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AN ALGORITHM FOR PREMIUM ADJUSTMENT WITH AVAILABLE DATA

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INTRODUCTION

An important part of performing a loss ratio type of rate adequacy study is the ability to restate historical earned premiums at the level implied by the present rate structure. Of course, the most straight-forward and desirable way of accomplishing this restatement is by the extension-of-exposures method. That is, the historical book of business is actually rerated by using today's rate book. With the power of present computers such a procedure is practical if the required exposure information exists in a reliable form. However, the practicing actuary may find that for many lines of insurance reliable exposure information in the required level of detail is not available. Further, the extension-of-exposures procedure requires specialized data processing talents, which may not always be readily available.

Clearly, an approximation method uses realistic assumptions, produces reasonable results, and uses a mathematical simulation of the earnings process is desirable. One approximation method, which has long been popular, is the so-called rectangular method, as explained in Kallop's[1] article on workers' compensation ratemaking. This will be referred to as the traditional method of premium adjustment. A second alternative would be to take the historical written premiums and attempt to approximate the earning process with adjustments for rate level changes. This procedure will be impractical for heavily audited lines where exposure earned premium may be available, but where writings are on a calendar year basis.

[1] Kallop, R., "A Current Look at Worker's Compensation Ratemaking," *PCAS*, LXII (1975), p. 62.

The method introduced in this article attempts to make efficient use of the minimum amount of information, earned premiums and rate change history, that must always be available. Given the available data, the basic concept is to build a straight line approximation to the historical rate of premium writings. This straight line approximating function is determined by the requirements that (1) actual earned premiums are produced by the model, (2) the straight line segments form a continuous curve, and (3) the rate of writings is expressed in terms of the base rate level. Additionally, this algorithm allows the actuary to introduce certain known qualitative information. This is accomplished by designing an "objective function" which is to be minimized, that chooses a "minimal" element from the family of continuous piecewise linear functions that satisfy the above three requirements.

The following discussion shows how the model may be used to obtain better approximations to restated earned premiums at present rates. The model of premium writings may also be useful in quantifying marketing results in terms of measuring an annualized rate of writing at a constant rate level, which is directly proportional to exposure writings. In terms of corporate planning models, if future expected earned premiums are projected and future rate change strategy plotted, this algorithm will produce a required "rate of writings" that allows agents' writings to be monitored month by month to determine if marketing performance is actually fulfilling standards required to meet corporate earned premium projections. With regard to fire insurance, the exposure related premium writings resulting from the algorithm can be modified to reflect increasing amounts of insurance in the final adjusted earned premiums.

BASIC CONSTRUCTION OF THE ALGORITHM

The mathematics of the algorithm can be conveniently developed in terms of linear algebra. For purposes of exposition it is preferable to present a detailed example of the calculations. An object of this demonstration is to familiarize the reader with the idea of choosing a "best" element from a family of approximations as a useful actuarial tool which can easily be modified to meet a particular problem. A mathematical appendix presents the algorithm in terms of matrix algebra, thus making it simple to program the calculations using a mathematical programming language such as APL.

The algorithm exploits analytic expressions for the premium earning process which have recently been made available. For example, if ERCON $(y_0, y_1; x_0, x_1; t)$ represents the contribution to earnings during the time period from x_0 to x_1 of the writings during the time period from y_0 to y_1, Ross[2] presents the following formula:

$$(1)\ \text{ERCON}(y_0, y_1; x_0, x_1; t) = \int_{x_0-t}^{x_0} \frac{(x - x_0 + t)}{t} g(x)\,dx + \int_{x_0}^{x_1-t} g(x)\,dx + \int_{x_1-t}^{x_1} \frac{(x_1 - x)}{t} g(x)\,dx,$$

where t = term of the policies, and $g(x) = \begin{cases} f(x) \text{ if } y_0 \leq x \leq y_1, \\ 0 \text{ otherwise} \end{cases}$

and $f(x)$ is the exposure related rate of premium writings at time x. This formula assumes that $x_0 \leq x_1 - t$.

Miller and Davis[2] also give formulas for the earning process which will yield the following expression for the earned contributions from a period of writings:

$$(2)\ \text{ERCON}(y_0, y_1; x_0, x_1; t) = \frac{1}{t}\int_{x_0}^{x_1} \int_{a(x)}^{b(x)} f(x - y)\,dy\,dx,$$

where $a(x) = \min(\max(x - y_0, 0), t)$, $b(x) = \max(\min(x - y_1, t), 0)$ and (y_0, y_1) is the period of premium writings, (x_0, x_1) is the earning period, and t is the term of the policies. The proof that these two expressions are actually equivalent is recommended as an exercise for the mathematically inclined reader.

Other formulations for the same process may also be derived. Whichever expression is used, the actuary is always faced with the same problem: he must come up with a rate of exposure (or premium) writings. This rate is a handy theoretical concept which makes the analytic formulas work; unfortunately, it cannot be observed or measured under any practical situations. The best data actuaries can come up with is aggregate writings, i.e., $\int_{y_0}^{y_1} f(y)\,dy$ for some time period (y_0, y_1). Even if such writings are properly related to exposures, we are still faced with the problem of conjuring up the associated rate function $f(x)$ to continue the analysis.

[2] Miller, D. J., and Davis, G. E., "A Refined Model for Premium Adjustment," *PCAS*, LXIII (1976), p. 117.

The approach taken in the design of this algorithm is to start from the assumption that premium writings for a time period can be described by the linear rate of writings function $f(x) = Ax + B$. A different pair of parameters (A, B) is allowed for each writing period, with the continuity condition that the different line segments must meet at common end points of the writing periods. The condition that the model must produce the collected earned premiums makes use of the analytic expressions of the earnings process. With the assumption $f(x) = Ax + B$, we can use an equation such as (2) and calculate:

$$\frac{1}{t}\int_{x_0}^{x_1}\int_{\min(\max(x-y_0,0),t)}^{\max(\min(x-y_1,t),0)} (A(x-y)+B)\,dydx$$
$$= AH(y_0, y_1; x_0, x_1; t) + BG(y_0, y_1; x_0, x_1; t).$$

That is, for each earning and writing period, we obtain numerical coefficients for the unknown parameters A and B of the model. The explicit formulation of H and G is not given because the argument is complicated by the limits of integration. The calculation for any specific (x_0, x_1) and (y_0, y_1) is quite straight-forward. The general formula of H and G is not as easily written and its detailed development adds nothing to the basic demonstration (see Appendix 2). In a practical situation, it is best to program a routine that can handle the necessary logic for limits of integration.

Suppose the data is given as in Table 1, showing earned premiums for the three years 1974, 1975, 1976 and the rate change history for the years 1973 through 1976. We are assuming that the policy term is one year. Table 2 shows the organization of the given data and the results of using formula (1) to calculate the coefficients of the parameters A_i and B_i. Since there are four periods of written premium, there are eight parameters $(A_i, B_i; i = 1,2,3,4)$ to be determined.

TABLE 1

PREMIUM AND RATE CHANGE HISTORY

Accident Year	Earned Premium	Rate Change History	
1974	1600	4/1/73	+15%
1975	1820	7/1/74	+10%
1976	1860	1/1/75	−5%
		5/1/76	+20%

According to Table 2, the contribution of writings from 1/1/74 to 6/30/74 to the earnings of 1974 can be expressed as $.45833A_2 + .375B_2$. To obtain these calculations, consider a time line with 1/1/73 as 0.0 and 1/1/74 as 1.0. The contribution of the writings of one-year policies from 1/1/74 (1.0) to 6/30/74 (1.5) to the earnings of 1974 (the interval 1.0 to 2.0) can be written, using (1):

$$\text{ERCON}\,(1.0, 1.5; 1.0, 2.0; 1) = \int_{1.0}^{1.5} (A_2x + B_2)(2 - x)\,dx =$$

$$A_2 \int_{1.0}^{1.5} (2x - x^2)\,dx + B_2 \int_{1.0}^{1.5} (2 - x)\,dx = (.45833)\,A_2 + (.375)\,B_2.$$

That is, $H(1.0, 1.5; 1.0, 2.0; 1) = .45833$ and $G(1.0, 1.5; 1.0, 2.0; 1) = .375$.

Once Table 2 has been calculated, one can immediately write down three expressions for the historical earned premiums of the three years. For example, the written premiums generating 1975 earned premiums were written at three different rate levels — 1.150 from 1/1/74 to 6/30/74, 1.265 from 7/1/74 to 12/31/74, and 1.202 from 1/1/75 to 12/31/75. Thus, the total earned premium for 1975 of $1,820 must satisfy the relationship:

$$1820 = A_2\,((.16667)(1.15) + (.6667)(1.265)) + B_2\,((.125)(1.15) + (.375)(1.265)) + A_3\,(1.16667)(1.202) + B_3\,(.5)(1.202).$$

Similar expressions can be written for 1974 and 1976 earned premiums yielding the three equations:

$$1600 = .38255A_1 + .57031B_1 + .79062A_2 + .58938B_2,$$

$$1820 = 1.03505A_2 + .61813B_2 + 1.40234A_3 + .601B_3,$$

$$1860 = 1.60266A_3 + .601B_3 + 2.19296A_4 + .65433B_4.$$

In addition to these three equations, we require that our linear approximation must be continuous. This means that the line segments must meet at their end points, i.e.,

$$A_1 \cdot (1) + B_1 = A_2 \cdot (1) + B_2,$$

$$A_2 \cdot (2) + B_2 = A_3 \cdot (2) + B_3,$$

and $A_3 \cdot (3) + B_3 = A_4 \cdot (3) + B_4.$

TABLE 2

CONTRIBUTION OF WRITINGS TO EARNINGS PERIODS — COEFFICIENTS OF PARAMETERS

Parameters	Writings Period	Rate Change	Cumulative Change	Earnings Period 1974 H	1974 G	1975 H	1975 G	1976 H	1976 G
A_1, B_1	1/1/73 to 3/31/73	1.000	1.000	.00521	.03125	0	0	0	0
	4/1/73 to 12/31/73	1.150	1.150	.328125	.46875	0	0	0	0
A_2, B_2	1/1/74 to 6/30/74	1.000	1.150	.458333	.375	.16667	.125	0	0
	7/1/74 to 12/31/74	1.100	1.265	.208333	.125	.66667	.375	0	0
A_3, B_3	1/1/75 to 12/31/75	.950	1.202	0	0	1.16667	.5	1.3333	.5
A_4, B_4	1/1/76 to 4/31/76	1.000	1.202	0	0	0	0	.876543	.27778
	5/1/76 to 12/31/76	1.200	1.442	0	0	0	0	.790123	.22222

At this point, we have a series of six equations in eight unknown parameters which may be written in the convenient form:

(3)

$$
\begin{array}{rrrrrrrrcl}
.3826A_1 + & .5703B_1 + & .7906A_2 + & .5894B_2 & & & & & = & 1600 \\
& & 1.0351A_2 + & .6181B_2 + & 1.4023A_3 + & .601B_3 & & & = & 1820 \\
& & & & 1.6027A_3 + & .601B_3 & & & = & 1860 - 2.1930A_1 - .6543B_4 \\
A_1 + & B_1 - & A_2 - & B_2 & & & & & = & 0 \\
& & 2A_2 + & B_2 - & 2A_3 - & B_3 & & & = & 0 \\
& & & & 3A_3 + & B_3 & & & = & 3A_4 + B_4
\end{array}
$$

This represents a system of six equations in the six unknowns (A_i, B_i; $i = 1, 2, 3$), which can be readily solved; each of (A_i, B_i; $i = 1, 2, 3$) can be written as a linear function of A_4 and B_4. The result should be interpreted as a two parameter family of continuous, piecewise linear functions. That is, for any values of A_4 and B_4, we will obtain values for (A_i, B_i; $i = 1, 2, 3$) that will yield the given earned premiums. The solutions for our problem are as follows in terms of the two parameters A_4, B_4:

$$
\begin{aligned}
(4)\quad A_1 &= -145{,}170.26 + 318.52A_4 + 98.63B_4 \\
B_1 &= 116{,}462.05 - 252.59A_4 - 78.21B_4 \\
A_2 &= 37{,}994.28 - 82.88A_4 - 25.69B_4 \\
B_2 &= -66{,}702.48 + 148.82A_4 + 46.11B_4 \\
A_3 &= -9{,}286.07 + 19.95A_4 + 6.27B_4 \\
B_3 &= 27{,}858.21 - 56.85A_4 - 17.80B_4
\end{aligned}
$$

Given this description of the family of curves representing the rate of premium writings, it remains for the actuary to choose that particular approximation that seems most appropriate for the situation. Probably the most popular choices, if there is no better information available, would be certain optimal members of the family, such as the "smoothest" or the "flattest". These optimal members can easily be found by methods of ordinary calculus, as the following will show.

Recall that the A_i's of the model are the slopes of the line segments representing the rate of written premiums in each year. Thus, the "smoothest" member of the family can be obtained by minimizing the sum of squares function (which we refer to as the "objective function"):

(5a)

$$S(A_i, B_i; i = 1,\ldots,4) = (A_1 - A_2)^2 + (A_2 - A_3)^2 + (A_3 - A_4)^2.$$

Likewise, the "flattest" member of the family is obtained by minimizing the sum of squares function:

(5b)

$$S(A_i, B_i; i = 1,\ldots,4) = A_1^2 + A_2^2 + A_3^2 + A_4^2.$$

Of course, many other choices are possible for the objective function, including the weighting of its components. For instance, if the actuary has qualitative information that writings for 1973 were relatively flat and a new marketing program started in 1974, he may prefer to design the following objective function:

(5c)

$$S(A_i, B_i; i = 1,\ldots,4) = KA_1^2 + A_2^2 + A_3^2 + A_4^2,$$

where K is chosen as some arbitrary large constant. This procedure will force A_1 to be very small in order to minimize the function.

To continue with the demonstration, assume we have decided the flattest member should be chosen. Then the objective function can be rewritten in terms of the free parameters A_4 and B_4 by using the relationships (4). To minimize the resulting $S(A_4, B_4)$, we take the partial derivatives of $S(A_4, B_4)$ with respect to A_4 and B_4, set the resulting linear equations equal to zero, and solve for A_4 and B_4. The procedure can be conveniently written as follows, by use of the chain rule for differentiation:

$$\frac{\partial}{\partial A_4} S(A_4, B_4) = 2A_1 \frac{\partial A_1}{\partial A_4} + 2A_2 \frac{\partial A_2}{\partial A_4} + 2A_3 \frac{\partial A_3}{\partial A_4} + 2A_4 = 0,$$

$$\frac{\partial}{\partial B_4} S(A_4, B_4) = 2A_1 \frac{\partial A_1}{\partial B_4} + 2A_2 \frac{\partial A_2}{\partial B_4} + 2A_3 \frac{\partial A_3}{\partial B_4} = 0.$$

Substituting for A_i and $\frac{\partial A_i}{\partial A_4}$, $\frac{\partial A_i}{\partial B_4}$, $i = 1, \ldots, 4$ by use of (4), yields the following system of equations in A_4 and B_4 only:

$$-49{,}574{,}336.13 + 108{,}725.20A_4 + 33{,}671.13B_4 = 0,$$

$$-15{,}352{,}848.09 + 33{,}671.13A_4 + 10{,}427.72B_4 = 0.$$

This system can be solved for A_4 and B_4, which in turn will yield values for all the A_i, B_i to produce the flattest writings curve. Thus, solving for A_4 and B_4 yields:

$$A_4 = -27.109,$$

$$B_4 = 1{,}559.846;$$

and for the remaining parameters,

$$A_1 = 48.079, \qquad B_1 = 1{,}309.508,$$

$$A_2 = 172.120, \qquad B_2 = 1{,}185.467,$$

$$A_3 = -51.189, \qquad B_3 = 1{,}632.085.$$

Note that solutions and coefficients have been rounded to three and two decimal places, respectively, so some rounding error will be evident if the reader checks these calculations.

Referring back to Table 2, one sees that the coefficients of the A_i, B_i necessary to produce the earned premiums implied by these writings rates have already been calculated. Hence, the earned premium for 1974 will be \$1,378 $((.00521 + .328125) \cdot (48.079) + (.03125 + .46875) \cdot (1{,}309.508) + (.45833 + .20833) \cdot (172.120) + (.375 + .125 \cdot (1{,}185.467))$. Likewise, earned premiums for 1975 and 1976 are \$1,492 and \$1,483, respectively. Note that these earned premiums are stated at the premium level in effect at 1/1/73 so they must be restated at the 12/31/76 rate level by multiplying by 1.442. The final cumulative rate level indices to obtain the adjusted earned premiums for this demonstration are shown on Table 3, column (5).

EVALUATION OF RESULTS

Table 3 shows the resulting adjusted earned premiums computed by the algorithm. Note two different minimal elements were considered. Columns (5) and (6) give results for the "flattest" approximating element, while columns (7) and (8) give the "smoothest" approximating element. Various patterns of premium levels were tested to obtain results which may be used to compare the traditional method and the new algorithm. Only the earned premium levels were varied, assuming the same rate level change history. Table 3a details the traditional method of obtaining earned premium adjustment factors from rate change history as explained in Kallop's paper.

Table 3 makes it very evident that the algorithm yields different premium adjustment factors for different patterns of premium volume. This behavior is more realistic than that assumed by the traditional method, which is not affected by premium volume fluctuations. The summary table shows the range of adjustment factors produced by various premium patterns. In most cases, the factors are very close to each other; however, the factors produced by the traditional method for 1976 may be as much as 2% overstated, depending on the actual premium pattern.

Table 4 presents the results of an investigation into the actual accuracy of the algorithm. Briefly, it is assumed that the rate of premium writings is known and can be described by the cubic equation:

$$r(t) = 500t^3 - 1{,}950t^2 + 1{,}150t + 2{,}800,$$

where $0 \leq t \leq 4$. Earned premiums and *actual* earned premium adjustment factors can be calculated for this writing pattern. This is done by means of a table similar in format to Table 2. The same rate history as used in the previous demonstration is assumed. Note that this model presents a fairly complicated writings pattern, as shown by the graph of Figure 1. Comparison of the premium adjustment factors produced by the traditional method shows that they are surprisingly accurate for 1974 and 1975. However, for 1976 premium writings, the rate of writings increases dramatically, resulting in 1976 earned premiums almost double those of 1975 earned premiums. As expected, the traditional adjustment factor for 1976 will overstate premium 4.5%. The algorithm using the smoothest straight line approximation does much better in this extreme case, with only a 1.6% overstatement of premium. Of course, when the rate of exposure writings are known, the adjustment factors can be determined exactly. However, in the absence of any knowledge of the exposure writing history, the algorithm comes up with a very reasonable approximation to writings, as shown in Figure 1, and greatly decreases any distortion to adjusted earned premiums.

TABLE 3

SENSITIVITY OF PREMIUM ADJUSTMENT FACTORS TO PATTERNS IN EARNED PREMIUM

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
		Adjusted Earned Premiums					
Exposure Year	Actual Earned Premium	Traditional Premium	Factor	Flattest Premium	Factor	Smoothest Premium	Factor
1974	1600	1989	1.243	1987	1.242	1987	1.242
1975	1820	2152	1.183	2152	1.182	2152	1.182
1976	1860	2158	1.160	2137	1.149	2139	1.149
1974	1600	1989	1.243	1988	1.243	1987	1.242
1975	2000	2364	1.183	2369	1.185	2368	1.184
1976	3000	3480	1.160	3429	1.143	3415	1.138
1974	1600	1989	1.243	1995	1.247	1997	1.248
1975	1200	1418	1.183	1420	1.183	1420	1.183
1976	900	1044	1.160	1037	1.152	1041	1.157
1974	2700	3356	1.243	3370	1.248	3373	1.249
1975	1820	2151	1.183	2154	1.184	2153	1.184
1976	1200	1392	1.160	1385	1.154	1392	1.160
1974	2700	3356	1.243	3374	1.250	3379	1.251
1975	1820	2151	1.183	2160	1.187	2158	1.186
1976	2600	3016	1.160	2966	1.141	2948	1.134
1974	2700	3356	1.243	3372	1.249	3376	1.250
1975	1820	2151	1.183	2157	1.185	2156	1.185
1976	1860	2158	1.160	2130	1.145	2126	1.143

Summary

Empirical Ranges Due to Premium Volume Patterns

Premium Adjustment Factors		% Deviation from (3)
1974 Range	1.242-1.250	−0.1 to +0.6
1975 Range	1.182-1.187	−0.1 to +0.3
1976 Range	1.134-1.160	−2.2 to 0.0

TABLE 3a

PREMIUM ADJUSTMENT FACTOR CALCULATION TRADITIONAL RECTANGULAR METHOD

Year = 1974

(1) Rate Change Date	(2) Manual Change	(3) (2) x (3 Prior) Cumulative Index	(4) Weights*	(5) (3) x (4) Product	Premium Adjustment Factor
1/1/73	Base	1.000	.03125	.03125	
4/1/73	1.150	1.150	.84375	.97031	
7/1/74	1.100	1.265	.1250	.15813	
				1.15969	1.24344

Year = 1975

(1) Rate Change Date	(2) Manual Change	(3) (2) x (3 Prior) Cumulative Index	(4) Weights*	(5) (3) x (4) Product	Premium Adjustment Factor
4/1/73	Base	1.150	.125	.14375	
7/1/74	1.100	1.265	.375	.47438	
1/1/75	.950	1.202	.500	.60100	
				1.21913	1.18281

Year = 1976

(1) Rate Change Date	(2) Manual Change	(3) (2) x (3 Prior) Cumulative Index	(4) Weights*	(5) (3) x (4) Product	Premium Adjustment Factor
1/1/75	Base	1.202	.82986	.99749	
5/1/76	1.20	1.442	.17014	.24534	
				1.24283	1.16026

*Weights are calculated as the fraction of the area of a square of side 1 intersected by 45° lines (angle determined by policy term of 1 year) which originate from point of rate change date. A detailed example of the procedure, with diagrams may be found in Kallop's paper referenced above, Appendix to Section B-2, Exhibit 1-B, "Factor Adjusting Calendar Year Premium to Level of Present Rates."

TABLE 4

TEST OF THE ALGORITHM

	Contribution of Writings to Earnings Period		
Earning Period	1.0 to 2.0	2.0 to 3.0	3.0 to 4.0
Writing Period (&)			
0.0 to 0.25	91.683	0	0
0.25 to 1.0	1304.15	0	0
1.0 to 1.5	826.458	256.51	0
1.5 to 2.0	202.865	564.323	0
2.0 to 3.0	0	654.17	795.833
3.0 to 3.33	0	0	763.323
3.33 to 4.0	0	0	1065.844
Actual Earned Premium	2799	1795	3411
Earned Premium @ 12/31/76 Rates	3497	2127	3785
Actual Premium Adjustment Factor	1.249	1.185	1.110
Traditional Premium Adjustment Factor	1.243	1.183	1.160
% Distortion	−0.5%	−0.2%	+4.5%
Smoothest Algorithm Premium Adjustment Factor	1.253	1.188	1.128
% Distortion	+0.3%	+0.3%	+1.6%

Rate of writings function is $r(x) = 500x^3 - 1950x^2 + 1150x + 2800$.

THE RANGE OF PREMIUM ADJUSTMENT FACTORS

For a given history of rate changes and earned premiums it is often of interest to determine the range of possible values the premium adjustment factors can assume. Such information is of special importance when using this algorithm because the rate change and earned premium history do not determine a unique rate of writings model. Rather, a complete family of such rate of writings functions is obtained, any member of which will produce the historical earned premium numbers. The following discussion will demonstrate the methods involved in obtaining the exact theoretical range of factors obtainable from the family of approximating functions.

The six equations of (4) fully describe all rate of premium writing models which are piecewise linear, continuous, and produce the earned premiums of Table 1. However, the parameters A_4 and B_4 appearing in these equations are not unrestricted; in other words, the rate of premium writings for the final year is not as completely arbitrary as may appear at first glance. The constraints that are put on A_4 and B_4 arise from the requirement that the rate of writings function be positive throughout its domain. Under this condition, the range of premium adjustment factors can be investigated by allowing A_4 and B_4 to vary through their set of admissible values.

The admissible range of the parameters A_4 and B_4 can be determined as follows. For the rate of writings function to be always positive the following four conditions must be satisfied for all t, $0 < t \leq 1$:

$$
\begin{aligned}
&A_1 t + B_1 \geq 0 \\
&A_2 (1 + t) + B_2 \geq 0 \\
&A_3 (2 + t) + B_3 \geq 0 \\
&A_4 (3 + t) + B_4 \geq 0.
\end{aligned}
$$

Of course, these conditions will be satisfied if and only if they are true for $t = 0$ and $t = 1$. This last observation makes it possible to restate the above conditions in terms of five inequalities:

$$
\begin{aligned}
B_1 &\geq 0 \\
A_1 + B_1 &\geq 0 \\
2A_2 + B_2 &\geq 0 \\
3A_3 + B_3 &\geq 0 \\
4A_4 + B_4 &\geq 0.
\end{aligned}
$$

Using the equations (4) the above inequality system can be written in terms of A_4 and B_4 alone. The following five inequalities then describe the constraints on the parameters A_4 and B_4:

$$
\begin{aligned}
(6)\quad 252.589\, A_4 + 78.213\, B_4 &\leq 116{,}462.054 \\
16.950\, A_4 + 5.267\, B_4 &\leq 9{,}286.071 \\
65.933\, A_4 + 20.421\, B_4 &\geq 28{,}708.206 \\
3\, A_4 + B_4 &\geq 0 \\
4\, A_4 + B_4 &\geq 0
\end{aligned}
$$

Expressions for the three premiums adjusted to present rates, as well as the premium adjustment factors, can also be written in terms of the parameters A_4 and B_4. By use of the coefficients of the A_i and B_i given in Table 2 and the final cumulative rate level index of 1.442 the adjusted earned premiums can be written as follows:

Accident Year	Adjusted Earned Premium Expression
1974	$1.48067\, A_1 + .721\, B_1 + .96133\, A_2 + .721\, B_2$
1975	$3.36472\, A_2 + .721\, B_2 + 1.68823\, A_3 + .721\, B_3$
1976	$1.92266\, A_3 + .721\, B_3 + 2.40332\, A_4 + .721\, B_4.$

Again by use of the equations (4) the above premium expressions become functions of A_4 and B_4 alone:

Accident Year	Adjusted Earned Premium Expression
(7a) 1974	$2624.1913 - 1.3974 A_4 - .4325\, B_4$
(7b) 1975	$2027.2253 + .2729 A_4 + .0848\, B_4$
(7c) 1976	$2231.7970 - .2284 A_4 - .0642\, B_4.$

At this point the question of the range of the premium adjustment factor for 1974, for example, has been recast as the problem of finding the maximum and minimum of the linear expression (7a) subject to the constraints arising from the system of linear inequalities (6). As stated, the question is almost a linear programming problem but for the fact that neither A_4 or B_4 are constrained to be non-negative. This problem can easily be remedied by writing $A_4 = A_4^+ - A_4^-$ where

$$
A_4^+ = \begin{cases} A_4 & \text{if } A_4 \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad A_4^- = \begin{cases} -A_4 & \text{if } A_4 \leq 0 \\ 0 & \text{otherwise,} \end{cases}
$$

and similarly for B_4. Thus the maximum value of 1974 adjusted earned premium, for example, is obtained by solving the following linear programming problem:

$$\text{Maximize} - 1.3974\, A_4^+ + 1.3974\, A_4^- - .4325\, B_4^+ + .4325\, B_4^-$$

subject to the constraints

$$\begin{aligned}
252.589\, A_4^+ - 252.589\, A_4^- + 78.213\, B_4^+ - 78.213\, B_4^- &\leq 116{,}462.054 \\
16.950\, A_4^+ - 16.950\, A_4^- + 5.267\, B_4^+ - 5.267\, B_4^- &\leq 9{,}286.071 \\
65.933\, A_4^+ - 65.933\, A_4^- + 20.421\, B_4^+ - 20.421\, B_4^- &\geq 28{,}708.206 \\
3\, A_4^+ - 3\, A_4^- + B_4^+ - B_4^- &\geq 0 \\
4\, A_4^+ - 4\, A_4^- + B_4^+ - B_4^- &\geq 0.
\end{aligned}$$

The six max and min problems of the above type can be solved by using standard computer routines available for solution of linear programming problems. The resulting theoretical ranges of adjusted earned premiums and premium adjustment factors are given in the following table.

Accident	Adjusted Earned Premium		Premium Adjustment Factor		
Year	Minimum	Maximum	Minimum	Maximum	Range
1974	$1976	$2018	1.235	1.261	.026
1975	2139	2156	1.175	1.185	.010
1976	2000	2180	1.075	1.172	.097

The range of 1976 premium adjustment factors (1.075 to 1.172) is wide enough to be disconcerting. However this range should be considered as the uncertainty inherent in the data processed by the algorithm. It is preferable that the analyst be aware of the limits on the information that can be extraced from his data. The alternative use of procedures that present one definite result, when an entire range is possible, can be highly misleading.

INCORPORATING ADDITIONAL DATA

As a final investigation of the algorithm, assume we also know the actual written premiums for the period 1973 to 1976. How would we make use of this additional information? The answer lies in the flexibility afforded by the design of the objective function. The actual premium writings implied by the model of Table 4 are:

1973	\$3169
1974	2216
1975	1743
1976	6482

(For example, $\$3160 = 1.0 \int_0^{0.25} r(x)dx + 1.15 \int_{0.25}^{1.0} r(x)dx.$)

For the linear model, the written premium for each year can be written in terms of the parameters A_i and B_i. (For 1974, the writings are calculated as:

$$1.150 \int_{1.0}^{1.5} A_2x + B_3dx + 1.265 \int_{1.5}^{2.0} A_2x + B_2dx.)$$

Hence, the writings are expressed as:

1973	$.57031A_1 + 1.1125B_1$
1974	$1.82563A_2 + 1.2075B_2$
1975	$3.00500A_3 + 1.2020B_3$
1976	$4.78048A_4 + 1.3620B_4$

Clearly, we want to minimize the deviation between the actual written premium and the written premium expressions of the linear model. That is, the proper objective function to be minimized is:

(8)

$$S(A_i, B_i; i = 1, \ldots, 4) = (3169 - .57031A_1 - 1.1125B_1)^2 + (2216 - 1.82563A_2 - 1.2075B_2)^2 + (1743 - 3.005A_3 - 1.202B_3)^2 + (6482 - 4.78048A_4 - 1.362B_4)^2.$$

The matrix techniques developed in the appendix make it a simple matter to find the solution to this particular problem. The correct entries are placed into the objective matrix as defined in appendix 1 that produces the objective function. Solving for A_i, B_i and calculating the corresponding adjusted earned premiums and premium adjustment factors yields:

	Adjusted Earned Premium	Adjustment Factors	% Distortion from Actual
1974	\$3498	1.250	+0.1%
1975	2129	1.186	+0.1%
1976	3773	1.106	−0.3%

This example provides a rather straightforward demonstration of the improvement in results due to the use of more information. Since the traditional method of premium adjustment is not flexible enough to take advantage of all available information, the techniques involved in this new algorithm offer the actuary a more responsive tool to aid in rate adequacy investigations.

SUMMARY AND CONCLUSION

The methods developed in this algorithm treat one source of distortion present in the procedure of restating earned premiums at present rates; the distortion due to fluctuating levels of premium volume. Of course, there may be other sources of distortion such as territorial or classification distributions. These can be treated by refining the data into units small enough to give a reasonable approximation of the effect of distributional shifts. The mathematics of the procedure has been explained to the extent that the reader can modify the individual parts of the algorithm, especially the objective function, to take maximum advantage of all information available.

The results of the algorithm have been compared with a simpler premium adjustment procedure which ignores the effect of premium volume. This is the rectangular method, also referred to as the traditional premium adjustment method. An empirical investigation shows that for the particular rate change history used, the adjusted premium factors can have a 2% range in variation due only to different patterns of yearly earned premiums. An example with severe premium fluctuation is presented in which the traditional premium adjustment method overstates premium by 4.5%. This distortion is significantly reduced by the smoothest straight line approximation to premium writings. In addition, the distortion is virtually eliminated by use of additional data. In this case, the appropriate adjustments to the algorithm's objective function were explained to take into account the exposure related written premiums that were assumed to be available.

The mechanics of obtaining the exact range of premium adjustment factors arising from the family of approximating functions are explained. It is important to realize that even for a fixed rate change history and earned premium pattern there is a range of results rather than a single answer. The choice of factors from within this range is accomplished by minimizing a sum of squares function.

APPENDIX I

RESTATEMENT IN LINEAR ALGEBRA

The specific calculations that we explained for the expository problem in the body of the paper will now be reformulated in terms of matrix manipulations. The economy of notation that is available in the linear algebra formulation is preferable if this procedure is to be used frequently as a computational tool.

Initially, the system of equations (3) can be rewritten as:

$$AX = BY \tag{9}$$

where

$$A = \begin{vmatrix} .3826 & .5703 & .7906 & .5894 & 0 & 0 \\ 0 & 0 & 1.0351 & .6181 & 1.4023 & .601 \\ 0 & 0 & 0 & 0 & 1.6027 & .601 \\ 1.0 & 1.0 & -1.0 & -1.0 & 0 & 0 \\ 0 & 0 & 2.0 & +1.0 & -2.0 & -1.0 \\ 0 & 0 & 0 & 0 & 3.0 & 1.0 \end{vmatrix}$$

$$X = \begin{vmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ A_3 \\ B_3 \end{vmatrix}, \quad B = \begin{vmatrix} 1600 & 0 & 0 \\ 1820 & 0 & 0 \\ 1860 & -2.193 & -.6543 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3.0 & 1.0 \end{vmatrix}, \text{ and } Y = \begin{vmatrix} 1 \\ A_4 \\ B_4 \end{vmatrix}$$

Thus, the solution (4) becomes simply:

$$X = (A^{-1}B)Y \tag{10}$$

For convenience, define the matrix C to be the 6 x 3 matrix $C = A^{-1}B$. Then, it will be helpful in further manipulations to use the augmented matrices:

$$\hat{C} = \begin{vmatrix} 1 & 0 & 0 \\ & C & \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \text{and } \hat{X} = \begin{vmatrix} 1 \\ X \\ A_4 \\ B_4 \end{vmatrix}$$

In this case,

$$\hat{X} = \hat{C}Y. \tag{10a}$$

The heart of the calculation lies in the formulation of the objective function as a quadratic form. For instance, in order to produce the "flattest" form of the objective function (5a), we could write:

$$S(A_i, B_i) = \hat{X}^T(\Theta^T \Theta)\hat{X} \text{ where} \tag{11}$$

$$\Theta = \begin{vmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{vmatrix}$$

The calculation should be viewed as follows:

$$\Theta\hat{X} = (A_1, A_2, A_3, A_4)^T$$

and $(\Theta X)^T(\Theta X) = (A_1, A_2, A_3, A_4) \begin{vmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{vmatrix} = \sum_{i=1}^{4} A_i^2$, exactly the form of (5a).

It is evident that, in the form (11), any of a large class of objective functions can be obtained simply by choosing Θ properly. For example, in order to obtain the objective function selecting the smoothest member of the family of approximating functions, choose Θ to be as follows:

$$\Theta = \begin{vmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \end{vmatrix}$$

Clearly, in an interactive computer session (such as APL provides) much information on the range of results can be gained with minimum effort by simply trying different Θ matrices in the calculation program.

Recall that, while $\hat{X}$ contains eight unknown parameters for our demonstration, we can reduce this to two by use of relation (10a). Thus,

$$S(A_i, B_i) = Y^T (\hat{C}^T \Theta^T \Theta \hat{C})\, Y = Y^T F\, Y$$

where the matrix $F = \hat{C}^T \Theta^T \Theta \hat{C}$ is the 3 x 3 matrix of coefficients of the quadratic form. In order to minimize this particular quadratic form, it suffices to set up the system of two linear equations in the unknown parameters A_4 and B_4. This process of taking partial derivatives can also be accomplished by matrix multiplication. If we let

$$E_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad \text{and } D = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

we can then write the system resulting from setting partial derivatives equal to 0 as:

$$(DFD^T) \begin{vmatrix} A_4 \\ B_4 \end{vmatrix} = -DFE_1.$$

Hence,
$$\begin{vmatrix} A_4 \\ B_4 \end{vmatrix} = (DFD^T)^{-1} (-DFE_1),$$

and once A_4 and B_4 have been determined, the other parameters follow as a result of equation (11).

That is, $(A_1, B_1, A_2, B_2, A_3, B_3)^T = C(1, A_4, B_4)^T$

and, to obtain earned premiums implied by these writings parameters, we form a matrix based on factors used to obtain the A and B matrices of equation (9).

The procedure is as follows. Let P be the vector of adjusted earned premiums. Then,

$$P = KA$$

where $A = (A_1, B_1, A_2, B_2, A_3, B_3, A_4, B_4)^T$

contains the solutions of the parameters $(A_i, B_i; i = 1, \ldots, 4)$, and

$$K = \begin{vmatrix} .333335 & .500 & .6667 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & .83337 & .5 & 1.1667 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.3333 & .5 & 1.6667 & .5 \end{vmatrix}$$

Note that K is formed directly from the entries of Table 2. Since the earned premiums P are at the rate level of January 1, 1973, earned premiums at present rates can be obtained by taking (1.442)P.

Finally, for the example analysis of Table 4, the actual written premium is known as well as the earned premium. The object is to force the written premium implied by the model to be as close to actual written premium as possible. If the mathematical tools described in this appendix have been implemented, most likely in the form of a computer program, the solution to this particular problem is easily obtained by merely changing the entries of the Θ matrix. All the matrix calculations remain the same, and the appropriate Θ matrix needed to obtain the objective function (8) is given by:

$$\Theta = \begin{matrix} 3169 & -.57031 & -1.1125 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2216 & 0 & 0 & -1.82563 & -1.2075 & 0 & 0 & 0 & 0 \\ 1743 & 0 & 0 & 0 & 0 & -3.005 & -1.202 & 0 & 0 \\ 6482 & 0 & 0 & 0 & 0 & 0 & 0 & -4.78048 & -1.362 \end{matrix}$$

Of course, the example computations have been developed for a case with policy term of one year and three years of earned premiums. The form of the matrix equations does not change for different policy term or number of years of earned premium. Thus, the same matrix equations will handle all the calculations for a problem with policy term of three years and four years of earned premium data. Note that this problem will involve fourteen unknown parameters, which can be reduced to a parametric family of approximations described by four free parameters. Thus, the objective function can be written in terms of four unknowns and minimized by a procedure of taking four partial derivatives. The problem is reduced to writing a program flexible enough to handle any combination of policy term and years of earned premium.

APPENDIX 2

GENERALIZED EARNINGS CONTRIBUTIONS FORMULA

Let the rate of writing function be given by $f(x) = Ax + B$. Then in order to calculate the contributions to earnings of the period from x_0 to x_1 of the writings from y_0 to y_1, we can use a number of different formulas. This appendix will develop the formula due to Ross[3] where

$$\mathrm{ERCON}\,(y_0, y_1; x_0; x_1; t) = \int_{x_{0-t}}^{x_0} \frac{x - x_0 + t}{t}\, g(x)\, dx$$

$$+ \int_{x_0}^{x_{1-t}} g(x)\, dx + \int_{x_{1-t}}^{x_1} \frac{x_1 - x}{t}\, g(x)\, dx,$$

$$g(x) = \begin{cases} f(x) \text{ if } y_0 \leq x \leq y_1 \\ 0 \text{ otherwise} \end{cases}$$

and t is the policy term. It is assumed that $x_0 \leq x_1 - t$.

The three integrals in this formula can be evaluated for limits of integration a, b and $f(x) = Ax + B$ as follows:

$$\int_a^b \frac{x - x_0 + t}{t}\, (Ax + B)dx = A \cdot h_1^{(1)}(a, b) + B \cdot h_1^{(2)}(a, b)$$

where

$$h_1^{(1)}(a, b) = \left(\tfrac{1}{6t}\right) (2(b^3 - a^3) + 3\,(b^2 - a^2)\,(t - x))$$

$$h_1^{(2)}(a, b) = \left(\tfrac{1}{2t}\right) (b^2 - a^2) + 2\,(t - x_0)\,(b - a));$$

$$\int_a^b Ax + Bdx = A \cdot h_2^{(1)}(a, b) + B \cdot h_2^{(2)}(a, b)$$

where

$$h_2^{(1)}(a, b) = (\ 1/2\ )\,(b^2 - a^2)$$

$$h_2^{(2)}(a, b) = b - a;$$

[3] Ross, J. P., "Generalized Premium Formulae," *PCAS*, LXII (1975), p. 50.

$$\int_a^b \frac{(x_1 - x)}{t} \; Ax + Bdx = A \cdot h_3^{(1)}(a, b) + B \cdot h_3^{(2)}(a, b)$$

where

$$h_3^{(1)}(a, b) = \left(\frac{1}{6t}\right)(2(b^3 - a^3) + 3x_1 (b^2 - a^2))$$

$$h_3^{(2)}(a, b) = \left(\frac{1}{2t}\right)((b^2 - a^2) + 2x_1 (b - a)).$$

Define the following logical expression which is a function of the order of four points:

$$j(a, b, c, d) = \begin{cases} 1 \text{ if } a \leq b < c \leq d \\ 0 \text{ otherwise} \end{cases}$$

Using the function j define the following $H_i^{(k)}(a, b, c, d)$ functions for $i = 1, 2, 3$ and $k = 1, 2$:

$$H_i^{(k)}(a,b,c,d) = j(a,c,b,d) \cdot h_i^{(k)}(c, b) + j(a,c,d,b) \cdot h_i^{(k)}(c, d)$$

$$+ j(c,a,b,d) \cdot h_i^{(k)}(a,b) + j(c,a,d,b) \cdot h_i^{(k)}(a,d).$$

At this point we can write

$$\text{ERCON}(y_0, y_1; x_0, x_1; t) = A.H._1(y_0, y_1; x_0; x_1; t)$$

$$+ B.H_2(y_0, y_1; x_0; x_1; t)$$

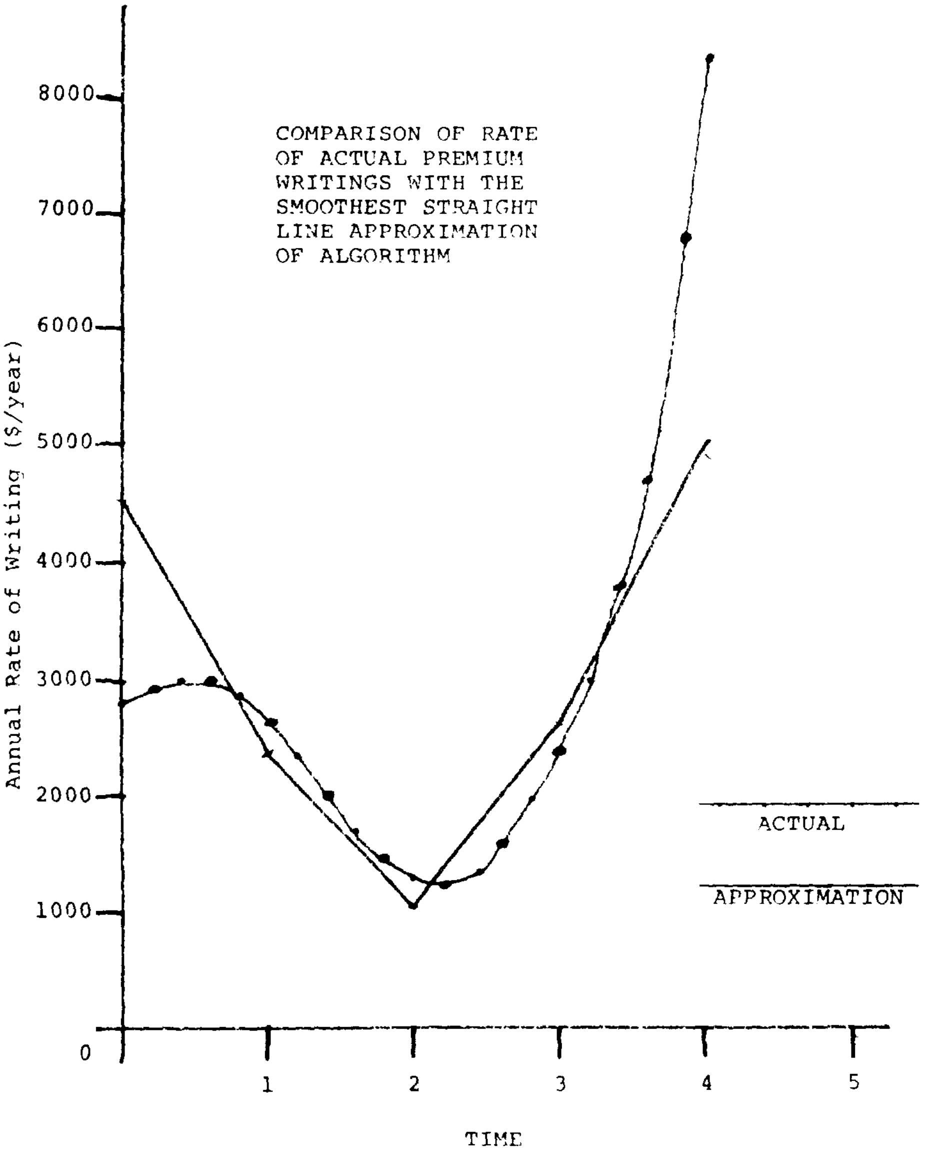
where

$$H_k(y_0, y_1; x_0, x_1; t) = H_1^{(k)}(x_0 - t, x_0, y_0, y_1) +$$

$$H_2^{(k)}(x_0, x_1 - t, y_0, y_1) + H_3^{(k)}(x_1 - t, x_1, y_0, y_1).$$

In this form, the formula can be readily programmed to produce the coefficients of A and B.

FIGURE 1:



Rate of Writing = $500t^3 - 1950t^2 + 1150t + 2800$