ESTIMATING CASUALTY INSURANCE LOSS AMOUNT DISTRIBUTIONS

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I. INTRODUCTION

It is often necessary to estimate probability distributions to describe the loss processes covered by insurance contracts. For example, in order that the premium charged for a particular contract be correct according to any reasonable premium calculation principle, it must be based upon the underlying loss process for the contract. Practically, it is impossible to know the true underlying loss process, but a reasonably accurate estimate of this process can provide the basis for a reasonably accurate premium. One may discuss the loss process for an individual insured with a single coverage provided by a single contract, or for a group of insureds with multiple coverages provided by many contracts.

This paper considers the estimation of individual loss amount (severity) distributions. The term "loss amount" is used to signify the total settlement value of a single loss event. The term "contract" will be used to define any particular situation: the context should make clear whether individual or group, single or multiple coverages and contracts are being discussed.

I assume that for a particular contract at any point in time, there exists a probability distribution governing the loss amount for any loss event occurring at that time. There may be different distributions at different times, and they all may be interrelated and mutually dependent upon the number of events and their times of occurrence (Bühlmann (1970), p. 54ff). The distribution of loss amounts over a contract period is a function of the point-of-time distributions, the number of events and their timings.

This paper concentrates upon probability model-building and statistical techniques for estimating and testing the model parameters. I describe a general procedure for selecting a "best" parameterized model based upon loss amount data. This solves only part of a broader problem, which is to estimate loss amount distributions for future coverage periods or (future) final-valued loss amount distributions for past coverage periods where the losses are not all settled or even known. To solve this broader problem it is necessary to specify models of the overall insurance loss processes, defining how the future relates to the past and how the individual insured relates to the whole insurance portfolio. These models may be very simple or very complex, very loose or very mathematically precise, but we implicitly create them whenever we specify these future/past, individual/whole relationships. I will argue that a key component of the broader problem's solution is the use of probability models for loss amount distributions. Although this paper ignores some of the broader issues such as trend, loss development, population structure (classification), etc., I believe that precise model-building and testing would also resolve many of the problems connected with these.

This paper extends the work of Weissner (1978), who estimated report lag distributions from truncated data, to the estimation of loss amount distributions using censored and truncated data. The particular techniques were developed both for the estimation of commercial liability increased limits factors and for excess-of-loss reinsurance pricing. I would like to thank the following people for their contributions to this paper: Charles Hachemeister, Russell John, Mark Kleiman, Aaron Tenenbein, and Edward Weissner.

II. MATHEMATICAL MODELS

There are compelling reasons to use mathematical models to describe insurance loss amount distributions. In general, a model is a simplified, idealized interpretation of reality. A mathematical model describes the behavior of a real system by use of mathematical symbols, functions and equations. All science is a continuing process of model-building and model-testing (Kuhn (1970)). Wagner (1969) describes the purpose of a model as follows:

> Constructing a model helps you put the complexities and possible uncertainties attending a decision-making problem into a logical framework amenable to comprehensive analysis. Such a model clarifies the decision alternatives and their anticipated effects . . . In short, the model is a vehicle for arriving at a well-structured view of reality. (p. 10)

But we must be careful: "The scientist who uses models in his reflections must always remain alert to the possibility that his questions are inspired only by properties of the model, having nothing directly to do with the subject matter itself" (Hanson (1971), p. 79). Because of this, some actuaries believe that we should not attempt to describe real loss amount distributions with mathematical models, but rather we should work with the raw data. This would be fine if all we wished to do was discuss sample realizations of historical loss amount distributions. However, as has been argued in the introduction, whenever we want to extend or compare our various information, we must use some kind of model. Since this is the case in insurance work when we want to predict future possibilities, it seems clear that we should use mathematical models to describe loss amount distributions.

However, we should not believe that we can build models which will completely describe reality.[1] What we can hope for instead is to discover models which describe the salient features of a real system with some degree of accuracy. Whenever we specify a model, we should test it to see if it adequately describes the real system it is meant to describe.

I suggest that the type of model we should construct for a given loss amount distribution should be a probability model with only as many parameters as warranted by the data against which we will be testing it. Too few parameters and it is unlikely to adequately describe any given loss amount distribution; too many parameters and it becomes difficult to understand, difficult to work with and difficult to specify and test. Some of the advantages of such a ''parsimonious'' probability model are as follows:

1. It can be easily understood. Its main characteristics can be clearly described and measured.
2. It can be easily manipulated. For instance, loss development and inflationary trends *might* be accounted for simply by adjusting a few parameters.
3. It can be easily extended to more general cases or to analogous cases in a consistent manner. For example, some knowledge of the distribution of loss amounts up to certain policy limits *might* indicate something about the tail of the unbounded distribution. Also, we might expect that the loss amount distributions for similar lines of insurance would have the same general form.

[1] Gödel's proof that any axiomatic system for the natural numbers must be incomplete (Gödel (1931), (1934) and Edwards (1967)) should lead us to suspect that if an abstract idealized mathematical system cannot be completely described, then any real system must be too complicated to be completely described by a model.

4. It can be easily restricted to particular cases in a consistent manner. For example, the distribution of Owners, Landlords and Tenants liability loss amounts for small grocery stores in Kansas *might* be a special case of the general countrywide distribution of Owners, Landlords and Tenants liability loss amounts. Also, the distribution of loss amounts for any particular policy limit *might* be a restriction of some general distribution of unlimited loss amounts.

5. It can be tested using explicit statistical methods. For example, the fit of any probability model to any set of loss amount data can be tested via the Kolmogorov-Smirnov statistic or by other statistical tests.

6. It can be used to compare or combine various contracts or sets of data. For example, for a given set of contracts, the probability models for various years can be explicitly compared in order to determine the effects of inflation. Or perhaps the relationship of the probability models for different contracts can be tested to see if it would be better to specify a single ''credible'' probability model for the group.

Many possible forms of probability models for loss amount distributions have appeared in the literature. The Bibliography is a fairly comprehensive listing of relevant English-language papers and books; Johnson and Kotz (1970) is especially useful.

The purpose of this paper is to describe a general procedure for selecting an adequate model for any particular loss amount distribution; I do not advocate any particular model. However, for illustration I will use the Pareto distribution of the second kind, also called the Lomax or Pearson Type VI distribution (Johnson and Kotz (1970), p. 233ff). Its cumulative distribution function (c.d.f.) for the random variable X is defined by:

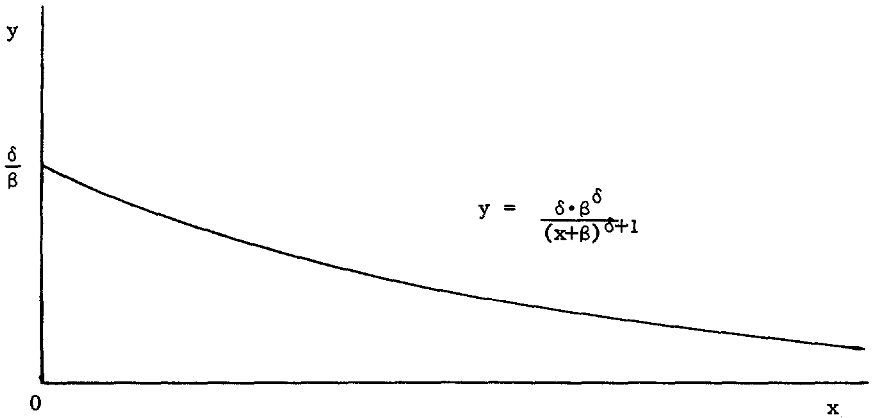
$$F(x \mid \beta, \delta) = \text{Prob}\,[X \leq x] = 1 - \left(\frac{\beta}{x + \beta}\right)^{\delta} \qquad \text{for all } x \geq 0 \qquad (2.1)$$

where $\beta > 0$, $\delta > 0$ are parameters.

This Pareto distribution is very easy to work with; a catalog of its main properties is given in Appendix A. The graph of its probability density function (p.d.f.) is shown in Figure 2.1.

FIGURE 2.1

PARETO DENSITY FUNCTION



In a slightly more complex form (to be discussed later), this model has been very useful both to myself and to the Insurance Services Office Increased Limits Subcommittee. It appears to more accurately account for large liability loss amounts than does the lognormal or other c.d.f.'s we have tested. Other investigators, such as Benckert and Sternberg (1957), Benktander and Segerdahl (1960), Mandelbrot (1964), Benckert and Jung (1974), Ramachandran (1974), and Shpilberg (1977) have found that this or the usual form of the Pareto describes fire loss data fairly well.

We will next consider how to estimate parameters for any probability model.

III. MAXIMUM LIKELIHOOD ESTIMATION

Suppose that we have postulated a probability model, such as the Pareto distribution (2.1), to describe a given loss amount distribution. The next step in our procedure should be to estimate values for the parameters of the model.

Suppose that we have a random sample of loss amounts and let us assume that they are properly adjusted to the level of the (future) distribution we are interested in. This is a strong assumption since, as already mentioned, we never see proper data because of the problems of individual loss reserve development, IBNR, and time and population differences. However, let us start here; later we will discuss a simple way of handling the time (trend) problem. We can estimate the parameters first via the method of maximum likelihood.

The situation is as follows:

1. We are given a sample $x_1, x_2, \ldots, x_n$ which we believe to be distributed according to a c.d.f. of a certain form, $F(x \mid \theta) = \text{Prob}\,[X \leq x]$, where X designates the random variable and $\theta = \{\theta_1, \ldots, \theta_r\}$ designates the indeterminate general parameter (actually, a set of individual component parameters) for the c.d.f.

2. We want to find a paramter $\hat{\theta}$ such that the model with $\hat{\theta}$ as the value of θ "best" describes the data.

The method of maximum likelihood chooses that parameter $\hat{\theta}$ which maximizes the likelihood function:

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) \tag{3.1}$$

where $f(x_i \mid \theta)$ is either the probability of x_i given θ or the p.d.f. evaluated at x given θ, depending upon whether or not the distribution function has a jump or is absolutely continuous at x_i.

Thus, $\hat{\theta}$ is "best" in the sense of being "most likely" given $x_1, x_2, \ldots, x_n$.

For example, if $F(x \mid \theta)$ is our Pareto c.d.f. (2.1), then $\theta = \{\beta, \delta\}$ and

$$L(\beta, \delta) = \prod_{i=1}^{n} \frac{\delta \cdot \beta^{\delta}}{(x_i + \beta)^{\delta+1}} \tag{3.2}$$

Maximum likelihood estimation is a standard statistical method and much is known in general about the properties of maximum likelihood estimates (MLE's). Kendall and Stuart (1967), Dudewicz (1976) and many other standard statistical texts discuss the general MLE properties, and many papers discuss particular examples. Appendix B outlines the general properties and describes how to calculate the estimates in the case of r parameters. Essentially, we can expect that for large samples, MLE's will be more accurate than any other estimates. Because of this it is surprising that the maximum likelihood method has not been used more often by actuaries. I believe the reason for this is that MLE's are usually difficult to calculate—we must have detailed data and usually we must use some fancy iterative technique to approximate the MLE's. However, with modern computers the method is much easier; even mini-computers can be programmed to approximate MLE's very quickly for many of the standard probability models.

The standard procedure is to set the first partial derivatives of the natural logarithm of (3.1) equal to zero. These first partials are of the form:

$$\frac{\partial \log L(\theta)}{\partial \theta_j} = \sum_{i=1}^{n} f(x_i \mid \theta)^{-1} \cdot \left(\frac{\partial f(x_i \mid \theta)}{\partial \theta_j} \right) \tag{3.3}$$

for $j = 1, 2, \ldots, r$

Setting these equal to zero gives us a system of r equations in the r unknowns $\theta_1, \ldots, \theta_r$ which we can solve by the Newton-Raphson iterative technique outlined in Appendix B.

We will consider a few examples in this paper to illustrate that MLE's can be much better than the standard method-of-moments estimates most often used in the actuarial literature. These are not presented as proofs; the proofs are in the statistical texts. These are simply illustrations.

Let us begin with the simple case of our Pareto c.d.f. (2.1). Suppose the data are the set of loss amounts listed in Appendix C (column 1); these 200 values are computer-generated pseudo-random Paretian values with parameters $\beta = 25{,}000$ and $\delta = 1.5$. These are realistic parameters for commercial liability losses. We can easily compare the MLE's and the method-of-moments estimates to these values. To compute the MLE's we must maximize the likelihood function (3.2). It is equivalent to maximize the loglikelihood:

$$\begin{aligned} \log L &= \log L(\beta, \delta) \\ &= n \cdot \log \delta + n\delta \cdot \log \beta - (\delta + 1) \sum_{i=1}^{n} \log (x_i + \beta) \end{aligned} \tag{3.4}$$

If $\log L$ has second partial derivatives with respect to β and δ existing throughout its range, then a necessary condition for a point $(\hat{\beta}, \hat{\delta})$ to maximize $\log L$ is that the first partials evaluated at $(\hat{\beta}, \hat{\delta})$ be equal to zero. The first partials are:

$$\frac{\partial \log L}{\partial \beta} = n\delta\beta^{-1} - (\delta + 1) \cdot \sum_{i=1}^{n} (x_i + \beta)^{-1} \tag{3.5}$$

$$\frac{\partial \log L}{\partial \delta} = n\delta^{-1} + n \cdot \log \beta - \sum_{i=1}^{n} \log(x_i + \beta) \tag{3.6}$$

The second partials are:

$$\frac{\partial^2 \log L}{\partial \beta^2} = -n\delta \cdot \beta^{-2} + (\delta + 1) \cdot \sum_{i=1}^{n} (x_i + \beta)^{-2} \tag{3.7}$$

$$\frac{\partial^2 \log L}{\partial \delta \cdot \partial \beta} = \frac{\partial^2 \log L}{\partial \beta \cdot \partial \delta} = n\beta^{-1} - \sum_{i=1}^{n} (x_i + \beta)^{-1} \tag{3.8}$$

$$\frac{\partial^2 \log L}{\partial \delta^2} = -n\delta^{-2} \tag{3.9}$$

Since $\beta > 0$ and $\delta > 0$, the second partials exist throughout the range of $\log L$. Thus, setting (3.5) = 0 and (3.6) = 0 defines a point $(\hat{\beta}, \hat{\delta})$ which may maximize $\log L$. We should check to be sure that $(\hat{\beta}, \hat{\delta})$ indeed gives a maximum (see Appendix B). The equations can be solved by a simple iterative technique such as the Newton-Raphson technique (see Appendix B).

For our Pareto example, the calculated MLE's and the implied tail probabilities for amounts greater than 100,000 and 1,000,000 (calculated via (2.1)) are displayed in Table 3.1.

TABLE 3.1

PARETO

	β	δ	Prob $[X > 100{,}000]$	Prob $[X > 1{,}000{,}000]$
Model	25,000	1.500	.089	.004
MLE	26,297	1.586	.083	.003
Method-of-Moments	56,042	2.371	.088	.001

The corresponding method-of-moments estimates β', δ' are obtained by solving the two equations:

$$\frac{\beta'}{\delta' - 1} = \text{sample mean} = 40{,}880 \tag{3.10}$$

$$\frac{\delta' \cdot \beta'^2}{(\delta' - 2) \cdot (\delta' - 1)^2} = \text{sample variance} = 10.683 \times 10^9$$

The method-of-moments implied probability that $X > 100{,}000$ is close to the true value, but the implied tail probability beyond 1,000,000 is understated.

One property of the Pareto distribution is that any non-central moment $E[X^k]$ for the unbounded c.d.f. exists only if $k < \delta$ (see Appendix A). The method-of-moments estimates, by assuming the existence of the variance and thus of $E[X^2]$, automatically forces $\delta' > 2$. Consequently, the method-of-moments estimates based upon Pareto data with the true $\delta < 2$ will always produce an estimated c.d.f. with relatively fewer large losses than the true model. Let us note here that values of δ less than 2 are typical for liability loss amount data.

Next we will consider how to test a probability model with estimated parameters against the sample data, and we will discuss how we may select final models and parameter values.

IV. MODEL TESTING AND PARAMETER SELECTION

Now suppose that we have postulated a probability model to describe a particular loss amount distribution and from sample data we have calculated MLE's of the parameters. The next step in our procedure should be to test the model and perhaps to modify the parameters for other considerations, such as credibility.

We know that any model cannot be a perfect descriptor of reality, so we should only be looking for one that is good enough for the use to which it is to be put. I suggest that the following two tests are useful for determining whether or not a particular probability model with specified parameters adequately describes a random sample from a particular loss amount distribution:

Test 1: The Kolmogorov-Smirnov Test

This is a standard statistical test that attempts to decide whether or not a given sample was generated according to a specified c.d.f. See Massey (1951), Kendall and Stuart (1967) or Conover (1971) for good general discussions of the test. The test statistic is the maximum absolute difference between the specified c.d.f. and the sample c.d.f. That is, the test statistic D_n is defined by:

$$D_n = \max \{|F(x_i^- \mid \theta) - S_n(x_i^-)|, |F(x_i^+ \mid \theta) - S_n(x_i^+)|\} \tag{4.1}$$

where $\{x_i\}$ is the ordered sample $x_1 \leq x_2 \leq \cdots \leq x_n$

$S_n(x_i^-) = \dfrac{i-1}{n}$; the value of the sample c.d.f. before the jump at x_i

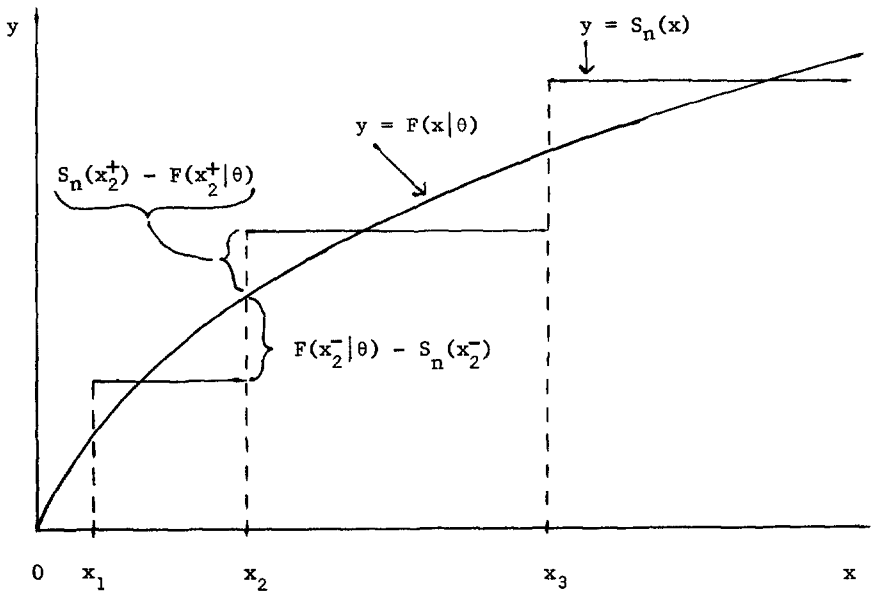
$S_n(x_i^+) = \dfrac{i}{n}$; the value after the jump

$F(x_i^- \mid \theta) = \lim_{x \to x_i^-} F(x \mid \theta)$; the limit from the left of the values of the specified c.d.f.

$F(x_i^+ \mid \theta) = \lim_{x \to x_i^+} F(x \mid \theta)$; the limit from the right.

FIGURE 4.1

K-S TEST



For any pre-specified confidence level, one rejects the hypothesis that the sample was generated according to a specified c.d.f. if the test statistic is greater than some critical value. Appendix C displays the K-S test of our Pareto c.d.f. with MLE parameters (Table 3.1) against the sample Pareto data. The K-S test statistic is .032 (Appendix C). Thus, using K-S test critical values from one of the aforementioned texts, we would not reject the hypothesis that the sample was generated by the specified Pareto c.d.f., if we were testing at a 5% significance level.[2]

[2] K-S test critical values should be smaller when the parameters are estimated from the sample. For example, see Lilliefors (1969) and Dropkin (1964).

The K-S test is more powerful than the chi-square test, since it takes into account the natural order of the data while the chi-square test ignores this order. See Massey (1951) and Conover (1971) for comparisons of the two tests. In fact, a problem with the K-S test is that in practice it seems to be too powerful for testing c.d.f.'s of loss amount distributions. At a 5% significance level it rejects any probability model yet tried for liability loss amount data. We will see an example of this later. Of course, the data we have may not be truly representative of the underlying loss amount distributions because of the previously mentioned problems (development, IBNR, etc.). I believe that we should continue to use the K-S test because its properties are well known, and the value of the K-S statistics can help us decide among different c.d.f.'s. However, we should have another test, to use in conjunction with the K-S test, which will not reject every probability model. The following test gives much useful information to an actuary.

Test 2: Expected Value Comparison (EVC) Test

This is a test of the expected value functions of the specified c.d.f. and the sample c.d.f. Define the following functions:

$$G(x \mid \theta) = \int_0^x X \cdot dF(X \mid \theta) + x \cdot (1 - F(x \mid \theta)) \tag{4.2}$$

$$G_n(x) = \frac{1}{n} \left\{ \sum_{x_i \le x} x_i + x \cdot (\text{number of } x_i > x) \right\} \tag{4.3}$$

A suitable EVC test statistic might be the vector of values:

$$\left\{ \frac{G(x_i \mid \theta) - G_n(x_i)}{G(x_i \mid \theta)} \right\} \tag{4.4}$$

where $x_1 \le x_2 \le \cdots \le x_n$ is again the sample.

This test statistic is simply the relative difference of the expected value functions $G(x \mid \theta)$ and $G_n(x)$ at each sample point. It is similar to the K-S test *vector* (not just a maximum). I don't know of any statistical work which investigates the properties of this statistic, but it is certainly a good statistic for actuaries who are interested in losses per layer. Appendix C (last column) displays the EVC statistic (as a percentage) for our Pareto c.d.f. with MLE parameters (Table 3.1) against the sample Pareto data. Note that for these 200 data points, the EVC statistic changes sign ten times and the largest absolute value is .0194 (−1.94%).

A reasonable decision rule might be: choose the probability model which has a low K-S test statistic and also has an EVC statistic which is close to the 0-vector and which has random-looking sign changes. We will see applications of this rule in the following sections.

After deciding which c.d.f. best describes a given sample of loss amounts, we should have further decision criteria which use additional information and knowledge to judge the reasonableness of the particular model. We want to know, for instance, that the particular model does not contradict our general knowledge of what loss amount distributions should look like based upon analogous data. Of the six advantages of mathematical models listed in section II, three have to do with extending to, restricting to, or comparing analogous cases. If we had a good broader model for how loss amount probability models should differ for different but similar contracts, we could test any particular c.d.f. against the general criteria. This is a deep problem in the realm of credibility theory, and it is certainly beyond the scope of this paper. However, in practice, since we actuaries do not yet have a comprehensive credibility model, we all use "actuarial judgment" to specify other pieces of the broader model.

Next we will consider some practical modeling and estimation problems and revise our basic probability model to handle them.

V. MODELING AND ESTIMATION PROBLEM 1: POLICY LIMITS BOUND THE LOSS AMOUNT DATA

For most lines of insurance, loss amounts are inherently bounded by policy limits. If the parameters of an unbounded c.d.f. are estimated from bounded data, then the c.d.f. with the estimated parameters may greatly understate the true tail of the loss amount distribution. This happens because the unbounded c.d.f. does not expect the tail of the loss amount distribution to be cut off by the policy limit.

For example, suppose our Pareto data in Appendix C (column 1) is limited to 200,000; we will then have 7 data points limited to the value 200,000. If the parameters are estimated by the method-of-moments formulas (3.10), we obtain the results displayed in Table 5.1. The "censored" MLE results will be derived later.

TABLE 5.1

PARETO (DATA LIMITED TO 200,000)

	β	δ	Prob [$X > 100{,}000$]	Prob [$X > 1{,}000{,}000$]
Model	25,000	1.500	.089	.004
Method-of-Moments	96,773	3.984	.059	.0001
"Censored" MLE	25,119	1.533	.085	.003

Since the lognormal model has been used so often for loss amount distributions, I thought that a lognormal example would also be instructive here. Appendix D lists 200 computer-generated pseudo-random lognormal values with parameters and tail probabilities given by $\mu = 9.0$ and $\sigma = 2.0$, where the parameterization used is the usual one with:

$$\text{Prob }[X \leq x] = \phi\left(\frac{\log x - \mu}{\sigma}\right) \tag{5.1}$$

where $\phi(y)$ is the normal (0, 1) c.d.f.

The standard method-of-moments estimates μ', σ' are obtained from the data limited to 200,000 by solving the two equations:

$$\exp\left\{\mu' + \frac{\sigma'^2}{2}\right\} = \text{sample mean} = 28{,}166 \tag{5.2}$$

$$(\exp\{\sigma'^2\} - 1) \cdot (\text{mean})^2 = \text{sample variance} = 2.204 \times 10^9$$

Solving these we obtain the results displayed in Table 5.2 as "Method-of-Moments I."

TABLE 5.2

LOGNORMAL (DATA LIMITED TO 200,000)

	μ	σ	Prob [$X > 100{,}000$]	Prob [$X > 1{,}000{,}000$]
Model	9.000	2.000	.104	.008
Method-of-Moments I	9.581	1.153	.047	.0001
Method-of-Moments II	8.950	1.897	.088	.005
"Censored" MLE	8.980	1.973	.100	.007

Alternative method-of-moments estimates μ'', σ'' can be obtained by considering the natural logarithms of the data limited to 200,000 to be normally distributed and taking the usual method-of-moments estimates for the normal distribution, e.g., μ'' = mean of the logs, etc. The results are displayed in Table 5.2 as "Method-of-Moments II." The "censored" MLE results will be derived later.

The c.d.f.'s with the method-of-moments estimated parameters underestimate the tail probabilities. These examples are important because this method has been used exactly as shown here so often in actuarial work.

Thus, our probability model must account for the effect of policy limits. When policy limits have been recognized in the actuarial literature, there seems to be a standardized model for liability losses. See Benktander and Segerdahl (1960), Lange (1969), Miccolis (1977) among others. They postulate that for liability loss amounts, for each particular type of business and type of coverage, there exists a unique underlying probability law dictating the distribution of loss amounts in the absence of policy limits; call this implied c.d.f. $F(x \mid \theta)$. The standard model hypothesizes that any policy limit c acts on the losses as a "censor" in the following sense: any loss which naturally would be greater than c is artificially limited to amount c. The bounded c.d.f. $F(x \mid \theta; c)$ for policy limit c can be written:

$$F(x \mid \theta; c) = \begin{cases} F(x \mid \theta) & \text{if } x < c \\ 1 & \text{if } x \geq c \end{cases} \tag{5.3}$$

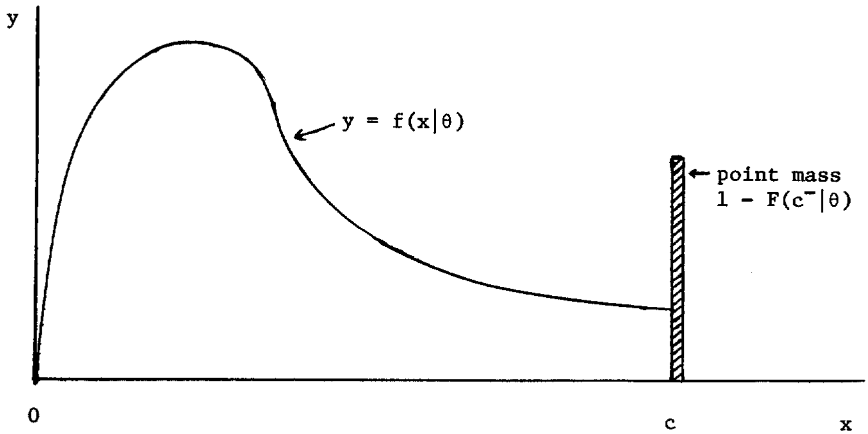
The probability function $f(x \mid \theta; c)$ is given by $f(x \mid \theta; c) = f(x \mid \theta)$ for $x < c$, and there is a discrete probability point mass $f(c \mid \theta; c) = 1 - F(c^- \mid \theta)$ at the point c, where $F(c^- \mid \theta) = \lim_{x \to c^-} F(x \mid \theta)$. The graph of $f(x \mid \theta; c)$ may be illustrated loosely by Figure 5.1.

There is no standard model for property loss amount distributions. See Benckert and Sternberg (1957), Bickerstaff (1972), Benckert (1962), Benckert and Jung (1974), Shpilberg (1977) among others. Many investigators have studied the distribution of the individual loss amount ratioed to the policy face amount for particular groups of fire insurance contracts. They have proposed either an inherently bounded c.d.f. such as the Beta distribution or some kind of censored model similar to the standard liability model.

We proceed with the censored model (5.3).

FIGURE 5.1

GRAPH OF $y = f(x|\theta;\, c)$



Suppose that $x_1, x_2, \ldots, x_n$ is a sample of loss amounts which we believe to be distributed according to a censored c.d.f. of the form (5.3), where c is the policy limit. We may reorder the x_i's so that the first $n - m$ are less than c and the remaining m are equal to c, i.e.,

$$x_1 \leq x_2 \leq \cdots \leq x_{n-m} < c,\ x_{n-m+1} = \cdots = x_n = c.$$

In this case, the likelihood function for the censored c.d.f. (5.3) can be written:

$$L(\theta;\, c) = \left\{ \prod_{i=1}^{n-m} f(x_i \mid \theta) \right\} \cdot \{1 - F(c^- \mid \theta)\}^m \tag{5.4}$$

We can continue from here to find MLE's for the parameters by using the standard techniques discussed in Appendix B. Note that we will be solving a system of equations of the form:

$$\begin{aligned}\frac{\partial \log L}{\partial \theta_j} = &\sum_{i=1}^{n-m} f(x_i \mid \theta)^{-1} \cdot \left(\frac{\partial f(x_i \mid \theta)}{\partial \theta_j}\right) \\ &- m \cdot \{1 - F(c^- \mid \theta)\}^{-1} \cdot \left(\frac{\partial F(c^- \mid \theta)}{\partial \theta_j}\right)\end{aligned} \tag{5.5}$$

for $j = 1, 2, \ldots, r$

This is simply (3.3) with the addition of a censorship term. (Remember that n in (3.3) corresponds to $n - m$ in (5.5).) *Note that the resulting MLE's are the parameters for the unbounded c.d.f.* $F(x \mid \theta)$.

We can illustrate this with our Pareto example. The Pareto censored c.d.f. is given by:

$$F(x \mid \beta, \delta; c) = \begin{cases} 1 - [\beta/(x + \beta)]^{\delta} & \text{for } 0 \leq x < c \\ 1 & \text{for } x \geq c \end{cases} \tag{5.6}$$

where $\beta > 0$, $\delta > 0$

The loglikelihood function is then:

$$\log L = (n - m) \cdot \log \delta + n\delta \cdot \log \beta - m\delta \cdot \log (c + \beta) - (\delta + 1) \cdot \sum_{i=1}^{n-m} \log (x_i + \beta) \tag{5.7}$$

Note that this is simply (3.4) with the addition of the censorship term: $m\delta \cdot \log \beta - m\delta \cdot \log (c + \beta)$. (To see this, note that n in (3.4) becomes $n - m$ in (5.7).) We can compute the first and second partials as before and use a Newton-Raphson iteration to approximate the MLE's. For our Pareto data in Appendix C (column 1) censored to 200,000, we obtain MLE's and estimated tail probabilities displayed in Table 5.1 as "censored" MLE. These are quite different from the method-of-moments estimates in Table 5.1 and are quite close to the true values.

The MLE's and estimated tail probabilities for our lognormal data in Appendix D censored at 200,000 are displayed in Table 5.2 as "censored" MLE. Again, these are quite different from the method-of-moments estimates in Table 5.2 and are quite close to the true values. Of course, we could compute correct method-of-moments estimates accounting for the policy limit censorship. But the equations that must be solved are much more complicated than the general equation (5.5).

Insurance loss amount data are usually from a mixture of contracts with different policy limits. Since the standard liability model postulates a single underlying distribution $F(x \mid \theta)$ for unbounded loss amounts for a particular type of business at a particular time, the data from all policy limits should be used simultaneously to estimate the model for this distribution. The maximum likelihood method allows us to do this very easily.

Suppose that $\{x_{ki}\}$ is a sample of loss amounts which we believe to be distributed according to the same unbounded c.d.f. except that for each k, the x_{ki}'s are censored by policy limit c_k. Again, we may reorder the x_{ki}'s so that for each k, the first $n_k - m_k$ x_{ki}'s are strictly less than c_k and the remaining m_k are equal to c_k. In this case, the general likelihood function for the total sample is simply the product of the likelihood functions for each policy limit:

$$L(\theta; c_1, \ldots, c_s) = \prod_{k=1}^{s} L(\theta; c_k) \tag{5.8}$$

Since the general likelihood function is a product, the loglikelihood and all its partial derivatives will be sums of the individual censored components. Writing it all out results in equations terrifying to behold, but whose solution is really quite straightforward in practice. For example, (5.5) becomes:

$$\begin{aligned} \frac{\partial \log L}{\partial \theta_j} = \sum_{k=1}^{s} \Bigg\{ & \sum_{i=1}^{n_k - m_k} f(x_i \mid \theta)^{-1} \cdot \left(\frac{\partial f(x_i \mid \theta)}{\partial \theta_j} \right) \\ & - m_k (1 - F(c_k^- \mid \theta))^{-1} \cdot \left(\frac{\partial F(c_k^- \mid \theta)}{\partial \theta_j} \right) \Bigg\} \end{aligned} \tag{5.9}$$

for $j = 1, 2, \ldots, r$

There may be a problem with the MLE's in this general censored case. Since the general likelihood function (5.8) is a product of likelihood functions with respect to different c.d.f.'s (because of different censorship points), the properties discussed in Appendix B may not hold. The theoretical results on the properties of MLE's have been derived for a likelihood function with respect to a single c.d.f. I have not seen any derivation of the properties of MLE's for the general likelihood function (5.8). However, in practice thus far we have noticed no strange behavior of the resulting $\hat{\theta}$.

We will see an example of data from mixed policy limits in section VIII when we consider the problem of having data from many dates of occurrence.

VI. MODELING AND ESTIMATION PROBLEM 2: THERE ARE MORE SMALL LOSSES THAN CAN BE PREDICTED BY THE USUAL MODELS

A problem encountered when we attempt to describe liability loss amount data by one of the usual c.d.f.'s, such as the Pareto or lognormal, is that there are more small loss amounts than the model predicts. Appendix E, Part 1, displays an ISO Owners, Landlords and Tenants bodily injury liability loss amount data summary for policy limit \$300,000 for policy year 1976 evaluated as of March 31, 1978. The number of losses below \$8,000 is more than predicted by any model that I or the ISO Increased Limits Subcommittee have tried. Forcing one of these probability models to fit the total distribution will cause the model to greatly understate the potential tail.

To account for the many small "nuisance claims," Hewitt and Lefkowitz (1979) worked with mixed c.d.f.'s such as:

$$\text{Prob}\,[X \leq x] = p \cdot G(x \mid \theta_G) + (1 - p) \cdot H(x \mid \theta_H) \tag{6.1}$$

where $0 \leq p \leq 1$

$G(x \mid \theta_G)$ is gamma with parameter θ_G
$H(x \mid \theta_H)$ is loggamma or lognormal with parameter θ_H

The rationale for this model is that there may be two distinct loss amount generating processes, where some losses are "regular" large losses and may be described by a c.d.f. $H(x \mid \theta_H)$, while others are "nuisance" small losses which may be described by a c.d.f. $G(x \mid \theta_G)$.

For a sample $x_1, \ldots, x_n$ generated according to this loss amount distribution, the loglikelihood function is:

$$\log L = \sum_{i=1}^{n} \log \{p \cdot g(x_i \mid \theta_G) + (1 - p) \cdot h(x_i \mid \theta_H)\} \tag{6.2}$$

where g and h are the relevant p.d.f.'s.

We can certainly calculate MLE's for this model, although one can see that the equations will be complicated and that we will have many parameter components θ_G, θ_H and p to consider simultaneously.

A much simpler alternative model may be used if we are primarily interested in the large losses and thus want to concentrate upon estimating the tail of the loss amount distribution. This model assumes that the overall distribution splits into two distinct pieces above and below some truncation point t. The overall c.d.f. $F(x)$ can be written as follows:

$$F(x \mid \theta_G, \theta_H, t, p) = \tag{6.3}$$

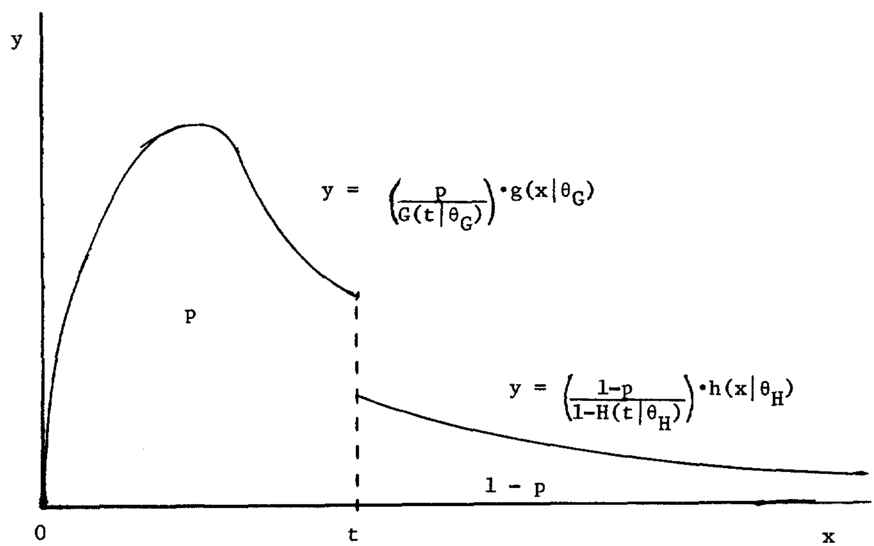
$$\begin{cases} \left(\dfrac{p}{G(t \mid \theta_G)}\right) \cdot G(x \mid \theta_G) & \text{for } x \leq t \\ p + \left(\dfrac{1-p}{1 - H(t \mid \theta_H)}\right) \cdot (H(x \mid \theta_H) - H(t \mid \theta_H)) & \text{for } x > t \end{cases}$$

where $0 \leq p \leq 1$ and $t \geq 0$
$G(x \mid \theta_G)$ = small loss amount c.d.f.
$H(x \mid \theta_H)$ = large loss amount c.d.f.

If $g(x \mid \theta_G)$ and $h(x \mid \theta_H)$ are the respective p.d.f.'s, the graph of the overall p.d.f. $f(x \mid \theta_G, \theta_H, t, p)$ is shown in Figure 6.1.

FIGURE 6.1

GRAPH OF $y = f(x|\theta_G, \theta_H, t, p)$



The picture is intentionally drawn so that the graphs do not match up at t, i.e., $f(x \mid \theta_G, \theta_H, t, p)$ is not necessarily continuous at t. Unless we are pricing small deductibles, we are primarily interested in $H(x \mid \theta_H)$ and need only gross estimates of $G(x \mid \theta_G)$. In this case, there is no need to try to force continuity at t. For commercial liability data, good values for t seem to lie between \$2,000 and \$8,000.

In practice, it is convenient to specify a value for t so that it is no longer an indeterminant parameter. Maximum likelihood estimation for this model then becomes very simple, because the parameters θ_G, θ_H and p may all be estimated separately. That is, the following lemma holds:

Lemma: Assume that for the model (6.3), t is fixed and θ_G and θ_H are disjoint sets.[3] Suppose that $x_1, \ldots, x_n$ is a sample generated according to the model and $x_i \leq t$ for $i = 1, 2, \ldots, m$ and $x_i > t$ for $i = m + 1, m + 2, \ldots, n$. Then:

1. $\hat{p} = (m/n)$ is the MLE for p
2. The MLE's $\hat{\theta}_G$ and $\hat{\theta}_H$ are obtained independently from the subsamples $\{x_1, \ldots, x_m\}$ and $\{x_{m+1}, \ldots, x_n\}$ respectively.

The proof of this lemma is obvious once we write out the loglikelihood function:

$$\log L(\theta_G, \theta_H, p) = \sum_{i=1}^{n} \log f(x_i \mid \theta_G, \theta_H, p) \tag{6.4}$$

$$= \sum_{i=1}^{m} \log \left\{\left(\frac{p}{G(t \mid \theta_G)}\right) \cdot g(x_i \mid \theta_G)\right\}$$

$$+ \sum_{i=m+1}^{n} \log \left\{\left(\frac{1-p}{1 - H(t \mid \theta_H)}\right) \cdot h(x_i \mid \theta_H)\right\}$$

$$= m \cdot \log p - m \cdot \log G(t \mid \theta_G) + \sum_{i=1}^{m} \log g(x_i \mid \theta_G)$$

$$+ (n - m) \cdot \log (1 - p) - (n - m) \cdot \log \{1 - H(t \mid \theta_H)\}$$

$$+ \sum_{i=m+1}^{n} \log h(x_i \mid \theta_H)$$

$$= m \cdot \log p + (n - m) \cdot \log (1 - p)$$

$$+ \sum_{i=1}^{m} \log g(x_i \mid \theta_G) - m \cdot \log G(t \mid \theta_G)$$

$$+ \sum_{i=m+1}^{n} \log h(x_i \mid \theta_H) - (n - m) \cdot \log \{1 - H(t \mid \theta_H)\}$$

---

[3] θ_G and θ_H have no elements in common.

Since the loglikelihood splits into three parts dependent upon p, θ_G, and θ_H respectively, then the first partial derivatives with respect to p, θ_G, and θ_H depend only upon those three parts respectively.

This model and the lemma allow us to split the Owners, Landlords and Tenants bodily injury liability loss data at $8,000, for example, and to estimate the distribution of the large loss amounts by:

1. estimating p from the relative number of large and small loss amounts;
2. estimating θ_H strictly from the large loss amounts.

If $H(x \mid \theta_H)$ is our Pareto c.d.f. censored at c = $300,000 (5.6), then (6.4) and (5.4) together say that $\theta_H = \{\beta, \delta\}$ may be estimated from the loglikelihood:

$$\begin{aligned}
\log L &= \sum_{i=m+1}^{n-m'} \log h(x \mid \beta, \delta) + m' \cdot \log \{1 - H(c \mid \beta, \delta)\} \\
&\quad - (n - m) \cdot \log \{1 - H(t \mid \beta, \delta)\} \qquad (6.5)\\
&= \sum_{i=m+1}^{n-m'} \log \left(\frac{\delta\beta^{\delta}}{(x_i + \beta)^{\delta+1}}\right) + m' \cdot \log \left(\left(\frac{\beta}{c + \beta}\right)^{\delta}\right) \\
&\quad - (n - m) \cdot \log \left(\left(\frac{\beta}{c + \beta}\right)^{\delta}\right) \\
&= (n - m - m') \cdot \log \delta + (n - m - m') \cdot \log \beta \\
&\quad - (\delta + 1) \cdot \sum_{i=m+1}^{n-m'} \log (x_i + \beta) \\
&\quad + m'\delta \cdot \log \beta - m'\delta \cdot \log (c + \beta) - (n - m)\delta \cdot \log \beta \\
&\quad + (n - m)\delta \cdot \log (t + \beta) \\
&= (n - m - m') \cdot \log \delta - (\delta + 1) \cdot \sum_{i=m+1}^{n-m'} \log (x_i + \beta) \\
&\quad - m'\delta \cdot \log (c + \beta) + (n - m)\delta \cdot \log (t + \beta)
\end{aligned}$$

where $n - m$ = total number of $x_i > t$

m' = number of loss amounts equal to c

$t < x_i < c$ for $i = m + 1, m + 2, \ldots, n - m'$

The MLE's of β, δ and p for our Pareto c.d.f. fit to the O. L. & T. data excess of $8,000 in Appendix E, Part 1, are displayed in Table 6.1 as "Pareto MLE I" where for the fitted Pareto, the tail probabilities are:

$$P[X > x \mid \hat{\beta}, \hat{\delta}, \hat{p}] = (1 - \hat{p}) \cdot \left(\frac{t + \hat{\beta}}{x + \hat{\beta}}\right)^{\hat{\delta}} \qquad \text{from (6.3)}$$

TABLE 6.1

PARETO FIT TO O. L. & T. DATA

	β	δ	p	P[$X > 100{,}000$]	P[$X \geq 300{,}000$]
Data	NA	NA	.950	.0016	.0004
Pareto MLE I	1,463	1.453	.950	.0016	.0003
Pareto MLE II	347	0.877	NA	.0069	.0027

Appendix E, Part 2, displays the results of the K-S test and the EVC test. The Pareto model, of course, fails the K-S test. But the EVC statistic does not look bad: there are eight sign changes, the last component is $-.0187$ (-1.87%) and the absolute maximum 0.056 (-5.60%) occurs in the same interval ($9,000 – $10,000) as the K-S maximum. The Pareto c.d.f. fits better at the upper end and the expected value functions coincide well. If we fit the Pareto model to the overall loss amount distribution, the results are worse. The results are displayed in Table 6.1 as "Pareto MLE II"; note how poorly this "untruncated" Pareto c.d.f. predicts the tail probabilities. Appendix E, Part 3, displays the K-S test and EVC test results for this "untruncated" Pareto. The results are opposite those which usually occur; in this particular case, both the tail probabilities and expected values are too high for the Pareto. The point here is that the "untruncated" results are misleading.

A word about the data. Remember that these are a full policy year of incurred loss amounts evaluated at 27 months and grouped by intervals. They are immature (individual loss reserve development), incomplete (IBNR), from loss events occurring over a two year period, and are not listed by individual loss amount for the MLE procedure. Thus any remarks regarding the approximation of the true underlying distribution are tentative. I decided to use undeveloped and incomplete data for this example so as not to get involved in the question of how to develop and complete it. Unsatisfactory though the data are, I hope that the example is illustrative.

A word about using data summarized by interval, as the O. L. & T. data. For c.d.f. $F(x \mid \theta)$, the likelihood function for interval data is based upon the discrete distribution of probability per interval. If the intervals are $(a_0, a_1]$, . . . , $(a_{s-1}, a_s]$ and a sample produces n_i losses in the ith interval, the true likelihood function is:

$$L(\theta) = \prod_{i=1}^{s} \left(\frac{F(a_i \mid \theta) - F(a_{i-1} \mid \theta)}{F(a_s \mid \theta) - F(a_0 \mid \theta)} \right)^{n_i} \tag{6.6}$$

Our MLE's based upon the O. L. & T. data, however, treated the data as if all the losses in each interval were concentrated at the average value for the interval. In Appendix E, Part 1, for example, the 58 losses in the interval 10,000 − 11,000 were assumed each to have value 10,430 and the "individual data point" loglikelihood function (6.5) was used. Our testing has shown that treating interval data this way gives good results as long as the intervals are fairly narrow.

This estimation technique for large loss data truncated below is also useful when dealing with excess-of-loss reinsurance coverage where the data are usually excess of some underlying retention. To illustrate its accuracy in estimating the "total distribution," we again turn to our Pareto data in Appendix C. Truncating below at 5,000 and censoring above again at 200,000, we calculate the MLE's and tail probabilities displayed in Table 6.2. The analogous estimates from the lognormal data in Appendix D truncated below at 5,000 and censored above at 200,000 are displayed in Table 6.3.

The lognormal estimates here are not as good as we have seen in previous cases. In both these examples, the corresponding method-of-moments estimates would be ridiculous if the lower truncation were not taken into account, and the formulas would be difficult if it were.

TABLE 6.2

PARETO (DATA TRUNCATED AT 5,000)

	β	δ	p	P[X > 100,000]	P[X > 1,000,000]
Model	25,000	1.500	.239	.089	.004
MLE	23,354	1.492	.235	.085	.004

TABLE 6.3

LOGNORMAL (DATA TRUNCATED AT 5,000)

	μ	σ	p	P[X > 100,000]	P[X > 1,000,000]
Model	9.00	2.000	.405	.104	.008
MLE	8.98	1.858	.370	.091	.005

VII. MODELING AND ESTIMATION PROBLEM 3: THE UNDERLYING LOSS AMOUNT DISTRIBUTIONS ARE NOT SMOOTH

Whenever we see detailed loss amount data, such as the Owners, Landlords and Tenants bodily injury liability loss interval data in Appendix E, Part 1, or individual loss amount data, we are immediately struck by the fact that the losses tend to cluster at certain round values such as $1,000, $10,000, $25,000, . . . , $100,000, etc. This clustering occurs even in mature loss amount data. Thus, it is apparent that any probability model which is to describe the data as exactly as possible cannot be a smooth c.d.f. such as the Pareto or lognormal.

An alternative to a smooth model might be a mixed c.d.f. similar to the Hewitt/Lefkowitz model (6.1) with $G(x \mid \theta_G)$ smooth and $H(x \mid \theta_H)$ discrete. If a mixed model is to fit loss amount data significantly better than a completely smooth model can, then it may need many parameters, perhaps one for each discrete cluster point. The likelihood equations (6.2) would be very difficult to solve. And even then would such a model provide any better prediction, through simple parameter changes, of future loss amount distributions? I believe that our data are inadequate to support such a model.

Even though our data seem to have cluster points, the last column of Appendix E, Part 2, shows that the Pareto c.d.f. describes the expected loss function fairly well. Remember that expected value is the most important component of most insurance premiums. The same exhibit shows how well the Pareto c.d.f. estimates the tail probabilities (see also Table 6.1); this is also an important aspect of insurance pricing. Since we apparently cannot specify a better model without great difficulty (remember the data problems), it looks as if we must be satisfied with smooth models for large loss amount distributions as long as the parameters are properly estimated.

VIII. MODELING AND ESTIMATION PROBLEM 4: THE DATA ARE FROM MANY OCCURRENCE DATES

To have enough data to be able to study loss amount distributions, we must of necessity use data from many accident occurrence dates. Suppose that we want to study the loss amount distribution at one point-of-time and suppose that we have specified a broader model which tells us how to trend the data from different occurrence dates to the level of this single date. Let us also suppose that besides being subject to various policy limits, our data are also larger than some common lower truncation value t. This situation is common in reinsurance, where our data are often excess of some specified retention level. Also this situation arises if we use a probability model such as (6.3) to separate large and small losses and we use a common split point t for all our data. In this case, for each occurrence date we must trend the value t along with the loss amounts and the policy limits to our single occurrence date. Thus our trended data have a mixture of trended lower truncation points and trended policy limits (censorship points). The method of maximum likelihood allows us to painlessly calculate parameter estimates for the single point-of-time model simultaneously from all these data.

Let us illustrate this situation with our Pareto c.d.f. Let the x_{ki}'s represent the trended loss amounts. Assume that they are ordered so that for each k, the x_{ki}'s are larger than the trended lower truncation point t_k and are censored at the trended policy limit c_k. Also assume that for each k, the first $n_k - m_k$ x_{ki}'s lie strictly between t_k and c_k and the remaining m_k are equal to c_k. We assume that except for lower truncation and upper censorship, the x_{ki}'s are subject to the same underlying Pareto c.d.f. $F(x \mid \beta, \delta)$. Then using equations (5.8) and (6.5), with a suitable change in notation, the general likelihood function is:

$$L(\beta, \delta) = \prod_{k=1}^{s} L(\beta, \delta; t_k, c_k) \tag{8.1}$$

where the component loglikelihood function for each k is:

$$\begin{aligned} \log L(\beta, \delta; t_k, c_k) = {} & (n_k - m_k) \cdot \log \delta \\ & - (\delta + 1) \cdot \sum_{i=1}^{m_k} \log (x_{ki} + \beta) \\ & - m_k\delta \cdot \log (c_k + \beta) + n_k\delta \cdot \log (t_k + \beta) \end{aligned} \tag{8.2}$$

As in (5.9), the partial derivatives of the general loglikelihood will be the sum of the partials of the components in (8.2) for each truncation/censorship combination. The computer programming is straightforward.

For example, let us use the Owners, Landlords and Tenants bodily injury liability loss amount data in Appendices E and F for policy years 1975 and 1976 for policy limits of \$300,000 and \$500,000, and adjust by a trend factor of 18.9% per annum to an occurrence date of July 1, 1980. Since the individual loss occurrence dates are unknown, we will simply assume the average occurrence dates: for policy year 1975 this is January 1, 1976 and for policy year 1976 it is January 1, 1977. It would, of course, be better to know the occurrence month of each loss amount. We will use original lower truncation points for each year of \$8,000.

The simultaneous estimates for β, δ for July 1, 1980 are $\beta = 4{,}955$ and $\delta = 1.473$. The K-S test and EVC test results are displayed in Appendix G for the Pareto with parameters β, δ and each set of trended data. Note that in this case, the assumption that the loss amounts from different policy limits have the same underlying distribution looks like it may be false. The reason for this tentative conclusion is that the fitted Pareto greatly understates the expected values for higher limits of the \$500,000 policy limit trended data. This can be seen by studying the EVC test statistic in the last columns of Appendix G; the final value for trended policy year 1975 \$500,000 policy limit data is $-.298$ (Part 2) and the final value for trended policy year 1976 \$500,000 policy limit data is $-.178$ (Part 4).

The ISO Increased Limits Subcommittee has had mixed results when testing this assumption of a common loss amount distribution underlying different policy limits (except, of course, for the censorship at each limit). It is apparent that much more testing (and more careful model-building) needs to be done.

IX. CONCLUSION

This paper has presented a general procedure for selecting an adequate model for any particular loss amount distribution. The point of view is that we must use models whenever we want to extend or compare our various information, and moreover, we should use mathematical models. The procedure for finding an adequate model is to:

1. specify a particular probability model;
2. estimate parameters via the method of maximum likelihood;
3. test the model and select final parameters.

We have also discussed how to account for policy limits as censors, for too many small losses, for probability cluster points, and finally for loss amount inflation trends. The estimation technique discussed has been the method of maximum likelihood.

The Bibliography lists, beyond the direct references, many English-language papers and books which study casualty loss amount distributions or related problems. I trust that other American actuaries will find these references of interest.

APPENDIX A

Pareto (Lomax, Pearson Type VI) Distribution

$$F(x) = \text{Prob}\,[X \leq x] = 1 - \left(\frac{\beta}{x + \beta}\right)^{\delta} \qquad \text{for } x \geq 0 \tag{A1}$$

where $\beta > 0$, $\delta > 0$ are parameters.

$$f(x) = \frac{\delta \cdot \beta^{\delta}}{(x + \beta)^{\delta+1}} \text{ density} \tag{A2}$$

$$E[X] = \frac{\beta}{\delta - 1} \qquad \text{exists if } \delta > 1 \tag{A3}$$

$$\text{Var}\,[X] = \frac{\delta \cdot \beta^2}{(\delta - 2) \cdot (\delta - 1)^2} \qquad \text{exists if } \delta > 2 \tag{A4}$$

$$E[X^k] \text{ does not exist for } k \geq \delta \tag{A5}$$

$$\text{Prob}\,[X - t \leq x \mid X > t] = 1 - \left(\frac{\beta + t}{x + (\beta + t)}\right)^{\delta} \qquad \text{for } x \geq 0 \tag{A6}$$

Thus, a Pareto distribution excess of a lower truncation t is a Pareto distribution with new "beta parameter" $\beta + t$.

If $Y = tX$ for some $t > 0$, then (A7)

$$\text{Prob}\,[Y \leq y] = 1 - \left(\frac{\beta t}{y + \beta t}\right)^{\delta} \qquad \text{for } y \geq 0$$

Thus, if t is a trend factor and Y is the inflated value of X, then Y also has a Pareto distribution with new "beta parameter" βt. For any limit c, notate the integral of X^k from 0 to c by:

$$E[X^k; c] = \int_0^c X^k dF(X) \tag{A8}$$

Lemma: For any censor c, if k is a non-negative integer and $k - \delta$ is not a non-negative integer, then the integral of X^k from 0 to c is given by:

$$E[X^k; c] = \frac{k! \cdot \beta^k}{(\delta - 1)(\delta - 2) \cdots (\delta - k)}$$

$$- \delta \left(\frac{\beta}{c + \beta}\right)^{\delta} \left\{ \frac{(c + \beta)^k}{\delta - k} - \binom{k}{1} \frac{\beta(c + \beta)^{k-1}}{\delta - k + 1} + \cdots \right. \tag{A9}$$

$$\left. + (-1)^i \binom{k}{i} \frac{\beta^i (c + \beta)^{k-i}}{\delta - k + i} + \cdots + (-1)^k \left(\frac{\beta^k}{\delta}\right) \right\}$$

Proof: (This lemma and proof are due to Mark Kleiman)

$$E[X^k; c] = \int_0^c x^k \cdot \delta \cdot \beta^{\delta} \cdot (x + \beta)^{-\delta-1} dx$$

$$= \int_0^c \delta\beta^{\delta}(x + \beta)^{-\delta-1}\{(x + \beta) - \beta\}^k dx$$

$$= \int_0^c \delta\beta^{\delta}(x + \beta)^{-\delta-1} \left\{ (x + \beta)^k - \binom{k}{1} \beta(x + \beta)^{k-1} + \cdots \right.$$

$$\left. + (-1)^i \binom{k}{i} \beta^i (x + \beta)^{k-i} + \cdots + (-1)^k \beta^k \right\} dx$$

$$= \delta \int_0^c \left\{ \beta^{\delta}(x + \beta)^{k-\delta-1} - \binom{k}{1} \beta^{\delta+1}(x + \beta)^{k-\delta-2} + \cdots \right.$$

$$\left. + (-1)^i \binom{k}{i} \beta^{\delta+i}(x + \beta)^{k-\delta-i-1} + \cdots + (-1)^k \beta^{\delta+k}(x + \beta)^{-\delta-1} \right\} dx$$

$$= \delta \left\{ \left(\frac{\beta^{\delta}}{k - \delta}\right) \cdot (x + \beta)^{k-\delta} - \binom{k}{1}\left(\frac{\beta^{\delta+1}}{k - \delta - 1}\right) (x + \beta)^{k-\delta-1} + \cdots \right.$$

$$+ (-1)^i \binom{k}{i}\left(\frac{\beta^{\delta+i}}{k - \delta - i}\right) (x + \beta)^{k-\delta-i} + \cdots$$

$$\left. + (-1)^k \left(\frac{\beta^{\delta+k}}{\delta}\right) (x + \beta)^{-\delta} \right\} \Bigg|_0^c$$

$$
\begin{aligned}
&= \delta \left\{ \left(\frac{\beta^{\delta}}{k-\delta}\right) [(c+\beta)^{k-\delta} - \beta^{k-\delta}] \right. \\
&\quad - \binom{k}{1}\left(\frac{\beta^{\delta+1}}{k-\delta-1}\right) [(c+\beta)^{k-\delta-1} - \beta^{k-\delta-1}] + \cdots \\
&\quad + (-1)^i \binom{k}{i}\left(\frac{\beta^{\delta+i}}{k-\delta-i}\right) [(c+\beta)^{k-\delta-i} - \beta^{k-\delta-i}] + \cdots \\
&\quad \left. + (-1)^k \left(\frac{\beta^{\delta+k}}{\delta}\right) [(c+\beta)^{-\delta} - \beta^{-\delta}] \right\} \\
&= \delta\beta^k \left\{ \frac{1}{\delta-k} - \binom{k}{1}\frac{1}{\delta-k+1} + \cdots \right. \\
&\quad \left. + (-1)^i \binom{k}{i}\frac{1}{\delta-k+i} + \cdots + (-1)^k \frac{1}{\delta} \right\} \\
&\quad - \delta\left(\frac{\beta}{c+\beta}\right)^{\delta} \left\{ \frac{(c+\beta)^k}{\delta-k} + \cdots \right. \\
&\quad \left. + (-1)^i \binom{k}{i}\frac{\beta^i (c+\beta)^{k-i}}{\delta-k+i} + \cdots + (-1)^k \frac{\beta^k}{\delta} \right\}
\end{aligned}
$$

We now want to prove that the first expression in braces in the last equality is equal to:

$$\frac{k!}{\delta(\delta-1)\cdots(\delta-k)} \qquad \text{if } \delta - k \text{ is not a negative integer}$$

This is proved by judicious use of the binomial theorem and from the definitions of Gamma and Beta functions.

$$
\begin{aligned}
&\frac{k!}{\delta(\delta-1)\cdots(\delta-k)} \\
&= \frac{\Gamma(k+1)\cdot\Gamma(\delta-k)}{\Gamma(\delta+1)} \qquad \text{if } \delta - k \text{ is not a negative integer} \\
&= \int_0^1 (1-x)^k x^{\delta-k-1}\,dx \\
&= \int_0^1 \left\{ 1 - \binom{k}{1} x + \cdots + (-1)^i \binom{k}{i} x^i + \cdots + (-1)^k x^k \right\} x^{\delta-k-1}\,dx
\end{aligned}
$$

$$= \int_0^1 \left\{ x^{\delta-k-1} - \binom{k}{1} x^{\delta-k} + \cdots + (-1)^i \binom{k}{i} x^{\delta-k+i-1} + \cdots \right.$$
$$\left. + (-1)^k x^{\delta-1} \right\} dx$$
$$= \left\{ \frac{x^{\delta-k}}{\delta - k} - \binom{k}{1} \frac{x^{\delta-k+1}}{\delta - k + 1} + \cdots \right.$$
$$\left. + (-1)^i \binom{k}{i} \frac{x^{\delta-k+i}}{\delta - k + i} + \cdots + (-1)^k \frac{x^\delta}{\delta} \right\} \Big|_0^1$$
$$= \frac{1}{\delta - k} - \binom{k}{1} \frac{1}{\delta - k + 1} + \cdots$$
$$+ (-1)^i \binom{k}{i} \frac{1}{\delta - k + i} + \cdots + (-1)^k \cdot \frac{1}{\delta}$$

Note: If $c < \infty$, then any integral $E[X^k; c]$ exists (is finite).

If $k - \delta$ is a non-negative integer, then $E[X^k; c]$ may be approximated for small $\epsilon > 0$ via:

$$E[X^k; c] \simeq \{E[X^{k-\epsilon}; c] + E[X^{k+\epsilon}; c]\}/2 \tag{A10}$$

So, the lemma evaluation formula may be used.

Corollary: For any censor c:

$$E[X; c] = \frac{\beta}{\delta - 1} \left\{ 1 - \left(\frac{\beta}{c + \beta}\right)^{\delta-1} \left(\frac{\delta c + \beta}{c + \beta}\right) \right\} \qquad \text{if } \delta \neq 1 \tag{A11}$$

APPENDIX B

Maximum Likelihood Estimation and Newton-Raphson Iteration

Given a sample $x_1, x_2, \ldots, x_n$ and general c.d.f. $F(x \mid \theta)$ with parameter $\theta = \{\theta_1, \ldots, \theta_r\}$ in some set Θ. The likelihood function is given by:

$$L(\theta) = \prod_{i=1}^{n} f(x_i \mid \theta) \tag{B1}$$

where $f(x_i \mid \theta)$ is either the probability of x_i given θ or the p.d.f. evaluated at x_i given θ, depending upon whether or not the distribution function has a jump or is absolutely continuous at x_i.

The important properties of MLE's are (Kendall and Stuart (1967), p. 38ff and Dudewicz (1976), p. 193ff):

1. Under very general conditions, MLE's are consistent. That is, the MLE $\hat{\theta}$ converges in probability to θ_0, the true value of θ, as the sample size increases.

2. Under very general conditions, MLE's are consistent asymptotically normally distributed and efficient. That is, $\hat{\theta}$ is asymptotically (as sample size $n \to \infty$) normally distributed with mean θ_0 and covariance matrix equal to the inverse of the Fisher information matrix:

$$\text{Cov }\hat{\theta} \simeq I(\theta_0)^{-1} \qquad \text{(approximately)} \tag{B2}$$

where

$$I(\theta_0) = -E\left[\left(\left(\frac{\partial^2 \log L(\theta)}{\partial\theta_j \cdot \partial\theta_i}\bigg|_{\theta=\theta_0}\right)\right)\right] \tag{B3}$$

The determinant of Cov $\hat{\theta}$ becomes minimal as $n \to \infty$.

For our Pareto example (2.1), for sample size n we have:

$$I(\beta, \delta) = \begin{pmatrix} \dfrac{n\delta}{\beta(\delta+2)} & \dfrac{-n}{\beta(\delta+1)} \\[2ex] \dfrac{-n}{\beta(\delta+1)} & \dfrac{n}{\delta^2} \end{pmatrix} \tag{B4}$$

So,

$$\text{Cov }(\hat{\beta}, \hat{\delta}) \simeq \frac{1}{n}\begin{pmatrix} \dfrac{\beta^2}{\delta}(\delta+1)^2(\delta+2) & \beta\delta(\delta+1)(\delta+2) \\[2ex] \beta\delta(\delta+1)(\delta+2) & \delta^2(\delta+1)^2 \end{pmatrix} \tag{B5}$$

Finding a $\hat{\theta}$ which maximizes $L(\theta)$ is equivalent to finding a $\hat{\theta}$ which maximizes

$$\log L = \log L(\theta) = \sum_{i=1}^{n} \log f(x_i \mid \theta) \tag{B6}$$

If $\log L$ has second partial derivatives with respect to the θ_j's existing throughout its domain Θ, then a necessary condition for a point $\hat{\theta}$ to maximize $\log L$ is that the first partials evaluated at $\hat{\theta}$ be equal to zero:

$$\frac{\partial \log L}{\partial \theta_j}\bigg|_{\theta=\hat{\theta}} = 0 \qquad \text{for } j = 1, 2, \ldots, r \tag{B7}$$

If the matrix of second partials evaluated at $\hat{\theta}$ is negative definite, then $\hat{\theta}$ indeed maximizes log L.

Assuming that the matrix of second partials will be negative definite, we must find $\hat{\theta}$ which satisfies the system of equations (B7). Newton-Raphson iteration allows us to find a sequence of vectors $\theta^{(1)}$, $\theta^{(2)}$, . . . which may converge to a solution for any system of equations such as (B7). The only condition necessary is that the partials of the equations in (B7) with respect to each θ_j must exist for each $\theta^{(m)}$. See Conte and de Boor (1972).

Expressions for the second partial derivatives of log L are somewhat unwieldly. So we will simplify the notation of (B7) to a more general case: we assume that our problem is to find a point $\hat{\theta} = \{\hat{\theta}_1, \ldots, \hat{\theta}_r\}$ which is a solution for the following system of r equations:

$$\begin{aligned} \Psi_1(\theta) &= 0 \\ \Psi_2(\theta) &= 0 \\ \vdots \quad & \quad \vdots \\ \Psi_r(\theta) &= 0 \end{aligned} \tag{B8}$$

And we assume that the partials of the Ψ_i's with respect to the θ_j's exist throughout the domain Θ.

Start by selecting an initial value $\theta^{(1)}$. Then, in general, $\theta^{(m+1)}$ is obtained from $\theta^{(m)}$ by solving the following system of r equations in r unknowns $\theta_1^{(m+1)}, \ldots, \theta_r^{(m+1)}$:

$$\begin{aligned} \sum_{j=1}^{r} (\theta_j^{(m+1)} - \theta_j^{(m)}) \cdot \frac{\partial \Psi_1(\theta^{(m)})}{\partial \theta_j} &= -\Psi_1(\theta^{(m)}) \\ \vdots \qquad\qquad & \qquad \vdots \\ \sum_{j=1}^{r} (\theta_j^{(m+1)} - \theta_j^{(m)}) \cdot \frac{\partial \Psi_r(\theta^{(m)})}{\partial \theta_j} &= -\Psi_r(\theta^{(m)}) \end{aligned} \tag{B9}$$

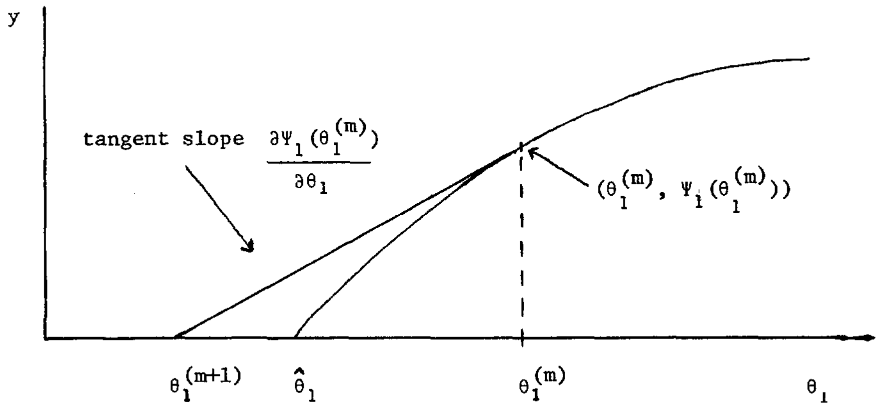
In the case that $r = 1$, we have the familiar solution:

$$\theta_1^{(m+1)} = \theta_1^{(m)} - \Psi_1(\theta_1^{(m)}) \cdot \left(\frac{\partial \Psi_1(\theta_1^{(m)})}{\partial \theta_1}\right)^{-1} \tag{B10}$$

The 1-dimensional case may be illustrated by Figure B1.

FIGURE B1

GRAPH OF $y = \Psi_1(\theta_1)$



The tangent through the point $(\theta_1^{(m)}, \Psi(\theta_1^{(m)}))$ intersects the θ_1-axis at the new θ_1 value, $\theta_1^{(m+1)}$. It should be clear that as long as $\Psi_1(\theta_1)$ is well-behaved, if $\theta_1^{(m)}$ is close to a zero $\hat{\theta}_1$ for $\Psi_1(\theta_1)$, then $\theta_1^{(m+1)}$ should be even closer, as in the figure.

In the general case, the solution to (B4) is obtained via Cramer's Rule as long as the matrix of second partials evaluated at $\theta^{(m)}$ is nonsingular (Herstein (1964), p. 288). For example, the case $r = 2$ is easily illustrated:

$$\theta_1^{(m+1)} = \theta_1^{(m)} - \frac{\Psi_1(\theta^{(m)}) \cdot \left(\frac{\partial\Psi_2(\theta^{(m)})}{\partial\theta_2}\right) - \left(\frac{\partial\Psi_1(\theta^{(m)})}{\partial\theta_2}\right) \cdot \Psi_2(\theta^{(m)})}{J(\theta^{(m)})} \tag{B11}$$

$$\theta_2^{(m+1)} = \theta_2^{(m)} - \frac{\Psi_2(\theta^{(m)}) \cdot \left(\frac{\partial\Psi_1(\theta^{(m)})}{\partial\theta_1}\right) - \left(\frac{\partial\Psi_2(\theta^{(m)})}{\partial\theta_1}\right) \cdot \Psi_1(\theta^{(m)})}{J(\theta^{(m)})}$$

where $J(\theta^{(m)})$ is the Jacobian evaluated at $\theta^{(m)}$:

$$J(\theta^{(m)}) = \left(\frac{\partial\Psi_1(\theta^{(m)})}{\partial\theta_1}\right) \cdot \left(\frac{\partial\Psi_2(\theta^{(m)})}{\partial\theta_2}\right) - \left(\frac{\partial\Psi_1(\theta^{(m)})}{\partial\theta_2}\right) \cdot \left(\frac{\partial\Psi_2(\theta^{(m)})}{\partial\theta_1}\right) \tag{B12}$$

This technique gives an iteration $\theta^{(1)}, \theta^{(2)}, \ldots$ which may be stopped when successive values $|\theta^{(m+1)} - \theta^{(m)}|$ are small enough according to some metric $|\theta|$.

APPENDIX C

Pareto

200 Psuedo-Random Values

$$\beta = 25{,}000 \qquad \delta = 1.5 \qquad P[X \leq x] = 1 - \left(\frac{\beta}{x + \beta}\right)^{\delta}$$

	K-S Test			EVC Test		
				Means		EVC
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	Statistic (%)
9	.001	.005	-.004	8	9	-.03
151	.009	.010	.004	150	150	.02
225	.013	.015	.003	223	223	-.03
282	.017	.020	-.003	279	279	-.03
295	.018	.025	-.007	292	292	-.01
670	.039	.030	.014	656	658	-.20
772	.045	.035	.015	754	757	-.34
773	.045	.040	.010	755	757	-.34
845	.049	.045	.009	824	827	-.37
910	.053	.050	.008	885	889	-.38
915	.053	.055	.003	890	893	-.38
1,035	.059	.060	.004	1,003	1,007	-.35
1,092	.062	.065	-.003	1,057	1,060	-.34
1,151	.066	.070	-.004	1,112	1,116	-.32
1,208	.069	.075	-.006	1,165	1,169	-.29
1,231	.070	.080	-.010	1,187	1,190	-.28
1,548	.087	.085	.007	1,479	1,481	-.19
1,616	.090	.090	.005	1,541	1,544	-.19
1,638	.091	.095	-.004	1,561	1,564	-.19
1,643	.092	.100	-.008	1,565	1,568	-.19
1,727	.096	.105	-.009	1,641	1,644	-.15
1,786	.099	.110	-.011	1,695	1,697	-.12
1,866	.103	.115	-.012	1,767	1,768	-.07
2,005	.110	.120	-.010	1,891	1,891	-.01
2,049	.112	.125	-.013	1,930	1,930	.01
2,081	.114	.130	-.016	1,958	1,958	.03
2,108	.115	.135	-.020	1,982	1,981	.05
2,127	.116	.140	-.024	1,999	1,998	.07
2,469	.133	.145	-.012	2,298	2,292	.29
2,626	.140	.150	-.010	2,434	2,426	.33
2,945	.155	.155	.005	2,706	2,697	.33
2,949	.155	.160	-.005	2,709	2,700	.33
2,958	.156	.165	-.009	2,717	2,708	.33
3,036	.159	.170	-.011	2,783	2,773	.34
3,048	.160	.175	-.015	2,793	2,783	.35
3,285	.170	.180	-.010	2,991	2,979	.40
3,356	.173	.185	-.012	3,049	3,037	.41
3,428	.177	.190	-.013	3,109	3,095	.43
3,593	.184	.195	-.011	3,244	3,229	.46
3,796	.193	.200	-.007	3,409	3,393	.48
3,936	.198	.205	-.007	3,521	3,505	.48
3,952	.199	.210	-.011	3,534	3,517	.48
4,424	.219	.215	.009	3,908	3,890	.45
4,753	.232	.220	.017	4,163	4,148	.34
4,849	.235	.225	.015	4,236	4,223	.31
4,961	.240	.230	.015	4,322	4,310	.27
4,979	.240	.235	.010	4,335	4,324	.26
5,114	.246	.240	.011	4,437	4,427	.23
5,129	.246	.245	.006	4,449	4,439	.23
5,135	.246	.250	-.004	4,453	4,443	.23

APPENDIX C

Pareto

200 Psuedo-Random Values

	K-S Test			EVC Test		
				Means		EVC
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	Statistic (%)
5.195	.249	.255	−.006	4.498	4.488	.23
5.331	.254	.260	−.006	4.600	4.589	.24
5.662	.266	.265	.006	4.845	4.834	.22
5.666	.266	.270	−.004	4.848	4.837	.22
5.966	.277	.275	.007	5.067	5.056	.20
6.044	.280	.280	.005	5.123	5.113	.20
6.274	.288	.285	.008	5.288	5.278	.17
6.418	.293	.290	.008	5.390	5.381	.16
6.675	.301	.295	.011	5.571	5.564	.12
7.048	.314	.300	.019	5.829	5.827	.03
7.064	.314	.305	.014	5.840	5.838	.03
7.117	.316	.310	.011	5.876	5.875	.02
7.287	.322	.315	.012	5.992	5.992	−.01
7.343	.323	.320	.008	6.030	6.031	−.01
7.352	.324	.325	.004	6.036	6.037	−.01
7.366	.324	.330	−.006	6.045	6.046	−.01
7.613	.332	.335	−.003	6.211	6.212	.00
8.184	.349	.340	.014	6.588	6.591	−.05
8.247	.351	.345	.011	6.629	6.633	−.06
8.261	.352	.350	.007	6.638	6.642	−.06
8.265	.352	.355	−.003	6.640	6.645	−.06
8.274	.352	.360	−.008	6.646	6.651	−.06
8.920	.371	.365	.011	7.059	7.064	−.07
8.924	.371	.370	.006	7.061	7.066	−.07
8.960	.372	.375	−.003	7.084	7.089	−.07
9.035	.374	.380	−.006	7.131	7.136	−.07
9.045	.374	.385	−.011	7.137	7.142	−.07
9.205	.379	.390	−.011	7.237	7.241	−.05
9.323	.382	.395	−.013	7.310	7.313	−.03
10.128	.404	.400	.009	7.799	7.800	−.01
10.494	.413	.405	.013	8.015	8.019	−.05
10.535	.414	.410	.009	8.040	8.044	−.05
10.580	.415	.415	.005	8.066	8.070	−.05
10.652	.417	.420	−.003	8.108	8.112	−.05
11.295	.433	.425	.013	8.478	8.485	−.09
11.584	.439	.430	.014	8.641	8.651	−.12
11.967	.448	.435	.018	8.854	8.870	−.18
12.036	.450	.440	.015	8.892	8.909	−.19
12.826	.467	.445	.027	9.319	9.351	−.34
12.961	.470	.450	.025	9.391	9.426	−.37
13.041	.472	.455	.022	9.433	9.470	−.39
13.185	.475	.460	.020	9.509	9.549	−.41
13.187	.475	.465	.015	9.510	9.550	−.41
13.321	.478	.470	.013	9.580	9.621	−.43
13.690	.486	.475	.016	9.772	9.817	−.46
13.846	.489	.480	.014	9.852	9.899	−.48
13.998	.492	.485	.012	9.929	9.978	−.49
14.373	.499	.490	.014	10.118	10.171	−.52
14.447	.501	.495	.011	10.155	10.209	−.53
14.648	.505	.500	.010	10.255	10.310	−.54
14.699	.506	.505	.006	10.280	10.336	−.54
14.766	.507	.510	−.003	10.314	10.369	−.54
15.007	.511	.515	−.004	10.432	10.487	−.53
15.305	.517	.520	−.003	10.577	10.631	−.52
15.415	.519	.525	−.006	10.630	10.684	−.51

APPENDIX C

Pareto

200 Psuedo-Random Values

	K-S Test			EVC Test		
				Means		EVC
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	Statistic (%)
15,909	.528	.530	.003	10,865	10,919	−.50
15,983	.529	.535	−.006	10,900	10,954	−.49
16,282	.534	.540	−.006	11,040	11,093	−.48
16,479	.538	.545	−.007	11,131	11,183	−.47
16,685	.541	.550	−.009	11,226	11,277	−.45
16,710	.542	.555	−.013	11,238	11,288	−.45
16,818	.543	.560	−.017	11,287	11,336	−.44
17,036	.547	.565	−.018	11,386	11,432	−.40
17,241	.551	.570	−.019	11,479	11,521	−.37
17,428	.554	.575	−.021	11,563	11,602	−.34
18,259	.567	.580	−.013	11,928	11,955	−.23
18,452	.570	.585	−.015	12,011	12,036	−.21
19,123	.580	.590	−.010	12,297	12,315	−.15
19,755	.589	.595	−.006	12,559	12,574	−.11
19,862	.590	.600	−.010	12,603	12,617	−.11
19,905	.591	.605	−.014	12,621	12,634	−.10
19,985	.592	.610	−.018	12,654	12,666	−.10
20,146	.594	.615	−.021	12,719	12,729	−.07
20,275	.596	.620	−.024	12,771	12,778	−.05
20,390	.598	.625	−.027	12,818	12,822	−.03
21,889	.617	.630	−.013	13,406	13,384	.16
22,017	.619	.635	−.016	13,455	13,431	.17
22,172	.621	.640	−.019	13,514	13,488	.19
22,309	.623	.645	−.022	13,566	13,537	.21
22,362	.623	.650	−.027	13,586	13,556	.22
22,369	.623	.655	−.032	13,588	13,559	.22
23,919	.642	.660	−.018	14,158	14,093	.46
23,919	.642	.665	−.023	14,158	14,093	.46
24,469	.648	.670	−.022	14,353	14,278	.53
25,241	.656	.675	−.019	14,622	14,532	.61
25,384	.658	.680	−.022	14,671	14,579	.63
27,539	.679	.685	−.006	15,386	15,268	.76
28,520	.688	.690	.003	15,696	15,578	.75
28,545	.688	.695	−.007	15,704	15,585	.75
28,712	.690	.700	−.010	15,756	15,636	.76
30,016	.701	.705	−.004	16,153	16,027	.78
32,430	.720	.710	.015	16,851	16,740	.66
33,821	.731	.715	.021	17,232	17,143	.52
34,131	.733	.720	.018	17,316	17,231	.49
34,177	.733	.725	.013	17,328	17,244	.48
34,448	.735	.730	.010	17,400	17,319	.47
34,947	.738	.735	.008	17,531	17,453	.44
35,422	.742	.740	.007	17,655	17,579	.43
35,987	.745	.745	.005	17,800	17,726	.41
37,488	.755	.750	.010	18,175	18,109	.36
37,641	.756	.755	.006	18,213	18,147	.36
37,975	.758	.760	.003	18,294	18,229	.35
38,361	.760	.765	−.005	18,387	18,322	.36
38,498	.761	.770	−.009	18,420	18,354	.36
39,750	.768	.775	−.007	18,715	18,642	.39
40,137	.770	.780	−.010	18,804	18,729	.40
40,987	.775	.785	−.010	18,998	18,916	.43
43,817	.789	.790	.004	19,615	19,524	.46
44,606	.793	.795	.003	19,780	19,690	.46
45,150	.795	.800	−.005	19,892	19,802	.46

APPENDIX C

Pareto

200 Psuedo-Random Values

	K-S Test			EVC Test		
				Means		EVC Statistic
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	(%)
47,319	.805	.805	.005	20,326	20,235	.45
48,160	.808	.810	.003	20,489	20,399	.44
53,161	.827	.815	.017	21,401	21,350	.24
53,909	.829	.820	.014	21,529	21,488	.19
57,601	.841	.825	.021	22,137	22,152	−.07
59,057	.845	.830	.020	22,365	22,407	−.19
59,908	.848	.835	.018	22,495	22,552	−.25
62,120	.854	.840	.019	22,825	22,917	−.40
63,423	.857	.845	.017	23,013	23,125	−.49
64,290	.859	.850	.014	23,136	23,260	−.53
64,539	.860	.855	.010	23,171	23,297	−.54
66,841	.865	.860	.010	23,487	23,631	−.61
66,928	.866	.865	.006	23,499	23,643	−.61
69,947	.872	.870	.007	23,894	24,051	−.65
71,658	.876	.875	.006	24,110	24,273	−.68
71,830	.876	.880	−.004	24,131	24,295	−.68
72,206	.877	.885	−.008	24,178	24,340	−.67
74,097	.881	.890	−.009	24,407	24,557	−.62
74,454	.881	.895	−.014	24,450	24,596	−.60
74,806	.882	.900	−.018	24,491	24,633	−.58
97,519	.914	.905	.014	26,774	26,905	−.49
97,704	.915	.910	.010	26,790	26,922	−.50
98,579	.915	.915	.005	26,864	27,001	−.51
98,812	.916	.920	−.004	26,884	27,021	−.51
117,157	.932	.925	.012	28,270	28,488	−.77
118,540	.933	.930	.008	28,363	28,592	−.81
120,717	.935	.935	.005	28,507	28,745	−.83
126,789	.939	.940	.004	28,890	29,139	−.86
137,105	.945	.945	.005	29,490	29,758	−.91
151,162	.952	.950	.007	30,216	30,531	−1.04
158,996	.955	.955	.005	30,582	30,923	−1.11
165,704	.957	.960	−.003	30,877	31,225	−1.13
166,837	.958	.965	−.007	30,925	31,270	−1.12
210,571	.969	.970	.004	32,498	32,801	−.93
217,732	.971	.975	−.004	32,712	33,016	−.93
243,729	.975	.980	−.005	33.413	33,666	−.76
253,630	.977	.985	−.008	33,652	33,864	−.63
370,910	.987	.990	−.003	35,733	35,623	.31
616,233	.994	.995	.004	37,978	38,076	−.26
1,176,968	.998	1.000	.003	40,100	40,880	−1.94

Beta is 26,297
Delta is 1.586
The Truncation Point is 0
The Censorship Point is 10,000,000
The Sample Size is 200
Kolmogorov-Smirnov Test Statistic is 0.0317

APPENDIX D

Lognormal

200 Pseudo-Random Values

$$\mu = 9 \qquad \sigma = 2 \qquad \text{Prob}\,[X \le x] = \phi\left(\frac{\log x - \mu}{\sigma}\right)$$

$$\text{where } \phi(y) = (2\pi)^{-1}\int_{-\infty}^{y} \exp\left\{-\frac{t^2}{2}\right\} dt$$

2	1,904	5,449	12,301	39,866
20	1,938	5,552	12,371	42,942
21	1,996	5,696	12,606	45,129
89	2,007	5,785	12,626	45,665
134	2,067	5,859	13,690	45,859
135	2,123	5,897	14,090	46,175
164	2,134	5,900	15,759	47,292
165	2,233	5,918	16,359	47,477
186	2,321	6,208	17,134	47,580
236	2,369	6,553	17,298	50,698
402	2,376	6,804	17,649	50,707
438	2,380	6,875	17,949	58,131
451	2,497	6,901	18,682	58,441
526	2,631	6,929	19,696	61,890
582	2,672	6,934	19,789	64,181
601	2,873	7,010	19,874	66,391
639	2,879	7,047	21,275	67,898
676	2,970	7,737	21,305	70,527
850	2,974	7,750	24,569	80,932
911	3,275	7,980	24,600	82,360
914	3,394	8,047	26,571	83,122
1,029	3,397	8,220	27,021	83,849
1,052	3,407	8,448	27,290	83,917
1,053	3,505	8,623	27,969	88,095
1,071	3,584	8,784	28,212	104,508
1,073	3,746	9,029	29,088	112,291
1,102	3,772	9,118	29,205	113,729
1,182	3,903	9,326	29,507	122,065
1,185	3,924	9,356	30,927	129,896
1,258	3,997	9,475	32,490	132,125
1,337	4,660	9,896	32,657	168,200
1,340	4,780	9,989	33,929	209,599
1,357	4,794	10,145	34,797	225,688
1,501	4,816	10,272	35,149	260,210
1,627	5,020	10,429	35,194	307,687
1,669	5,041	10,551	35,261	375,796
1,798	5,074	10,675	35,669	463,569
1,825	5,154	10,679	37,859	510,905
1,836	5,206	12,079	38,049	861,999
1,903	5,354	12,274	39,150	1,684,380

APPENDIX E

Part 1

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $300,000

Loss Amount	Number of Losses	Average Loss Amount
0- 250	10,075	88
250- 500	3,049	374
500- 1,000	3,263	783
1,000- 2,000	2,690	1,490
2,000- 3,000	1,498	2,543
3,000- 4,000	964	3,521
4,000- 5,000	794	4,777
5,000- 6,000	261	5,629
6,000- 7,000	191	6,600
7,000- 8,000	406	7,429
8,000- 9,000	114	8,500
9,000- 10,000	279	9,736
10,000- 11,000	58	10,430
11,000- 12,000	56	11,279
12,000- 13,000	47	12,572
13,000- 14,000	20	13,541
14,000- 15,000	151	14,965
15,000- 16,000	28	15,501
16,000- 17,000	16	16,471
17,000- 18,000	24	17,643
18,000- 19,000	9	18,713
19,000- 20,000	74	19,950
20,000- 21,000	16	20,445
21,000- 22,000	7	21,753
22,000- 23,000	12	22,658
23,000- 24,000	4	23,756
24,000- 25,000	70	24,960
25,000- 30,000	44	28,377
30,000- 35,000	30	33,888
35,000- 40,000	25	38,610
40,000- 45,000	15	43,106
45,000- 50,000	40	49,834
50,000- 55,000	3	51,146
55,000- 60,000	8	59,813
60,000- 65,000	4	64,247
65,000- 70,000	1	70,000
70,000- 75,000	8	75,000
75,000- 80,000	2	78,618
80,000- 85,000	5	82,425
95,000-100,000	12	99,404
100,000-110,000	3	104,556
110,000-120,000	1	120,000
120,000-130,000	5	124,463
140,000-150,000	4	150,000
150,000-160,000	4	155,128
160,000-170,000	1	161,000
170,000-180,000	2	173,398
190,000-200,000	2	197,495
230,000-240,000	2	233,449
240,000-250,000	1	250,000
250,000-260,000	1	252,800
270,000-280,000	1	273,747
280,000-290,000	1	287,540
300,000 and over	10	300,000
Total	24,411	2,279

APPENDIX E

Part 2

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $300,000

	K-S Test			EVC Test		
				Means		EVC Statistic
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	(%)
8,000	.000	.000	.000	0	0	.00
8,500	.072	.093	.072	481	500	−3.82
9,736	.217	.322	.124	1,534	1,620	−5.60
10,430	.283	.370	−.087	2,054	2,091	−1.77
11,279	.351	.416	−.065	2,633	2,626	.29
12,572	.436	.454	.021	3,415	3,381	.99
13,541	.488	.470	.034	3,936	3,910	.65
14,965	.551	.594	.081	4,618	4,664	−1.00
15,501	.572	.617	−.045	4,853	4,882	−.60
16,471	.605	.630	−.025	5,252	5,253	−.03
17,643	.640	.650	−.010	5,694	5,686	.13
18,713	.667	.657	.017	6,064	6,061	.05
19,950	.695	.718	.037	6,458	6,485	−.41
20,445	.705	.731	−.026	6,607	6,624	−.26
21,753	.729	.737	−.008	6,977	6,976	.02
22,658	.743	.747	.006	7,216	7,214	.03
23,756	.759	.750	.013	7,489	7,492	−.04
24,960	.775	.807	−.032	7,769	7,793	−.31
28,377	.812	.843	−.032	8,472	8,451	.24
33,888	.853	.868	−.015	9,389	9,314	.80
38,610	.877	.889	−.011	10,024	9,937	.86
43,106	.895	.901	.006	10,535	10,438	.91
49,834	.914	.934	−.019	11,173	11,106	.60
51,146	.917	.936	−.019	11,284	11,193	.81
59,813	.934	.943	−.009	11,924	11,747	1.49
64,247	.940	.946	−.006	12,204	12,001	1.66
70,000	.947	.947	.001	12,527	12,313	1.71
75,000	.952	.953	.005	12,779	12,579	1.57
78,618	.955	.955	.002	12,947	12,748	1.54
82,425	.958	.959	.003	13,113	12,920	1.47
99,404	.968	.969	.009	13,735	13,616	.87
104,556	.970	.971	.001	13,895	13,776	.85
120,000	.975	.972	.004	14,312	14,219	.65
124,463	.977	.976	.005	14,419	14,343	.52
150,000	.982	.980	.006	14,937	14,950	−.09
155,128	.983	.983	.004	15,026	15,056	−.20
161,000	.984	.984	.001	15,123	15,157	−.22
173,398	.986	.985	.002	15,312	15,360	−.31
197,495	.988	.987	.003	15,628	15,715	−.56
233,449	.991	.989	.004	16,009	16,187	−1.11
250,000	.991	.989	.003	16,157	16,377	−1.36
252,800	.992	.990	.002	16,181	16,407	−1.40
273,747	.993	.991	.002	16,346	16,613	−1.63
287,540	.993	.992	.002	16,446	16,737	−1.77
300,000	1.000	1.000	.000	16,530	16,839	−1.87

Beta is 1462.8
Delta is 1.4532
The Truncation Point is 8,000
The Censorship Point is 300,000
The Sample Size is 1,220
Kolmogorov-Smirnov Test Statistic is 0.1236

APPENDIX E

Part 3

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $300,000

	K-S Test			EVC Test		
				Means		EVC Statistic
X	Pareto CDF F(X)	Sample CDF S(X)	Maximum Difference	Pareto	Sample	(%)
88	.180	.413	−.233	79	88	−10.68
374	.473	.538	−.065	265	255	3.59
783	.645	.671	.107	440	444	−.91
1,490	.768	.781	.097	642	677	−5.50
2,543	.844	.843	.063	840	907	−7.94
3,521	.879	.882	.036	974	1,061	−8.89
4,777	.906	.915	.023	1,108	1,209	−9.09
5,629	.918	.926	−.008	1,183	1,281	−8.29
6,600	.928	.933	−.006	1,258	1,353	−7.59
7,429	.935	.950	−.016	1,315	1,409	−7.12
8,500	.942	.955	−.013	1,381	1,462	−5.86
9,736	.948	.966	−.018	1,449	1,518	−4.74
10,430	.951	.968	−.018	1,485	1,542	−3.85
11,279	.954	.971	−.017	1,525	1,568	−2.85
12,572	.958	.973	−.015	1,582	1,606	−1.55
13,541	.961	.974	−.013	1,621	1,633	−.71
14,965	.964	.980	−.016	1,675	1,670	.27
15,501	.965	.981	−.016	1,694	1,681	.75
16,471	.967	.982	−.015	1,727	1,700	1.58
17,643	.969	.983	−.014	1,765	1,721	2.47
18,713	.970	.983	−.013	1,798	1,740	3.20
19,950	.972	.986	−.014	1,834	1,761	3.94
20,445	.972	.987	−.014	1,847	1,768	4.28
21,753	.974	.987	−.013	1,883	1,786	5.14
22,658	.975	.987	−.013	1,906	1,798	5.68
23,756	.976	.988	−.012	1,933	1,812	6.29
24,960	.977	.990	−.014	1,962	1,827	6.89
28,377	.979	.992	−.013	2,037	1,860	8.71
33,888	.982	.993	−.011	2,143	1,903	11.22
38,610	.984	.994	−.010	2,223	1,934	12.99
43,106	.986	.995	−.010	2,291	1,959	14.49
49,834	.987	.997	−.009	2,382	1,992	16.38
51,146	.988	.997	−.009	2,399	1,997	16.77
59,813	.989	.997	−.008	2,500	2,024	19.03
64,247	.990	.997	−.008	2,547	2,037	20.01
70,000	.991	.997	−.007	2,604	2,053	21.16
75,000	.991	.998	−.007	2,650	2,066	22.03
78,618	.991	.998	−.006	2,681	2,074	22.63
82,425	.992	.998	−.006	2,713	2,083	23.23
99,404	.993	.998	−.005	2,842	2,118	25.48
104,556	.993	.999	−.005	2,877	2,126	26.11
120,000	.994	.999	−.005	2,974	2,148	27.78
124,463	.994	.999	−.005	3,000	2,154	28.20
150,000	.995	.999	−.004	3,135	2,184	30.33
155,128	.995	.999	−.004	3,160	2,190	30.70
161,000	.995	.999	−.004	3,187	2,195	31.14
173,398	.996	.999	−.004	3,242	2,205	32.00
197,495	.996	.999	−.003	3,340	2,223	33.45
233,449	.997	.999	−.003	3,468	2,246	35.23
250,000	.997	.999	−.003	3,521	2,256	35.94
252,800	.997	1.000	−.003	3,530	2,257	36.05
273,747	.997	1.000	−.002	3,593	2,267	36.88
287,540	.997	1.000	−.002	3,631	2,274	37.39
300,000	1.000	1.000	.000	3,665	2,279	37.82

Beta is 347.2
Delta is 0.8768

APPENDIX F

Part 1

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $500,000

Loss Amount	Number of Losses	Average Loss Amount
0- 250	3,977	83
250- 500	1,095	374
500- 1,000	1,152	774
1,000- 2,000	991	1,488
2,000- 3,000	594	2,520
3,000- 4,000	339	3,538
4,000- 5,000	307	4,770
5,000- 6,000	103	5,542
6,000- 7,000	79	6,477
7,000- 8,000	141	7,568
8,000- 9,000	52	8,674
9,000- 10,000	89	9,853
10,000- 11,000	23	10,420
11,000- 12,000	22	11,744
12,000- 13,000	23	12,551
13,000- 14,000	6	13,733
14,000- 15,000	51	14,960
15,000- 16,000	6	15,374
16,000- 17,000	5	16,700
17,000- 18,000	9	17,634
18,000- 19,000	3	18,801
19,000- 20,000	31	19,973
20,000- 21,000	2	20,502
21,000- 22,000	2	21,926
22,000- 23,000	5	22,530
23,000- 24,000	4	23,745
24,000- 25,000	31	24,968
25,000- 30,000	11	29,391
30,000- 35,000	18	34,249
35,000- 40,000	9	38,564
40,000- 45,000	4	43,718
45,000- 50,000	11	49,814
50,000- 55,000	3	52,333
55,000- 60,000	2	60,000
70,000- 75,000	9	74,750
75,000- 80,000	1	75,003
95,000-100,000	4	99,913
120,000-130,000	2	125,000
140,000-150,000	3	150,000
190,000-200,000	1	200,000
200,000-210,000	2	202,453
220,000-230,000	1	225,000
240,000-250,000	2	250,000
260,000-270,000	1	270,000
280,000-290,000	1	290,000
290,000-300,000	2	300,000
340,000-350,000	1	350,000
410,000-420,000	2	414,619
500,000 and over	0	500,000
Total	9,232	2,410

APPENDIX F

Part 2

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1975 as of March 31, 1978
Policy Limit $300,000

Loss Amount	Number of Losses	Average Loss Amount
0- 250	12,075	83
250- 500	3,420	381
500- 1,000	3,245	771
1,000- 2,000	2,623	1,509
2,000- 3,000	1,546	2,535
3,000- 4,000	877	3,557
4,000- 5,000	823	4,710
5,000- 6,000	308	5,471
6,000- 7,000	225	6,526
7,000- 8,000	384	7,451
8,000- 9,000	142	8,489
9,000- 10,000	279	9,792
10,000- 11,000	69	10,473
11,000- 12,000	76	11,711
12,000- 13,000	69	12,141
13,000- 14,000	30	13,750
14,000- 15,000	154	14,937
15,000- 16,000	32	15,574
16,000- 17,000	17	16,617
17,000- 18,000	33	17,601
18,000- 19,000	17	18,626
19,000- 20,000	91	19,907
20,000- 21,000	17	20,578
21,000- 22,000	9	21,900
22,000- 23,000	19	22,758
23,000- 24,000	12	23,667
24,000- 25,000	88	24,963
25,000- 30,000	65	28,364
30,000- 35,000	45	31,998
35,000- 40,000	41	39,018
40,000- 45,000	18	42,848
45,000- 50,000	61	49,721
50,000- 55,000	9	52,953
55,000- 60,000	6	59,568
60,000- 65,000	6	63,489
65,000- 70,000	6	67,598
70,000- 75,000	14	74,920
75,000- 80,000	2	78,260
80,000- 85,000	4	83,890
85,000- 90,000	7	88,705
90,000- 95,000	3	94,196
95,000-100,000	7	99,857
100,000-110,000	6	105,439
110,000-120,000	2	120,000
120,000-130,000	5	125,391
130,000-140,000	4	136,889
140,000-150,000	3	149,882
150,000-160,000	3	154,748
160,000-170,000	1	168,140
170,000-180,000	3	175,000
180,000-190,000	1	185,000
190,000-200,000	2	200,000
200,000-210,000	1	203,765
210,000-220,000	2	212,017
220,000-230,000	1	225,000
250,000-260,000	1	260,000
270,000-280,000	1	275,146
290,000-300,000	1	294,054
300,000 and over	6	300,000
Total	27,017	2,329

APPENDIX F

Part 3

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1975 as of March 31, 1978
Policy Limit $500,000

Loss Amount	Number of Losses	Average Loss Amount
0- 250	3,286	78
250- 500	837	389
500- 1,000	928	774
1,000- 2,000	687	1,498
2,000- 3,000	412	2,577
3,000- 4,000	267	3,593
4,000- 5,000	263	4,757
5,000- 6,000	89	5,569
6,000- 7,000	64	6,631
7,000- 8,000	108	7,543
8,000- 9,000	35	8,563
9,000- 10,000	83	9,904
10,000- 11,000	15	10,432
11,000- 12,000	22	11,667
12,000- 13,000	22	12,624
13,000- 14,000	15	13,517
14,000- 15,000	52	14,945
15,000- 16,000	11	15,364
16,000- 17,000	5	16,749
17,000- 18,000	15	17,650
18,000- 19,000	1	19,000
19,000- 20,000	27	19,918
20,000- 21,000	7	20,351
21,000- 22,000	1	22,000
22,000- 23,000	4	22,487
23,000- 24,000	1	24,000
24,000- 25,000	33	24,980
25,000- 30,000	11	27,915
30,000- 35,000	9	33,655
35,000- 40,000	18	38,794
40,000- 45,000	7	43,611
45,000- 50,000	6	49,917
55,000- 60,000	2	60,000
60,000- 65,000	5	63,075
65,000- 70,000	2	67,090
70,000- 75,000	9	74,294
75,000- 80,000	3	75,900
80,000- 85,000	3	82,016
85,000- 90,000	2	87,505
95,000-100,000	5	98,586
110,000-120,000	3	116,177
120,000-130,000	1	123,528
140,000-150,000	1	150,000
150,000-160,000	2	150,100
240,000-250,000	1	250,000
290,000-300,000	1	300,000
300,000-310,000	1	309,000
330,000-340,000	1	335,675
340,000-350,000	1	349,910
480,000-490,000	1	483,840
500,000 and over	3	500,000
Total	7,388	2,849

APPENDIX G

Part 1

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1975 as of March 31, 1978
Policy Limit $300,000
Trended to July 1, 1980 by 18.9% per annum

	K-S Test			EVC Test		
				Means		EVC
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	Statistic (%)
17,434	.000	.000	.000	0	0	.00
18,499	.066	.095	.066	1,028	1,064	−3.49
21,339	.211	.281	.116	3,466	3,635	−4.88
22,825	.272	.328	−.055	4,591	4,702	−2.42
25,523	.365	.378	.038	6,425	6,517	−1.43
26,460	.393	.424	−.032	7,006	7,099	−1.32
29,965	.480	.445	.056	8,975	9,116	−1.58
31,945	.521	.467	.076	9,962	10,216	−2.54
32,553	.532	.570	.065	10,250	10,540	−2.83
36,213	.592	.582	.022	11,848	12,113	−2.24
38,358	.622	.604	.040	12,691	13,011	−2.52
40,593	.649	.615	.045	13,506	13,897	−2.90
43,383	.678	.676	.063	14,444	14,971	−3.65
44,846	.692	.687	.016	14,905	15,445	−3.63
47,727	.716	.693	.029	15,756	16,347	−3.75
49,597	.731	.706	.037	16,273	16,920	−3.98
51,579	.744	.714	.039	16,793	17,503	−4.23
54,401	.762	.773	.048	17,489	18,311	−4.70
61,814	.800	.816	.027	19,105	19,995	−4.66
69,734	.830	.846	−.016	20,563	21,451	−4.32
85,031	.871	.874	.025	22,822	23,803	−4.30
93,379	.887	.886	.013	23,830	24,858	−4.31
108,357	.908	.926	.023	25,355	26,570	−4.79
115,401	.916	.932	−.016	25,973	27,088	−4.29
129,817	.929	.936	−.008	27,087	28,061	−3.60
138,363	.935	.941	−.005	27,668	28,604	−3.38
147,316	.941	.945	−.004	28,223	29,136	−3.23
163,273	.949	.954	−.005	29,104	30,022	−3.15
170,553	.952	.955	−.003	29,466	30,357	−3.03
182,822	.956	.958	−.002	30,028	30,907	−2.93
193,316	.960	.963	−.003	30,468	31,349	−2.89
205,281	.963	.965	−.002	30,929	31,797	−2.80
217,619	.966	.969	−.003	31,366	32,234	−2.77
229,784	.969	.973	−.005	31,763	32,608	−2.66
261,517	.974	.975	.001	32,671	33,456	−2.41
273,266	.976	.978	−.002	32,967	33,755	−2.39
298,323	.978	.981	−.002	33,542	34,308	−2.28
326,640	.981	.983	−.001	34,112	34,856	−2.18
337,243	.982	.985	−.003	34,308	35,041	−2.14
366,429	.984	.985	−.001	34,803	35,489	−1.97
381,379	.985	.987	−.002	35,035	35,709	−1.92
403,172	.986	.988	−.002	35,351	35,986	−1.80
435,861	.988	.989	−.002	35,780	36,379	−1.68
444,066	.988	.990	−.002	35,880	36,467	−1.64
462,049	.989	.991	−.003	36,092	36,647	−1.54
490,344	.990	.992	−.002	36,400	36,893	−1.35
566,620	.992	.993	−.001	37,117	37,505	−1.04
599,628	.992	.993	−.001	37,386	37,748	−.97
640,834	.993	.994	−.001	37,692	38,023	−.88
653,792	1.000	1.000	.000	37,782	38,101	−.85

Beta is 4955.2
Delta is 1.4728
The Truncation Point is 17,434
The Censorship Point is 653,792
The Sample Size is 1,496
Kolmogorov-Smirnov Test Statistic is 0.1159

APPENDIX G

Part 2

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1975 as of March 31, 1978
Policy Limit $500,000
Trended to July 1, 1980 by 18.9% per annum

	K-S Test			EVC Test		
				Means		
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	EVC Statistic (%)
17,434	.000	.000	.000	0	0	.00
18,661	.076	.078	.076	1,179	1,227	−4.02
21,584	.222	.263	.143	3,657	3,921	−7.20
22,734	.269	.297	−.028	4,525	4,768	−5.37
25,425	.362	.346	.065	6,363	6,660	−4.67
27,510	.421	.395	.075	7,629	8,024	−5.17
29,457	.469	.429	.074	8,708	9,201	−5.67
32,569	.533	.545	.104	10,258	10,979	−7.03
33,482	.549	.569	−.020	10,677	11,396	−6.73
36,501	.596	.580	.027	11,965	12,696	−6.11
38,465	.623	.614	.043	12,731	13,520	−6.20
41,407	.658	.616	.044	13,788	14,656	−6.30
43,408	.678	.676	.062	14,452	15,424	−6.73
44,352	.687	.692	.011	14,751	15,730	−6.63
47,945	.718	.694	.026	15,818	16,837	−6.44
49,006	.726	.703	.032	16,112	17,161	−6.51
52,303	.749	.705	.046	16,976	18,140	−6.86
54,438	.762	.779	.057	17,498	18,769	−7.27
60,836	.796	.804	.017	18,907	20,183	−6.75
73,344	.842	.824	.038	21,154	22,640	−7.02
84,544	.870	.864	.046	22,759	24,615	−8.15
95,042	.890	.879	.026	24,016	26,044	−8.45
108,784	.909	.893	.029	25,394	27,701	−9.08
130,758	.930	.897	.037	27,153	30,055	−10.69
137,460	.934	.908	.037	27,609	30,743	−11.35
146,210	.940	.913	.031	28,157	31,544	−12.03
161,910	.948	.933	.035	29,034	32,911	−13.35
165,409	.950	.940	.017	29,213	33,145	−13.46
178,737	.955	.946	.015	29,847	33,948	−13.74
190,700	.959	.951	.013	30,362	34,589	−13.92
214,849	.965	.962	.015	31,271	35,775	−14.40
253,186	.973	.969	.011	32,448	37,230	−14.74
269,205	.975	.971	.006	32,867	37,730	−14.80
326,896	.981	.973	.010	34,117	39,404	−15.50
327,114	.981	.978	.008	34,121	39,410	−15.50
544,827	.991	.980	.013	36,928	44,270	−19.88
653,792	.993	.982	.013	37,782	46,459	−22.97
673,406	.993	.984	.011	37,914	46,809	−23.46
731,539	.994	.987	.010	38,274	47,718	−24.67
762,561	.995	.989	.008	38,449	48,133	−25.19
1,054,436	.997	.991	.008	39,707	51,391	−29.42
1,089,653	1.000	1.000	.000	39,825	51,705	−29.83

Beta is 4955.2
Delta is 1.4728
The Truncation Point is 17,434
The Censorship Point is 1,089,653
The Sample Size is 448
Kolmogorov-Smirnov Test Statistic is 0.1433

APPENDIX G

Part 3

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $300,000
Trended to July 1, 1980 by 18.9% per annum

	K-S Test			EVC Test		
				Means		EVC Statistic
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	(%)
14,663	.000	.000	.000	0	0	.00
15,580	.065	.094	.065	886	916	−3.43
17,845	.199	.324	−.125	2,846	2,969	−4.31
19,118	.260	.371	−.111	3,826	3,829	−.09
20,673	.325	.418	−.092	4,924	4,807	2.39
23,044	.408	.456	−.049	6,423	6,188	3.66
24,819	.459	.473	−.014	7,427	7,153	3.70
27,430	.522	.597	−.075	8,754	8,529	2.57
28,412	.543	.620	−.078	9,214	8,925	3.14
30,189	.576	.633	−.057	9,996	9,600	3.97
32,338	.612	.653	−.041	10,867	10,387	4.42
34,298	.640	.661	−.021	11,600	11,067	4.60
36,565	.669	.722	−.053	12,383	11,836	4.42
37,473	.679	.735	−.056	12,679	12,089	4.66
39,871	.704	.741	−.037	13,419	12,725	5.17
41,530	.719	.750	−.031	13,897	13,156	5.34
43,542	.736	.754	−.017	14,444	13,658	5.45
45,749	.753	.811	−.058	15,008	14,201	5.37
52,011	.792	.848	−.056	16,426	15,383	6.35
62,113	.836	.872	−.036	18,288	16,922	7.47
70,768	.863	.893	−.030	19,582	18,027	7.94
79,008	.883	.905	−.023	20,627	18,909	8.32
91,340	.904	.938	−.034	21,936	20,078	8.47
93,746	.907	.941	−.033	22,163	20,226	8.74
109,630	.926	.947	−.022	23,479	21,168	9.84
117,757	.933	.951	−.018	24,054	21,597	10.21
128,302	.940	.951	−.011	24,720	22,118	10.53
137,467	.946	.958	−.012	25,239	22,563	10.60
144,098	.950	.960	−.010	25,585	22,842	10.72
151,076	.953	.964	−.011	25,926	23,124	10.81
182,197	.964	.974	−.010	27,208	24,252	10.87
191,640	.966	.976	−.010	27,537	24,500	11.03
219,947	.972	.977	−.004	28,397	25,177	11.34
228,127	.974	.981	−.007	28,616	25,365	11.36
274,934	.980	.984	−.004	29,683	26,252	11.56
284,333	.981	.988	−.007	29,866	26,399	11.61
295,095	.982	.988	−.006	30,065	26,532	11.75
317,820	.984	.990	−.006	30,453	26,794	12.01
361,987	.987	.992	−.005	31,102	27,231	12.45
427,886	.990	.993	−.004	31,883	27,774	12.89
458,223	.991	.994	−.004	32,186	27,974	13.09
463,355	.991	.995	−.004	32,234	28,003	13.13
501,748	.992	.996	−.004	32,572	28,193	13.45
527,029	.992	.997	−.004	32,775	28,297	13.66
549,867	1.000	1.000	.000	32,947	28,372	13.88

Beta is 4955.2
Delta is 1.4728
The Truncation Point is 14,663
The Censorship Point is 549,867
The Sample Size is 1,214
Kolmogorov-Smirnov Test Statistic is 0.1251

APPENDIX G

Part 4

ISO Owners, Landlords and Tenants Bodily Injury Liability

Loss Amount Data: Policy Year 1976 as of March 31, 1978
Policy Limit $500,000
Trended to July 1, 1980 by 18.9% per annum

	K-S Teat			EVC Test		
				Means		EVC Statistic
X	Pareto CDF $F(X)$	Sample CDF $S(X)$	Maximum Difference	Pareto	Sample	(%)
14,663	.000	.000	.000	0	0	.00
15,899	.086	.114	.086	1,181	1,235	−4.61
18,060	.210	.309	−.100	3,017	3,150	−4.41
19,099	.259	.360	−.100	3,812	3,867	−1.46
21,526	.357	.408	−.051	5,486	5,422	1.17
23,005	.407	.458	−.052	6,400	6,298	1.59
25,172	.468	.471	.010	7,616	7,471	1.91
27,420	.522	.583	−.062	8,750	8,659	1.03
28,179	.538	.596	−.059	9,107	8,976	1.44
30,609	.584	.607	−.024	10,172	9,956	2.13
32,322	.611	.627	−.016	10,861	10,628	2.14
34,460	.642	.634	.015	11,658	11,426	2.00
36,608	.669	.702	.035	12,397	12,212	1.49
37,578	.680	.706	−.026	12,713	12,502	1.66
40,187	.707	.711	−.004	13,512	13,268	1.80
41,294	.717	.721	.007	13,831	13,589	1.75
43,522	.736	.730	.015	14,439	14,209	1.59
45,763	.753	.798	−.045	15,011	14,814	1.31
53,871	.802	.822	−.021	16,804	16,450	2.11
62,775	.839	.862	−.023	18,396	18,031	1.98
70,684	.863	.882	−.019	19,571	19,124	2.28
80,130	.885	.890	−.006	20,757	20,242	2.48
91,304	.904	.914	.014	21,932	21,468	2.12
95,921	.910	.921	−.011	22,361	21,863	2.23
109,973	.926	.925	.005	23,505	22,972	2.27
137,009	.946	.945	.020	25,215	24,988	.90
137,472	.946	.947	−.001	25,240	25,013	.90
183,129	.964	.956	.017	27,242	27,416	−.64
229,111	.974	.961	.018	28,642	29,433	−2.76
274,934	.980	.967	.020	29,683	31,242	−5.25
366,578	.987	.969	.020	31,163	34,256	−9.93
371,074	.987	.974	.018	31,222	34,394	−10.16
412,400	.989	.976	.015	31,716	35,482	−11.87
458,223	.991	.980	.015	32,186	36,587	−13.68
494,880	.992	.982	.011	32,515	37,311	−14.75
531,538	.992	.985	.010	32,810	37,954	−15.68
549,867	.993	.989	.008	32,947	38,235	−16.05
641,512	.994	.991	.005	33,543	39,240	−16.99
759,951	.995	.996	.004	34,150	40,279	−17.95
916,445	1.000	1.000	.000	34,769	40,965	−17.82

Beta is 4955.2
Delta is 1.4728
The Truncation Point is 14,663
The Censorship Point is 916,445
The Sample Size is 456
Kolmogorov-Smirnov Test Statistic is 0.1003

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