing company, the cedent, against payments of such claims as would threaten his solvency, and, secondly, to secure the cedent a result of his risk business ('*earnings*') as even as possible.[1] (Emphasis supplied.)

To purchase reinsurance economically means to select a form suitable to the needs of the company, with a retention high enough to control costs, yet low enough to minimize loss experience fluctuations over the years.

There are basically two types of treaty reinsurance: (a) pro rata or proportional reinsurance, which calls for the equal sharing of premiums and losses, and (b) excess of loss. Much reinsurance sold today is on an excess of loss form. Coverage can apply (a) per occurrence to an individual insured or (b) per event to a group of insureds. Event reinsurance is termed catastrophe reinsurance. Excess of loss can also be time dependent, as opposed to occurrence dependent. For example, aggregate or stop loss reinsurance is used to restrict total claims incurred for typically an annual period either (a) on a per risk basis or (b) for a collection of risks.

Excess of loss per risk or per occurrence reinsurance is very popular today. Coverage usually is divided into several layers. According to Reinarz,[2] layers are either "exposed" or "unexposed." An exposed or working layer is expected to have reasonably predictable frequency/severity characteristics. If a moderate sized hospital company issues $1 million policy limits and its appropriate retention is $250,000, the layer $250,000 xs 250,000 could be a working layer ("xs" means "in excess of a retention of"). This narrow layer with substantial premium per annum should be self-funding over a three-to-five-year time horizon according to reinsurance practice. A layer of $500,000 xs 500,000 also would be exposed since any single loss could attach, but the layer would not work as often. Presumably, there would not be enough premium in the second layer to sustain full layer losses (an unbalanced condition); hence, the reinsurer should have highly variable accident year results. This layer would be expected to be self funding over a much longer time horizon. Since chronological stabilization is more valuable here for the cedent (and riskier to the reinsurer), rates for this layer would include a higher profit and risk charge than for the layer $250,000 xs 250,000.

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[1] S. Bjerreskov, "On the Principles for the Choice of Reinsurance Method and for the Fixing of Net Retention for an Insurance Company," *International Congress of Actuaries*, 1954.

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[2] R. Reinarz, *Reference Book of Property and Liability Reinsurance Management*, Mission Publishing, Fullerton, Cal., 1969.

Of course, an employed physician with separate limits may have attended the claimant negligently while he was hospitalized. Although, in my example, an individual policy would not pay beyond $1 million, the claimant might recover $1.5 million because both policies would be expected to contribute. Reinsurance can protect against these multiple claims through a "clash cover." Two policies with losses from the same occurrence would be subject to one retention (e.g., $250,000).

Stability is enhanced as large losses are truncated, as far as the insurer is concerned, at a cost of modest premium outlay. Modest premium outlay is important in these days of high investment returns on funds withheld. The environment is one of knowledgeable buyer dealing with knowledgeable seller so transactions are free of rate and form regulation. This places great pressure on the negotiators to form an equitable alliance.

Reinsurance Loss Loadings

Reinsurance actuaries believe contracts exhibiting low risk should be priced at low expected reward, and conversely, high risk reinsurance should be priced with a high expected reward. When we divide the variance in a loss portfolio between insurer and reinsurer, we have a two-person game. More determined attempts to minimize variance on the retained portfolio concomitantly bring about more costly reinsurance. European actuarial literature discusses this.

Lambert[3] notes that the reinsurance loading generally increases according to the form of reinsurance—(a) pro rata, (b) excess of loss, and (c) stop loss or aggregate excess respectively. Vajda[4] demonstrates that for a given level of premium, the reinsurer's variance is minimized if the form is pro rata:quota share. Borch[5] notes that stop loss reinsurance minimizes the variance of the portfolio retained by the ceding insurer.

It is no wonder that most reinsurance sold for capacity, stability, and catastrophe protection today is of the excess of loss form. The form functions well and in an era where investment income on retained funds is extremely important, excess of loss reinsurance is in some sense optimal. Pro rata requires a large premium outlay. Stop loss reinsurance is heavily loaded for profit and contin-

[3] H. Lambert, "Contribution to the Study of . . . Collective Risk Theory" (French), *ASTIN Bulletin #2*, 1963.

[4] S. Vajda, "Minimum Variance Reinsurance," *ASTIN Bulletin #2*, 1963.

[5] K. Borch, "An Attempt to Determine the Optimum Amount of Stop Loss Reinsurance," *16th International Congress of Actuaries*, 1960.

gency. Furthermore, it does not return cash quickly.

Utility Theory

Very little has been written in the U.S. about the quantitative study of (a) relative costs of various reinsurance forms and (b) methods of establishing retentions. One text, however, by Reinarz,[6] illustrates several pragmatic approaches that can be taken. If the excess of loss form is chosen, a cost effective retention can be viewed in light of (a) the reinsurer's loss loading, (b) minimizing the variation in retained loss ratio, (c) reinsuring where claims frequency drops off, and others. These are judgmental approaches calling for the actuary or reinsurance purchaser to guess at relative effectiveness. Can the consequences of the decision be measured objectively in advance?

In European literature, beginning in the 1960's, we see risk theory being applied in the insurance context. Retention and reinsurance programs are selected to help constrain the probability of ruin. For large companies more concerned with stable earnings growth, a fraction of surplus can be placed at risk. Stockholder or policyholder (in a mutual company) disappointment will certainly precede financial ruin or insolvency.

For those interested in risk theory, I suggest reading Gerber,[7] Seal,[8] Bühlmann,[9] Philipson,[10] Wilhelmsen,[11] Bjerreskov,[12] Pentikäinen,[13] Wooddy,[14] and

[6] R. Reinarz, *op. cit.*

[7] H. Gerber, *An Introduction to Mathematical Risk Theory,* Huebner Monograph #8, Richard Irwin, Homewood, Ill., 1979.

[8] H. Seal, *Stochastic Theory of a Risk Business,* John Wiley & Sons, New York, N.Y., 1969.

[9] H. Bühlmann, *Mathematical Methods in Risk Theory,* Springer-Verlag, Berlin, 1970.

[10] C. Philipson, "A Review of the Collective Theory of Risk," supplement to *ASTIN Bulletin, Vol. V,* (from *Skandinavisk Aktuarietidskrift,* 1968).

[11] L. Wilhelmsen, "On the Stipulation of Maximum Net Retentions in Insurance Companies," *International Congress of Actuaries,* 1954.

[12] S. Bjerreskov, *op. cit.*

[13] T. Pentikäinen, "On the Reinsurance of an Insurance Company," *International Congress of Actuaries,* 1954; T. Pentikäinen, "Reserves of Motor-Vehicle Insurance in Finland," *ASTIN Bulletin,* 1962; T. Pentikäinen, "On the Reinsurance of an Insurance Company," *op. cit.*

[14] J. Wooddy, "Part 5 Study Notes—Risk Theory," Education and Examination Committee of the Society of Actuaries.

Beard et al.[15] The methods they note generally are complicated in theory, simplified in practice, and may not be as safety oriented as stated.

Game theory, developing at the same time, can be viewed in the insurance context.[16] Various players, employing competing strategies, obtain payoffs which they seek to maximize by some measure. Payoffs depend on each player's strategy but all strategies are interactive. The simplest is the two-person zero sum game where "my gain is your loss." Properly structured reinsurance programs can benefit both the seller and buyer. Reinsurance should be thought of as a partnership arrangement.

Traditional economic theory "at first glance" may not seem to apply to insurance. Businessmen seek to maximize profits. The purchase of insurance at a cost greater than expected losses is, therefore, an irrational business decision. The resulting reduction in profits is contrary to the businessman's primary motive. But Bernoulli[17] stated that a rational man does not seek to maximize gain but instead to maximize the expected *utility* of gain. Uncertainty creates anxiety. Supply and demand forces are altered. Current economic theory embraces utility theory.

Let us explore utility. Briefly, the utility of money, the value an individual places on an amount of money, varies depending on the individual's wealth. Different individuals view $1, $10, and $1000 differently. One thousand dollars to the beggar is worth substantially more than $1000 to the millionaire. To the beggar, it represents food, shelter, and warmth. To the millionaire, it may only cover repairs to his prestigious automobile.

Figure 1 illustrates one utility curve. Along the forty-five degree line each dollar is worth no less and no more than the previous one, an unrealistic situation. Instead, most likely, we should see a convex down curve. The value of additional dollars decreases generally over the length of the curve. There may be risk-taking sections of the curve, however, where we play unfair lotteries because of our aspirations. Siegel[18] writes about levels of aspiration.

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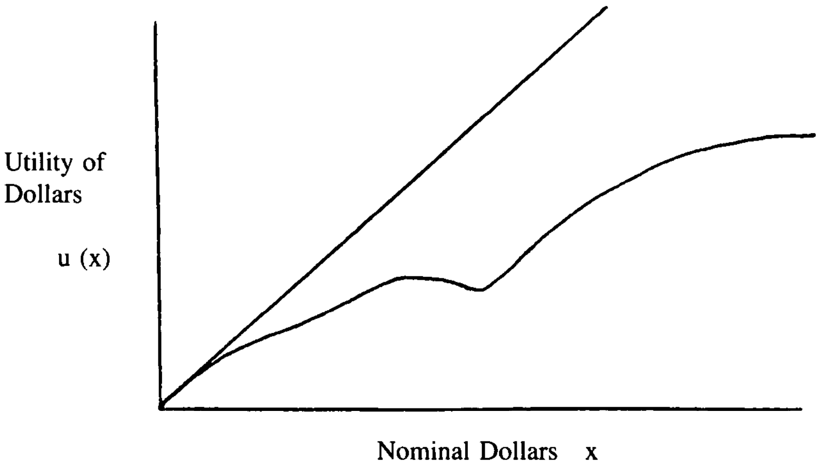
[15] R. Beard, T. Pentikäinen and E. Pesonen, *Risk Theory,* Methuen & Co., London, England, 1969.

[16] K. Borch, "Recent Developments in Economic Theory and Their Application to Insurance," *ASTIN Bulletin,* 1964.

[17] D. Bernoulli, "Exposition of a New Theory on the Measurement of Risk," translation of the original 1738 work, *Econometrica,* 1954.

[18] S. Siegel, "Level of Aspiration and Decision Making," *Psychological Review #64,* 1957.



We often see charities offering $10 tickets on a chance to win a new car. Although the game is unfair if ticket sales are brisk (the expected winnings are less than $10 per ticket sold), we might aspire to own that new car so we take a chance. The point is that the ticket price has lower utility than our aspiration to own the car. The equilibrium price or balance of indifference is what utility theory measures. In the case of insurance, how much premium is one willing to pay (the certain result) so as to escape an uncertain loss process? This is the mirror image of the car lottery example. In that example, you pay to gain (utility); for insurance, you pay not to lose (disutility).

Savage[19] gives an interesting history of utility and the papers written about it. Arrow[20] and Pratt[21] give accurate and meaningful interpretations of the concepts of risk aversion and risk preference.

It may appear that some insurers are nearly indifferent to risk. Only recently has the ISO varied profit and contingency loadings from the traditional 5%

[19] L. Savage, *The Foundations of Statistics*, John Wiley & Sons, New York, N.Y., 1954.

[20] K. Arrow, *Essays in the Theory of Risk Bearing*, Markham Publishing Co., Chicago, Ill., 1971.

[21] J. Pratt, "Risk Aversion in the Small and in the Large," *Econometrica #32*, 1964.

generally used. A $10,000 premium ($500 profit and contingencies loading) OL&T large risk at 25/75 limits was priced for profit and contingencies indifferently to a $10,000 neurosurgeon at $1/3 million limits.

Insurer underwriting practices reflect preferences. Certain insureds are desirable, as evidenced in Bailey's paper on "Skimming the Cream."[22] (Automobile classes weren't homogeneous.) Just as this example demonstrates risk preference, we see FAIR plans with loss-free insureds. Insurers obviously prefer not to insure these policyholders at the voluntary market price.

Utility theory is not abstract, incapable of practical use. Insurers can and do specify preferences. Utility theory quantitatively handles preferences.

The Utility Function

Figure 2 illustrates four families of utility functions.

Logarithmic utility was first suggested by Bernoulli.[23] It implies decreasing risk aversion. The family can be particularly useful for insurers if they become more risk prone or daring as they develop more wealth over time.

Quadratic utility also may be useful for insurance companies. Markowitz[24] shows that if a decision maker maximizes expected utility and always prices on a best mean-minimum variance principle, he will develop a Pareto-optimal portfolio. This occurs only if his utility function is quadratic. Borch[25] demonstrates that stop loss reinsurance should be preferred for insurers exhibiting quadratic utility toward risk.

Quadratic utility curves have two drawbacks, however. First, the curves only increase up to a wealth level of $b/2$ (see Figure 2). Second, it can be demonstrated that these curves imply an increasing aversion to risk as wealth increases. So the larger the insurer gets, the more likely he will raise prices and reinsure more of his business. In my experience, insurers do not behave in this manner; thus, quadratic utility curves are not very useful for insurance companies.

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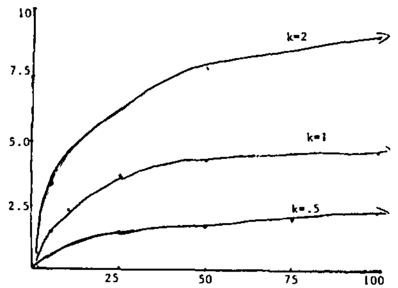
[22] R. Bailey, "Any Room Left for Skimming the Cream?" *PCAS*, XLVII, 1960.

[23] D. Bernoulli, *op. cit.*

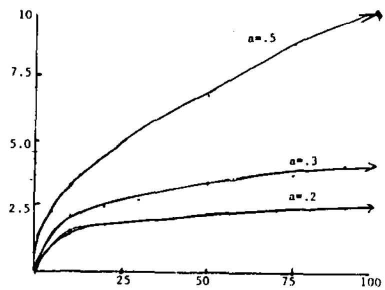
[24] H. Markowitz, *Portfolio Selection*, John Wiley & Sons, New York, N.Y., 1959.

[25] K. Borch, *op. cit.*

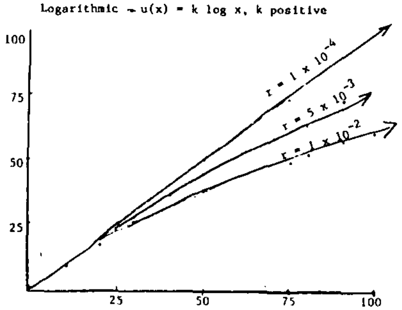
FOUR FAMILIES OF UTILITY FUNCTIONS



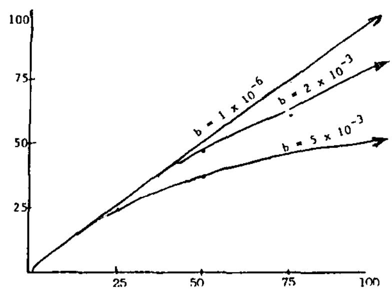
Logarithmic – $u(x) = k \log x$, k positive



Fractional Power – $u(x) = x^a$, x positive, a between--(0,1)



Exponential – $u(x) = 1/r\ (1-e^{-rx})$, r positive



Quadratic – $u(x) = x-bx^2$, b positive, x less than ½b

A third family of utility functions is termed exponential. Gould[26] shows that consumers choose deductibles consistent with those that exponential utility dictates. Shpilberg and DeNeufville[27] note that results are not particularly sensitive to the family of utility functions used; so, since the exponential is easiest to work with, use it. Cozzolino[28] and Freifelder[29] have developed ratemaking models relying exclusively on exponential utility.

The fourth family is termed fractional power.

Until recently insurance was largely priced on expected value (after expenses). Depending on the insurer, a level of underwriting return or total return was targeted. This implied the utility function $u(x) = x$, the forty-five degree line from Figure 1. (This linear function is a special case of the exponential utility family where, as r approaches zero, the quantity $(1/r) \times (1 - \exp(-rx))$ approaches x. An r of 0 would mean no risk aversion, or indifference.)

Utility functions generally are concave down in the first quadrant. "More is better" so the curve is increasing, but the rate of climb slows since added dollars are worth slightly less than prior dollars. Mathematically, the first derivative $u'(x)$ is greater than zero, but the second derivative $u''(x)$ is negative. If we calculate $-u''(x)/u'(x)$ as an index of risk preference, then only for the exponential family does everything cancel, and we are left with r: constant risk aversion. Wealth is immaterial. The reader can verify that logarithmic utility has decreasing risk aversion with wealth.

The constant risk aversion of the exponential family makes pricing a multiplicity of insureds over time easier. Decisions can be made independent of order or time. Other families of functions rely on wealth for pricing purposes, and all decisions must be made in light of others. The exponential function is both clean and aesthetically appealing.

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[26] J. Gould, "The Expected Utility Hypothesis and the Selection of Optimal Deductible for a Given Insurance Policy," *Journal of Business*, April, 1969.

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[27] D. Shpilberg and R. DeNeufville, "Use of Decision Analysis for Optimizing Choice of Fire Protection and Insurance: An Airport Study," *Journal of Risk and Insurance*, College of Business, University of Georgia, Athens, March, 1975.

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[28] J. Cozzolino, "A Method for the Evaluation of Retained Risk," *Journal of Risk and Insurance XLV #3*, College of Business, University of Georgia, Athens, 1978.

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[29] L. Freifelder, *A Decision Theoretic Approach to Insurance Ratemaking*, Huebner Monograph #4, Richard Irwin, Homewood, Ill., 1976.

We could assume that for the long run r can vary; call it a different r each year. Then, exponential utility can displace logarithmic utility's prime appeal. Risk aversion could decline periodically with increasing wealth with no complications in application.

One final and most important item. Assuming exponential utility and given particular reinsurance terms for a book of business, an insurer can determine an indifference price such that the insurer does not care whether it is ceding the business or keeping it. The cedent must be willing to make a fair offer of reinsurance to the reinsurer. The striking price for reinsurance can be determined using exponential utility theory as a guide. The retention can be the most cost effective one of a group tested.

The Mathematics of Utility

Suppose an insured with wealth a is given a choice of self-insuring completely a loss process X or paying a gross premium G for full coverage. Assume the insured has a linear utility attitude so that $u(x) = bx + d$.

To determine G, we solve the general equation $u(a - G) = E(u(a - X))$. The utility of net wealth after insurance must equate to the expectation of the utility of wealth without insurance. From our expression $bx + d$, we substitute $a - G$ and $a - X$ respectively for x, and get:

$$\begin{aligned} b(a - G) + d &= E(b(a - X) + d) \\ &= b(a - E(X)) + d \\ &= b(a - m) + d, \text{ where } E(X) = m, \text{ the mean expected losses} \\ G &= m \end{aligned}$$

Recall I said linear utility implied risk indifference. In this case an insured would pay no more than expected losses to relieve himself of the uncertain loss process.

Now suppose the insured's utility function is exponential so $u(x) = (1/r)(1 - \exp(-rx))$. Let us modify this somewhat. Let us make the process X negative so the function relates to losses. Let us also negate the entire expression and speak of the disutility (Du) of losses (See Cozzolino). In this case, G is given by:

$$\begin{aligned} Du\,(a - G) &= E(u(a - X)) \\ -(1/r)\,(1 - \exp(r(a - G))) &= E(-(1/r)\,(1 - \exp(r(a - X)))) \\ &= -(1/r)\,(1 - E\,(\exp(r(a - X)))) \\ \exp(r(a - G)) &= E(\exp(r(a - X))) \end{aligned}$$

$$\begin{aligned} \exp(-rG) &= E(\exp(-rX)) \\ G &= -(1/r)\ln E(\exp(-rX)) \\ G &= (1/r)\ln E(\exp(rX)) \quad \text{translated back!} \end{aligned}$$

To make this arithmetically workable, we can take the claim size distribution and separate it into n partitions, if necessary, each with probability p_i. Then if we assume a uniform distribution over the interval (x_i, x_{i+1}), the risk-adjusted severity is given by the following formula:

$$G = \frac{1}{r}\ln\left[\sum_{i=1}^{n} \frac{p_i}{x_{i+1} - x_i} \cdot \frac{(\exp(r^{x_{i+1}}) - \exp(r^{x_i}))}{r}\right]$$

It is now only necessary to bring in the frequency distribution. Let k represent the number of claims. Then the risk premium adjusted for frequency and severity equals:

$$G' = \frac{1}{r}\ln\left(\sum_{k=0}^{\infty} p(k)\exp(krG)\right)$$

In the case where frequency is Poisson distributed with parameter k, we have $G' = (k/r)(\exp(rG) - 1)$. If frequency is distributed according to the negative binomial with parameters p and b, (mean $b(1 - p)/p$, variance $b(1 - p)/p^2$) then $G' = (b/r)\ln(p/(1 - (1 - p)\exp(rG)))$.

At this point an illustration is in order. Suppose a property owner has a utility function $u(x) = \exp(-.005x)$. Further suppose there is a 1 in 10 chance of a property loss whose distribution is $f(x) = .10\ (.01\exp(-.01x))$. Then expected loss is given by:

$$E(X) = (.90)(0) + .10\int_0^\infty x\,(.01\exp(-.01x))\,dx = 10$$

Risk-adjusted premium, G' is given by:

$$\begin{aligned} u(a - G') &= .90\,u(a) + \textstyle\int_0^\infty u(a - x)\,f(x)dx \\ -\exp(-.005(a - G')) &= -.90\exp(-.005a) \\ &\quad - .10\textstyle\int_0^\infty \exp(-.005(a - x))\,(.01\exp(-.01x))dx \\ \exp .005G' &= .90 + (.10)\,(2) \\ G' &= 200\ln(1.10) \\ G' &= 19.06 \end{aligned}$$

The insured is willing to pay almost double expected losses because of the danger in the frequency/severity distributions coupled with his risk averseness.

A Test Case

Assume a hospital company writes only policy limits of \$5 million. According to recent ISO increased limit studies, losses can be modeled by a shifted Pareto distribution.[30]

The following chart provides a representative example of average severities:

Policy Limit	Average Loss	Average Allocated Loss Expense	Sum
$ 250,000	$ 54,402	$15,000	$ 69,402
$5,000,000	$112,227	$15,000	$127,227

Further, assume a claim frequency of .006 against 16,667 occupied beds, producing 100 expected claims. If acquisition; general expenses; taxes, licenses, fees; and profit amount to 25%, premium volume at $5 million limits should be:

100 (127,227)/.75 = $16,963,600

Expected losses in the $4,750,000 xs 250,000 layer (excluding pro rata allocated loss adjustment expenses) equal:

100 (127,227 − 69,402) = $5,782,500 and

divided by the 10 claims over $250,000 implied by the shifted Pareto, yields an average loss in this layer of $578,250.

Now let us view the reinsurer's loss distribution. If we move the y-axis of the gross loss distribution over to the right to $250,000; we have a decreasing reinsurance loss function defined on the interval (0; $4,750,000). Let us assume it is nearly exponential. (For ease in calculus the tail is included.)

A characteristic of the exponential is that the mean, $578,250 here, is the reciprocal of the value r, so $r = 1.729 \times 10^{-6}$. The loss function is then given by:

$$f(x) = .10\ (.000001729 \exp(-.000001729x));\ x \text{ positive.}$$

Mean losses are given by:

$$E(X) = 100((.90)(0) + .10 \int_0^{4,750,000} .000001729 \exp(-.000001729\, x)dx)$$

$$= 100\ (0 + (.10)\ (578,250))$$

$$= \$5.782,500$$

[30] Insurance Services Office, "Report of the Increased Limits Subcommittee: A Review of Increased Limits Ratemaking," 1980.

Suppose the insurer has a utility function given by

$$u(x) = -\exp(-.00000025x)$$

Then,

$$-\exp(-.00000025\,(a - (G'/100))) = .90\,u(a) +$$

$$\int_0^{4,750,000} \exp(-.00000025(a - x))\,(.10)\,(.000001729\,\exp(-.000001729))dx$$

Dividing through by $u(a)$ gives

$$\exp(.00000025\,(G'/100))$$

$$= .90 + .10\,(.000001729) \int_0^{4,750,000} \exp((.00000025 - .000001729)x)dx$$

$$= .90 + .10(.000001729)\,(675,000)$$

$$= 1.0167$$

Finally, we have

$$\frac{G'}{100} = \frac{\ln(1.0167)}{.00000025} = 66,248$$

or

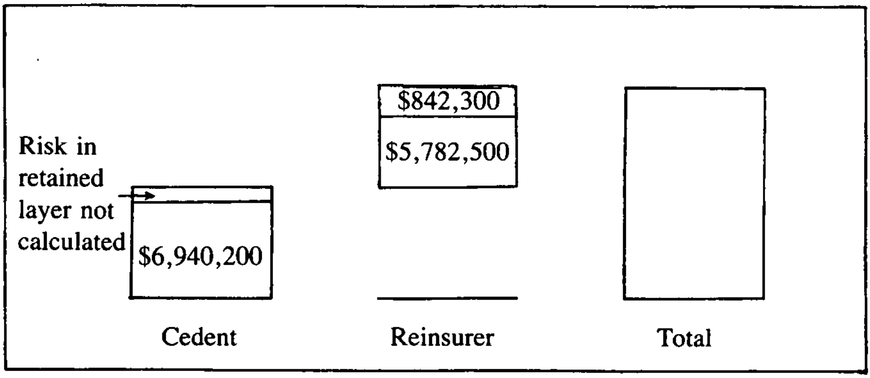
$$G' = \$6,624,800$$

(The appendix gives the framework of a more complete mathematical/statistical analysis.)

In this example, the reinsured should be willing to cede $4,750,000 xs 250,000 for $6,624,800 − 5,782,500 or $842,300 more than expected losses. If the reinsurer has the same utility function or is less risk averse, a deal can be struck. The reinsurer might express this as $16,963,600 (57,825/127,227) = $7,710,000 less a 14% ceding commission or $6,630,600. The $842,300 loading would have to cover all reinsurer operating expenses, including service, and a profit/risk charge.

This retention pricing example probably is not optimal. Other retention levels should be studied. Diagramatically, Figure 3 shows the first attempt loss costs.

An approach would be to minimize the sum of "Total" subject to a restriction on the risk proneness of the ceding company and a reasonably risk averse function for the reinsurer.



Each layer of loss by size will have certain frequencies. The insurer will calculate his risk load for each layer. The reinsurer, viewing the same data, may have similar pure premiums, but in any event, will also calculate risk loads. Depending on the optimism and risk proneness of the reinsurer, the ceding company may find layers cheaper (in terms of utility) to cede than retain. Other considerations will impact the purchased retention level (beyond the scope of this paper). Neither party has to know or attempt to negotiate the other's risk propensity. The bottom line will determine whether reinsurance is purchased, and at what level.

"r" Values

The question always rises, "How can management specify their risk aversion function?" Kalcek and McIntyre[31] begin to explore this. They suggest risk capital can be determined as: (a) 1 to 5% of annual working capital, (b) 1 to 3% of total assets, (c) 3 to 5% of annual earnings, or (d) 0.1 to 0.5% of annual sales. The rules of thumb come from the manufacturing environment; insurers might substitute cash on hand and cash flow for working capital. Other measures could be invented. Suppose we set a value on risk capital of x. We have a desire to

[31] K. Kalcek and W. McIntyre, *Financial Executive*, April, 1977.

bet x in annual adverse claim variability, but only lose it with small probability. Small companies must be aggressive with x as a percent of base capitalization, large companies would tend to select at range minimums. Risk capital may be defined loosely as an amount of money an insurer is prepared to lose in the case of unusual adverse claim variability.

From our exponential disutility (*Du*) function, we can take the first derivative. The slope constantly increases. Suppose we set our risk-reward level at 10:1. One dollar is worth the risk of 10. By analogy, horse race handicappers don't generally bet on "sure things." If a win ticket for \$2 will pay \$2.20, they won't bet \$10 to win \$11. The risk-reward is judgmentally poor.

Suppose, for a small company, x must be \$1 million. We can then calculate the r value.

$$Du(x) = -(1/r)\,(1-\exp(-rx))$$

$$\frac{d}{dx}\,Du(x) = -\exp\,(-rx)$$

$$10 = -\exp\,(-1{,}000{,}000\,r)$$

$$(\ln 10)/1{,}000{,}000 = r$$

$$r = .000002$$

We can also use a polling technique. By interviewing management, we can determine risk propensity. Ask what premium management would charge for several loss/no-loss situations, then graph expected payoffs (abscissa) against premium (ordinate). For example, "How much would you pay for a lottery ticket with a .001 chance of winning \$1 million?" Although the expected value is \$1000, the risk avoider might pay only \$500. If the question were asked, however, in the disutility context where there is a .001 chance of losing \$1 million, he might say \$1500. By getting premiums for a wide variety of expected payoffs, utility or disutility curves can be constructed.

A third method for determining r is to perform price and resulting earnings studies based on a variety of r values near zero. A company's earnings target, coupled with a business mix, can lead to an implied r value.

Risk Assessment

Once an appropriate excess of loss retention is determined, underwriters and actuaries can meet to discuss pricing techniques. Proposed treaty rates must be assessed both analytically and judgmentally. The pricing method previously described is completely analytic once utility is specified. This price indication can be compared with both empirical and exposure methods. Empirically, the company would have a history of observed losses per exposure unit by layer (after trend and development/IBNR). The gross price for insurance can also be layered by exposure. National Council ELPF's, ISO increased limit factors, and property distributions such as published by Salzmann[32] are useful.

If no credible past data exist, reinsurance collective experience and judgment are used to rate the account. It is of great benefit to reinsure or quote on many state doctor and hospital companies. Each lacks complete credibility, but collective experience fills whatever gaps exist. When a totally new risk presents itself, such as in 1973 New Jersey no-fault excess of loss coverage, reinsurers price by analogy. This no-fault should be similar to a combination of first party long-tail workers' compensation and automobile liability/medical payments.

Only after actuarial, claims, and underwriting personnel have evaluated the company's experience does a responsible quotation emerge.

Conclusion

It is not surprising that reinsurance has received little mathematical attention until lately in the *Proceedings*. Until recently, there have been but a handful of actuarial practitioners in the field. Mathematical and statistical tools, such as utility theory, were not studied in the U.S. for application to reinsurance. Utility theory, I believe, is a key to understanding which reinsurance forms make sense and what retentions are desirable.

Throughout history, reinsurance has operated along traditional lines. Excess of loss reinsurance is very popular today. The burning question is, "What retention is appropriate for my business and how much should reinsurance be worth to me?" This essay primarily attempts to seek an analytical solution to an otherwise judgmental decision. (Two examples were given, an individual property risk and a portfolio of hospital bed exposures.) By setting limits on retained loss variability (as measured by utility) a natural consequence is excess cession, and furthermore excess pure loss cost and risk charge. No attempt has been made to define a corporate utility function but several curves have been noted and insight given in how to interpret and use them.

My thanks go to William Weimer for extending my example. He eliminated the constraint that reinsurance frequency of loss be constant and I am grateful to him for the mathematics expressed in the appendix.

[32] R. Salzmann, "Rating by Layer of Insurance," *PCAS* L, 1963.

APPENDIX—FREQUENCY IN UTILITY CALCULATIONS

We can formalize the mathematical structure of the hospital example stated earlier. Specifically, we can eliminate the assumption of a constant number of excess $250,000 claims. Our choice of a Poisson frequency distribution will provide an elegant path to follow. In a collective risk theory framework, this will be a derivation using a particular frequency distribution and a particular severity distribution. We hope that after reviewing this example, the reader will gain more insight into the general formulas stated in the mathematical section and will be able to apply them with distributions of his or her choice.

We make the following assumptions:

Total losses (each is the excess of $250,000 portion)

$$X = X_1 + X_2 + \cdots + X_N$$

Frequency of claims distribution: Poisson (h) with $h = 10$

$$P(N = n) = \exp(-h)\, h^n/n!;\ n = 0,1,2, \ldots$$

Severity of claims distribution: Exponential with mean = $5,782,500

$$f(x) = s \exp(-sx);\ x > 0 \text{ and } s = .000\,001\,729$$

Utility function:

$$u(x) = -\exp(-rx);\ r = .000\,000\,25$$

Initial Net Worth = a

With these assumptions, the hospital company should be willing to pay an amount G' for a $250,000 excess of loss cover, where G' satisfies the equation:

$$u(a - G') = E(u(a - X)).$$

The "no memory" property of $u(x)$ leaves us with:

$$\exp(rG') = E(\exp(rX))$$

$$= E(E(\exp(rX)/N))$$

$$= \sum_n P(N = n)\, E(\exp(r(X_1 + X_2 + \cdots + X_n)))$$

$$= \sum_n P(N = n)\, (s/(s - r))^n$$

$$= \sum_n (\exp(-h)h^n/n!)\,(s/(s-r))^n$$

$$= \exp(-h) \sum_n (hs/(s-r))^n/n!$$

Solving for G' gives $G' = (h/r)\,((s/(s-r)) - 1)$.
Replacing h, s, and r with their selected values leaves $G' = \$6{,}761{,}325$.

We see that by letting the frequency vary, we are introducing more uncertainty into our problem, and the premium G' has gone up from \$6,624,800. (Actually, some of the increase, \$41,578, is due to the severity distribution no longer being truncated.)