A NOTE ON LOSS DISTRIBUTIONS

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Abstract

This paper presents a generalized notation in order to represent several actuarial rating values which are derived from loss distributions. Four functions are defined and then used to define various rating values such as Table M charges and savings, loss elimination ratios, increased limit factors, and excess loss premium factors. The notation has been adapted from a notation originally presented by R. J. Finger. Using this manner of presentation, a more unified approach to actuarial uses of loss distributions is possible. The paper should be of particular value to students of the Society.

I. INTRODUCTION

The topic of loss distributions has been and continues to be an important area of actuarial study. Many papers have been presented to this Society which discuss various actuarial applications utilizing loss distributions. Some of these papers appear on the CAS examination syllabus. In addition, the Actuarial Education and Research Foundation is currently preparing a textbook on loss distributions. It is the purpose of this paper to define some elementary functions which utilize an underlying loss distribution and then use these functions to generalize the derivation of several actuarial rating values. Using this manner of presentation, a more unified approach to actuarial uses of loss distributions is possible.

The term loss distribution is intended to be a general term. It could represent a per claimant loss distribution, a per occurrence loss distribution, a per risk annual loss ratio distribution, etc. The generality and wide application of the elementary functions result, in part, from the variety of types of specific loss distributions and probability models which could be considered in various areas of ratemaking. It should be noted that the functions presented are "distribution-free" in the sense that no particular probability law is assumed. In this paper we will use the terms "claim" and "loss" interchangeably.

II. ELEMENTARY FUNCTIONS

In this section, we give definitions for four elementary functions which utilize an underlying loss distribution and which will be used throughout the paper. We will use t to denote a loss variable and $f(t)$ to represent the probability density function (p.d.f.) of t. The domain of the functions is the non-negative real numbers and their range is the closed unit interval. In this paper we will use only continuous random variables; however, the discrete case is easily substituted. We now proceed with our definitions.

1. *Cumulative Distribution Function*
   This function represents the probability that a given loss size will be less than or equal to x.

$$F(x) = \int_o^x f(t)\, dt$$

2. *Basic Loss Function*
   This function represents the percentage of total losses generated by all claims which are smaller than some specified value x.

$$X1(x) = \frac{1}{\alpha}\int_o^x t\, dF(t),$$

$$\text{where} \quad \alpha = \int_o^\infty t\, dF(t) = \text{mean of the distribution.}$$

3. *Primary Loss Function*
This function represents the percentage of total losses generated by the aggregate amount of the first x dollars of each claim (the whole claim amount, if less than or equal to x).

$$X2(x) = \frac{1}{\alpha}\int_o^x t\, dF(t) + \frac{x}{\alpha}\int_x^\infty dF(t)$$

$$= X1(x) + \frac{x}{\alpha}[1 - F(x)]$$

4. *Excess Loss Function*
This function represents the percentage of total losses generated by the aggregate amount of the dollars of loss which exceed x per claim.

$$X3(x) = \frac{1}{\alpha}\int_x^\infty (t - x)\, dF(t)$$

$$= 1 - X1(x) - \frac{x}{\alpha}[1 - F(x)]$$

$$= 1 - X2(x)$$

III. FREQUENCY, SEVERITY, AND PURE PREMIUM

We would like to have an expression for the pure premium and its components in terms of the elementary functions. But first we need to make the following definitions.

R = retention (or deductible) amount
$E[n]$ = zero retention (or full coverage) frequency. Bickerstaff[1] calls this "absolute" frequency.
$p(R)$ = pure premium at retention level R

[1] D. R. Bickerstaff, "Automobile Collision Deductible and Repair Cost Groups: The Lognormal Model," *PCAS* LIX (1972), p. 68.

$$g(R) = \text{frequency at retention level } R = [1 - F(R)] \cdot E[n]$$
$$s(R) = \text{severity at retention level } R = \alpha \cdot X3(R) \,/\, [1 - F(R)]$$

Then,

$$\begin{aligned} p(R) &= g(R) \cdot s(R) \\ &= [(1 - F(R)) \cdot E[n]] \cdot [\alpha \cdot X3(R) \,/\, (1 - F(R))] \\ &= E[n] \cdot \alpha \cdot X3(R) \end{aligned}$$

We can define expected excess and expected primary losses as follows:

$$\begin{aligned} \text{expected primary losses} &= Ep = u \cdot p(0) - u \cdot p(R) = u \cdot E[n] \cdot \alpha \cdot X2(R) \\ \text{expected excess losses} &= Ee = u \cdot p(R) = u \cdot E[n] \cdot \alpha \cdot X3(R) \\ \text{expected losses} &= E = Ep + Ee = u \cdot p(0) = u \cdot E[n] \cdot \alpha \end{aligned}$$

where u = number of exposure units.

With these definitions and those of the preceding section, we are now ready to discuss some specific applications.

IV. LOSS ELIMINATION RATIOS

The Straight Deductible Loss Elimination Ratio

In his paper on automobile collision deductibles, Bickerstaff[2] defines a "first-dollar" loss elimination ratio (LER) as follows:

$$\begin{aligned} LER(D) &= \frac{D \cdot G(D) + \alpha \cdot H(D)}{\alpha} \\ &= H(D) + (D/\alpha)G(D) \\ &= (1/\alpha) \int_o^D t\, dF(t) + (D/\alpha)[1 - \int_o^D dF(t)] \\ &= X1(D) + (D/\alpha)[1 - F(D)] \\ &= X2(D) \end{aligned}$$

Bickerstaff takes $f(t)$ to be the lognormal p.d.f. and goes on to show that an adjustment to the loss cost, α, must be made in order to reflect an upper bound to the unlimited lognormal distribution. This is in recognition of the

[2] *ibid.*

practical fact that there exists a finite dollar bound, L, on the actual cash value of a vehicle. This adjustment can be calculated as:

$$\text{“adjustment”} = \frac{\alpha \cdot J(L) - L \cdot G(L)}{\alpha}$$

$$= J(L) - (L/\alpha)[1 - F(L)]$$

$$= 1 - X1(L) - (L/\alpha)[1 - F(L)]$$

$$= X3(L)$$

We can now compute the net cost per claim ($NCPC$) for a given deductible D as follows:

$$\begin{aligned} NCPC(D,L) &= \alpha - \alpha \cdot X2(D) - \alpha \cdot X3(L) \\ &= \alpha[1 - X2(D) - X3(L)] \\ &= \alpha[X3(D) - X3(L)] \end{aligned}$$

(Note that $NCPC$ does not equal severity, as defined in the previous section.)

If we expand this formula and make the modifications Bickerstaff suggests, we can obtain a "complete" formula for net loss cost (i.e., pure premium).

$$\begin{aligned} NCPC(D,L) &= \alpha - \alpha \cdot X2(D) - \alpha \cdot X3(L) \\ &= \alpha - \alpha \cdot X1(D) - D[1 - F(D)] - \alpha[1 - X1(L)] \\ &\quad + L[1 - F(L)] \end{aligned}$$

We now substitute $\alpha(1 + r)^{n-1}$ for α, Ld^{n-1} for L, and multiply by AC_n to obtain the formula for net loss cost,

$$AC_n\{\alpha(1 + r)^{n-1} - \alpha(1 + r)^{n-1} X1(D) - D[1 - F(D)] - \alpha(1 + r)^{n-1} \\ [1 - X1(Ld^{n-1})] + Ld^{n-1}[1 - F(Ld^{n-1})]\},$$

which the reader can verify is equivalent to the Bickerstaff formula.

Snader[3] gives a discrete formula for the straight deductible LER which can be generalized to $X2(D)$. This is straightforward and is left to the reader.

The Franchise Deductible Loss Elimination Ratio

The franchise deductible requires the insured to pay for losses less than or equal to the deductible amount, but when a loss exceeds the deductible, the

[3] R. H. Snader, "Fundamentals of Individual Risk Rating and Related Topics," CAS Study Note, Part III, p. 60.

loss is paid in full. The formula for the loss elimination ratio is:

$$LER(D) = \frac{1}{\alpha} \int_o^D t \, dF(t)$$

$$= X1(D).$$

The Disappearing Deductible Loss Elimination Ratio

The discrete formula for this type of deductible is given by Snader.[4] Since the derivation of the equivalent form in terms of the elementary functions is rather cumbersome, only the formula will be given. This type of deductible is a straight deductible up to losses of amount D, there is a decreasing amount of deductible from D to an amount A (at which $D = 0$), and no deductible for losses in excess of A.

$$LER\ (D;A) = X1(A) - \frac{A}{A - D} [X1(A) - X1(D)] + \frac{A}{A - D} (D/\alpha)[F(A) - F(D)]$$

V. EXPERIENCE RATING

D *Ratios*

Bailey[5] tells us that "any experience rating plan which uses a loss limitation must cope with D ratios." These ratios are necessary in order to split expected losses into expected primary and expected excess losses. However, there are several types of loss limitation which can be used to emphasize frequency of loss. One type is illustrated by the maximum single loss (*MSL*) limitation used in general liability.[6] In this case we have:

$$\begin{aligned} D \text{ ratio} &= Ep/E \\ &= [u \cdot E[n] \cdot \alpha \cdot X2(M)]/[u \cdot E[n] \cdot \alpha] \\ &= X2(M) \end{aligned}$$

where M = maximum single loss limitation.

[4] *ibid.*, Part III, p. 61.

[5] R. A. Bailey, "Experience Rating Reassessed," *PCAS* XLVIII (1961), p. 60.

[6] G. N. Alff, "Liability Experience Rating: Concepts and Structure," CPCU Journal, March, 1979, p. 44.

In workers' compensation, individual losses are split into primary and excess portions by the use of a formula in conjunction with additional dollar limits on multiple claimant cases, disease cases, etc. Currently, the formula primary portion of a loss is dependent on the size of the ground-up loss and, hence, is variable. This situation is similar to the deductible provisions of several of the crop-hail insurance policy forms.[7] In these cases, the formula given previously for Ep will not hold and, hence, we cannot obtain $X2(M)$ as a representation of the D ratio. Thus, this formula is not applicable when discussing workers' compensation experience rating, or any other plan not using a constant loss limitation, but it can be helpful in those plans which do use a constant loss limitation. This will also be the case with the excess ratio which we will discuss next.

Excess Ratios

In his paper on experience rating credibilities, Perryman[8] defines the excess ratio, r, to be the ratio of expected excess losses to expected losses. Hence,

$$\begin{aligned} r &= Ee/E \\ &= [u \cdot E[n] \cdot \alpha \cdot X3(M)]/[u \cdot E[n] \cdot \alpha] \\ &= X3(M) \end{aligned}$$

It should be emphasized that this formula is only valid when losses are limited by a constant amount such as an MSL limitation (see previous section). Since the excess ratio plays a part in two of Perryman's credibility formulae, we can see that, all other things being equal, the same forces which impact the loss distributions will also affect credibility values based on these formulae.

Values of g

These values are of more historical than practical interest; however, readings are currently on the examination syllabus which discuss the concept of a g value. The necessity for a g value arises from the possibility that primary credibility may exceed unity for sufficiently large values of the excess ratio, r, under Perryman's Formula II. By substituting $K_E = K \cdot (1 - W) + WgS$ for K when $Q \leq E$, we guarantee that primary credibility will not exceed unity.

[7] R. J. Roth, "The Rating of Crop-Hail Insurance," *PCAS* XLVII (1960), p. 108.

[8] F. S. Perryman, "Experience Rating Plan Credibilities," *PCAS* LVIII (1971), p. 143.

Perryman[9] defines g as:

$$g = \max \{r\}$$
$$= \max \{X3(M)\}$$

where r varies by classification.

Since g is a function of the excess ratio, this formula is valid only for constant amount loss limitations. It should be clear that, for a fixed M, the values of $X3(M)$ and hence g will increase under inflation and must be adjusted to reflect current conditions. Uhthoff[10] gives a good discussion of the impact of inflation upon these values and the implications of a failure to adjust certain experience rating values under changing conditions.

VI. RETROSPECTIVE RATING

Table M Charge

Snader[11] defines the "charge" (or excess pure premium ratio) at entry ratio r to be:

$$\phi(r) = \int_r^\infty (t - r)dF(t) / \int_o^\infty t\, dF(t)$$

$$= (1/\alpha) \int_r^\infty (t - r)dF(t)$$

$$= X3(r)$$

where r = entry ratio
= actual losses ÷ expected losses
= actual loss ratio ÷ expected loss ratio.

Since the Table M charge is based on a ratio to expected losses, we must multiply by the permissible loss ratio, E', to obtain a ratio to (standard) premium. Thus, the percentage charge (applicable to standard premium) for a maximum loss ratio is $E' \cdot X3(r)$ (exclusive of loss adjustment expenses).

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[9] *ibid.*

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[10] D. R. Uhthoff, "The Compensation Experience Rating Plan—A Current Review," *PCAS* XLVI (1959), p. 285.

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[11] Snader, *op. cit.*, Part II, p. 52.

Table M Saving

Snader[12] defines the "saving" at entry ratio r to be:

$$\begin{aligned}\psi(r) &= \int_o^r (r - t)dF(t) / \int_o^\infty t\,dF(t)\\ &= (r/\alpha)\int_o^r dF(t) - (1/\alpha)\int_o^r t\,dF(t)\\ &= (r/\alpha)F(r) - X1(r)\\ &= 1 - X1(r) - (r/\alpha)[1 - F(r)] + (r/\alpha) - 1\\ &= X3(r) + (r/\alpha) - 1\\ &= \phi(r) + (r/\alpha) - 1\end{aligned}$$

If $\alpha = 1$, then we obtain an important relationship between the charge and saving, namely,

$$\psi(r) = \phi(r) + r - 1$$

Excess Loss Ratio

Snader[13] defines the excess loss ratio for a given injury type and loss limitation, l, as follows:

$$\begin{aligned}e^*(l) &= y - r^*x\\ &= (1/\alpha)\int_l^\infty t\,dF(t) - (l/\alpha)\int_l^\infty dF(t)\\ &= (1/\alpha)\int_l^\infty (t - l)\,dF(t)\\ &= X3(l)\end{aligned}$$

where, $y = (1/\alpha)\int_l^\infty t\,dF(t)$

$$r^* = l/\alpha$$

$$x = \int_l^\infty dF(t).$$

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[12] *ibid.*, Part II, p. 54.

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[13] *ibid.*, Part II, p. 55.

Harwayne[14] describes a method of obtaining countrywide excess loss ratios using statewide tables of excess loss ratios based on ratios to the mean. If we let $r = l/\alpha$, then we can substitute r for l and obtain:

$$\begin{aligned} e^*(l) &= e^*(r) \\ &= X3(r) \end{aligned}$$

It should be noted that Skurnick[15] calls the excess loss ratio a loss elimination ratio (denoted k). This should not be confused with a deductible loss elimination ratio which is the complement of the excess loss ratio.

Excess Loss Premium Factor

We can now obtain the excess loss premium factor (*ELPF*) for a dollar loss limitation per claim under a retrospective rating plan (net of any expense items). Since $X3(l)$ is a ratio of excess losses to expected losses, we can transform this into a ratio of excess losses to premium by multiplying by the permissible loss ratio, E'. Hence,

$$\begin{aligned} ELPF &= E' \cdot X3(l) \\ &= E' \cdot X3(r) \end{aligned}$$

where $r = l/\alpha$.

VII. REINSURANCE

Excess of Loss Coverage

The term burning ratio (*BR*) could be used to describe the ratio of expected excess losses to expected losses. This can be written as:

$$\begin{aligned} BR &= Ee/E \\ &= [u \cdot E[n] \cdot \alpha \cdot X3(R)]/[u \cdot E[n] \cdot \alpha] \\ &= X3(R) \end{aligned}$$

In order to apply this ratio to subject premium we must multiply by the permissible loss ratio, E', underlying the primary rates. Compare this to our discussion of *ELPF*'s.

Ferguson[16] refers to burning cost (*BC*) as the ratio of unmodified excess

[14] F. Harwayne, "Accident Limitations for Retrospective Rating," *PCAS* LXIII (1976), p. 1.

[15] D. Skurnick, "The California Table L," *PCAS* LXI (1974), p. 117.

[16] R. E. Ferguson, "An Actuarial Note on Loss Rating," *PCAS* LXV (1978), p. 50.

losses to subject premium. Let us change this definition to include the modifications to excess losses which Ferguson discusses (e.g., trend and loss development factors). Then we see that,

$$\begin{aligned}(BC) \cdot P &= (BR) \cdot E' \cdot P \\ BC &= E' \cdot (BR) \\ &= E' \cdot X3(R)\end{aligned}$$

where P = subject premium.

In other words, burning cost is similar to an *ELPF* in retro rating and burning ratio is similar to the excess loss ratio.

In practice, the reinsurer will not accept unlimited exposure and thus the burning ratio would have to be modified for a reinsurer limit, L, as follows:

$$\begin{aligned}BR &= (1/\alpha) \int_R^{R+L} (t - R)\, dF(t) + (L/\alpha) \int_{R+L}^{\infty} dF(t) \\ &= (1/\alpha) \int_R^{\infty} (t - R)\, dF(t) - (1/\alpha) \int_{R+L}^{\infty} (t - R)\, dF(t) + (L/\alpha) \int_{R+L}^{\infty} dF(t) \\ &= \mathrm{X3}(R) - (1/\alpha) \int_{R+L}^{\infty} [t - (R + L)]\, dF(t) \\ &= X3(R) - X3(R + L) \\ &= X2(R + L) - X2(R)\end{aligned}$$

In his review of Ferguson, Patrik[17] gives a formula for expected aggregate losses excess of R with limit L as:

$$\begin{aligned}\text{“expected losses”} &= \int_R^{R+L} (t - R)\, dF(t) + L \int_{R+L}^{\infty} dF(t) \\ &= \alpha[X3(R) - X3(R + L)] \\ &= \alpha[X2(R + L) - X2(R)]\end{aligned}$$

Dividing this formula by α yields the above formula for the burning ratio with limit, L.

Stop Loss Coverage

In a previous section we discussed the Table M charge. This equals the percentage of expected losses which is expected to be incurred above a selected

[17] G. Patrik, discussion of "An Actuarial Note on Loss Rating" by R. E. Ferguson, *PCAS* LXV (1978), p. 56.

maximum loss ratio, r. We can show that this charge is equivalent to the charge necessary for a stop loss (aggregate excess) reinsurance contract. The only conceptual difference is in the definition of "claim" and "risk." Specifically, we can define "risk" to be a primary insurer with an underlying reinsurance program and "claim" to be an annual (aggregate) recoverable for net losses which exceed a specified loss ratio. Thus,

$$\begin{aligned} \text{stop loss ratio} &= \int_r^\infty (t - r)\, dF(t) / \int_o^\infty t\, dF(t) \\ &= (1/\alpha) \int_r^\infty (t - r)\, dF(t) \\ &= X3(r) \end{aligned}$$

where r = annual net loss ratio.

If there is a percentage participation (p) by the reinsured on excess losses and a reinsurer limit of $100L\%$, we would have

$$\text{stop loss ratio} = (1 - p)[X3(r) - X3(r + L)].$$

A stop loss premium factor can be obtained as the product of the permissible loss ratio, E', underlying the subject premium and the stop loss ratio, or $E' \cdot X3(r)$ (compare to the previous section).

VIII. EXCESS RATING

Increased Limit Factors

Miccolis[18] shows that increased limit factors can be obtained from a claim size distribution. If we let

$$\begin{aligned} E[g(x;k)] &= \int_o^k t\, dF(t) + k[1 - F(k)] \\ &= \alpha \cdot X2(k), \end{aligned}$$

then we can obtain a formula for an increased limit factor, $I(k)$.

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[18] R. S. Miccolis, "On the Theory of Increased Limits and Excess of Loss Pricing," *PCAS* LXIV (1977), p. 27.

$$I(k) = \frac{E[g(x;k)]}{E[g(x;b)]}$$

$$= \frac{\alpha \cdot X2(k)}{\alpha \cdot X2(b)}$$

$$= \frac{X2(k)}{X2(b)}$$

Miccolis goes on to show that risk-adjusted increased limit factors can be obtained as:

$$I_r(k) = \frac{E[g(x;k)] + \lambda \cdot E[g(x;k)^2]}{E[g(x;b)] + \lambda \cdot E[g(x;b)^2]}$$

$$= \frac{\alpha \cdot X2(k) + \lambda \cdot E[g(x;k)^2]}{\alpha \cdot X2(b) + \lambda \cdot E[g(x;b)^2]}$$

$$= \frac{X2(k) + \lambda \cdot E[g(x;k)^2]/\alpha}{X2(b) + \lambda \cdot E[g(x;b)^2]/\alpha}$$

IX. MISCELLANEOUS

Relative Trend

In his paper on basic limits trend factors, Finger[19] defines the term relative trend (*RT*) to be the ratio of basic limits trend to total limits trend. In order to obtain a working formula, Finger defines the average relative trend (*ART*), for a particular period of time, which is the percentage increase in basic limits losses divided by the percentage increase in total limits losses. That is,

$$ART(r) = \frac{B(vr) - B(r)}{B(r)} \div \frac{T(vr) - T(r)}{T(r)}$$

where, r = basic limit ÷ mean
$v = (1 + i)^{-1}$
i = total limits trend over the period of time.

[19] R. L. Finger, "A Note on Basic Limits Trend Factors," *PCAS* LXIII (1976), p. 106.

We will define:

$$
\begin{aligned}
B(r) &= E[n] \cdot \alpha \cdot X2(r) \\
E(r) &= E[n] \cdot \alpha \cdot X3(r) \\
T(r) &= B(r) + E(r) = E[n] \cdot \alpha \cdot [X2(r) + X3(r)] = E[n] \cdot \alpha
\end{aligned}
$$

Under inflation, r will decrease to vr and α will increase to $\alpha(1 + i)$. Hence,

$$
\begin{aligned}
B(vr) &= E[n] \cdot \alpha \cdot (1 + i) \cdot X2(vr) \\
E(vr) &= E[n] \cdot \alpha \cdot (1 + i) \cdot X3(vr) \\
T(vr) &= E[n] \cdot \alpha \cdot (1 + i)
\end{aligned}
$$

We can now compute a working formula for ART.

$$
\begin{aligned}
ART(r) &= \frac{E[n] \cdot \alpha \cdot (1 + i) \cdot X2(vr) - E[n] \cdot \alpha \cdot X2(r)}{E[n] \cdot \alpha \cdot X2(r)} \\
&\div \frac{E[n] \cdot \alpha \cdot (1 + i) - E[n] \cdot \alpha}{E[n] \cdot \alpha} \\
&= \frac{(1 + i) \cdot X2(vr) - X2(r)}{i \cdot X2(r)}
\end{aligned}
$$

If we take the limit of ART as $i \rightarrow 0$, we obtain the relative trend prior to the inflation of the period of time assumed. Thus,

$$
\begin{aligned}
RT(r) &= \lim_{i \rightarrow 0} ART(r) \\
&= \lim_{i \rightarrow 0} (1/i) \cdot \frac{(1 + i) \cdot X2(vr) - X2(r)}{X2(r)} \\
&= \frac{X1(r)}{X2(r)} \qquad \text{(using L'Hôpital's Rule)}
\end{aligned}
$$

Depending on the particular application, either $RT(r)$ or $ART(r)$ may be needed.

X. CONCLUSION

We have discussed several original papers which have presented material to this Society relating to loss distributions. All of these papers are currently on the examination syllabus. However, this is not to imply that these are the only papers which utilize loss distributions. There are other papers currently in the

Proceedings, and there will most likely be future papers, dealing with this topic. Many of these papers could be analyzed using the generalized notation presented here. If this paper assists in the development of a clearer framework from which to understand the uses of loss distributions in casualty actuarial work, then the goal of the paper will have been reached. An appendix is included which gives a summary of the formulae presented. From this summary, it is clear that several rating concepts are mathematically (actuarially?) equivalent. The notation for the elementary functions is similar to that derived by R. J. Finger.

APPENDIX

SUMMARY OF FORMULAE

1.	Straight deductible *LER*	$X2(D)$
2.	Franchise deductible *LER*	$X1(D)$
3.	Disappearing deductible *LER*	see Section IV
4.	*D* ratio	$X2(M)$
5.	Excess ratio	$X3(M)$
6.	*g* value	$\max \{X3(M)\}$
7.	Table M charge	$X3(r)$
8.	Table M saving	$(r/\alpha)F(r) - X1(r)$
9.	Excess loss ratio	$X3(r)$
10.	Excess loss premium factor	$E' \cdot X3(r)$
11.	Burning ratio	$X3(R)$
12.	Burning cost	$E' \cdot X3(R)$
13.	Stop loss ratio	$X3(r)$
14.	Stop loss premium factor	$E' \cdot X3(r)$
15.	Increased limit factor	$X2(k)/X2(b)$
16.	Average relative trend	see Section IX
17.	Relative trend	$X1(r)/X2(r)$