A REFINED MODEL FOR PREMIUM ADJUSTMENT

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INTRODUCTION

Loss ratio ratemaking is an important actuarial technique, especially for actuaries working with less sophisticated data than is available to a rating bureau; and despite the movement toward pure premium ratemaking, loss ratio ratemaking is still essential for several lines of business.

The premium adjustment factors used in the loss ratio method are familiar to most actuaries at least to the extent of Ralph Marshall's[1] or Roy Kallop's[2] descriptions in their papers on Workers' Compensation ratemaking. These papers contain an adequate discussion of the mechanics for calculating premium adjustment factors, but the conceptual background is sketchily drawn and the method used in those papers assumes a constant level of exposures.

Jim Ross, in *Generalized Premium Formulae*[3], presents mathematical expressions which fit the parallelogram approach in a variety of situations. He introduces to the Proceedings a description of the mathematical theory underlying the use of the parallelogram approach. Unlike the previously cited authors, he allows for changing levels of exposures but he does not address their impact on rate level indications or modify the geometrical model to accommodate their representation.

After reviewing the geometry of the traditional two-dimensional model, this paper will introduce a third dimension which will allow for the geometrical representation of levels of exposures; the mathematics fitting this model will be explored. Finally, an example will illustrate the practical application of this model while examing the impact of changing levels of exposure on rate level indications.

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[1] Marshall, R. M., *Workmen's Compensation Insurance Ratemaking*, (Revised, 1961), Casualty Actuarial Society

[2] Kallop, R., *A Current Look at Workers' Compensation Ratemaking*, PCAS LXII (1975)

[3] Ross, J. P., *Generalized Premium Formulae*, PCAS LXII (1975)

TRADITIONAL MODEL

In the usual technique a square is drawn to represent each calendar year's earned exposure (Figure 1). The horizontal or x-axis is always identified as time but the vertical or y-axis is not described. A little reflection will show that the second dimension represents the portion of the policy term expired, ranging from zero to one.

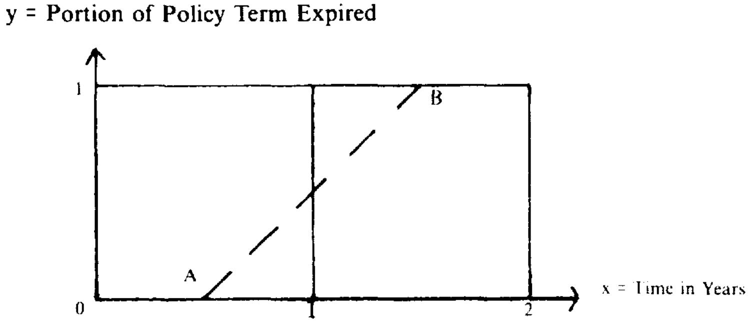

Figure 1

In terms of this model any point (x_t, o) represents the writing of exposures at time x_t because, with $y = o$, exposures are completely unearned. As we move forward in time the exposures written at time x_t are uniformly earned until at time $x_t + k$ (where k is the term of the policy) the exposures are fully earned. This pattern is shown in the geometrical configuration by a diagonal line connecting (x_t, o) and $(x_t + k, 1)$. For example, the earning of exposures on annual policies written at time ½ would be described by the line AB in Figure 1. All other exposures on annual policies, regardless of the time written, will follow a pattern of earning described by lines parallel to AB.

By assuming that exposures are written continuously over time, each square is viewed as being covered by a collection of diagonal lines. It is important to note, for use later in the paper, that any point (x_t, y_t) can be traced to the end point $(x_t - ky_t, o)$ of the diagonal line on which it lies.

In the application of the parallelogram method, the particular diagonal lines drawn mark the boundaries of areas of earned exposures where different rate levels are in effect. The various areas, taken as a percent of the total, are used as weights applied against the various rate levels to produce an average rate level for that period's earned exposures. The ratio of the current rate level to this average rate level is used to modify the period's earned premium. (For an example see pages 76 and 104 of Roy Kallop's paper.)

The method has been presented with an example using only annual term policies. In general where policies of several terms are involved we find it easiest to handle the adjustments for each term of policy separately. Once the separate adjustments are made, simply adding the individual results gives the total adjusted earned premium.

THREE-DIMENSIONAL MODEL

We have seen that the two dimensional model deals with time and the portion of policy term earned. The possibility of varying levels of exposures was addressed by Jim Ross who introduced a function, $f(x)$, representing the rate of exposure writing at time x. In the two dimensional model $f(x)$ cannot be shown. By introducing a third dimension we can account for changing levels of exposure. In the three dimensional model the x-axis and y-axis are defined as before; the z-axis will be defined as the level of exposures.

In order to make the model and the mathematics compatible we will let $z = g(x, y)$ define the level of exposures. Thus each value of z is a function of time and the portion of the policy term earned. It should be clear that $g(x, o)$ is the rate of exposure writing at time x and thus $g(x, o) = f(x)$. Using the relationship noted in the two-dimensional model that any point in the plane (x_t, y_t) can be traced to the point $(x_t - ky_t, o)$ we establish that $g(x_t, y_t) = g(x_t - ky_t, o)$ with the condition that all policies are held to full term. (The assumption of no cancellations will generally be acceptable. If the rate of cancellations is significant in a particular situation the relationship of $g(x, y)$ to $f(x)$ can be appropriately modified. For example, if on the average 10% of written exposures are cancelled during the term of the policy, we can approximate this situation by letting $g(x, y) = (1 - .1y)\, f(x - ky)$.)

Figure 2 shows, in three dimensions, the standard assumption of level exposures. The plane ABCD is comparable to line AB in figure 1 except that in three dimensions we are able to show the level of exposures. Because the value of z is the same throughout, the volumes will be proportional to the areas in the traditional model and the same weights will be obtained. Note that no dimensions are placed on the z-axis. In practice we can graduate the z-axis to absolute amounts of exposure or we can index the exposures to the level of exposures at any convenient time.

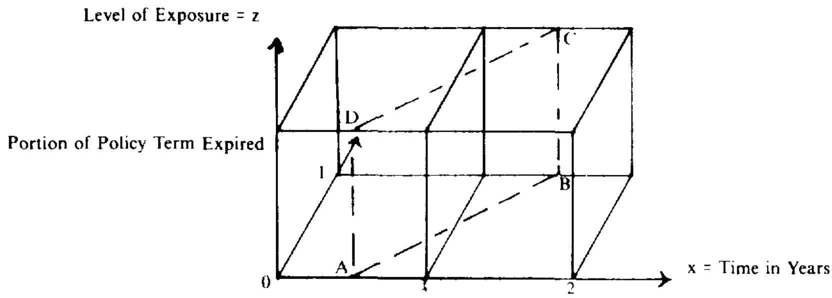

Figure 2

MATHEMATICAL DEVELOPMENT

We have defined f(x) as the rate of exposure writing at time x and g(x, y) as f(x — ky) where k is the policy term in years. The range of y, the portion of policy earned, is such that $0 \leq y \leq 1$ and we require always $g(x, y) \geq 0$.

Since g(x, y) is a density function, its integral describes an amount of exposures. For example, the written exposures between time x_0 and time x_1 may be expressed as:

$$\int_{x_0}^{x_1} g(x, o)dx = \int_{x_0}^{x_1} f(x)dx \tag{1}$$

By integrating over all values of y at a fixed time x_t, we can evaluate the rate of exposures being earned per unit of time. Thus,

$$\int_{0}^{1} g(x_t, y)dy = \int_{0}^{1} f(x_t - ky)dy \tag{2}$$

This differs from the usual notion of "in-force" which is not consistently on an annual basis; e.g., a three year policy would be tallied as three annual exposures in a usual accounting of in-force although only one annual exposure is being earned at any time during the policy period. The traditional concept of in-force would be obtained within this three dimensional model by introducing a factor for the policy term as shown in formula (3).

$$\text{In-force} = k \int_0^1 g(x_t, y)\,dy \tag{3}$$

Since formula (2) gives the rate of exposures being earned at time x_t, its integral over time describes earned exposures. The expression for earned exposures between time x_0 and time x_1 is shown as formula (4).

$$\int_{x_0}^{x_1} \int_0^1 g(x, y)\,dy\,dx = \int_{x_0}^{x_1} \int_0^1 f(x - ky)\,dy\,dx \tag{4}$$

Before proceeding with an example showing the practical application of these formulas we will set forth two additional relationships that can be seen within the three-dimensional framework.

First, noting that $y = 1$ indicates points at which exposures are expiring, the integral of z at $y = 1$ over a time interval x_0 to x_1 will give the level of exposures expiring in that interval. The mathematical notation for this integral is as follows:

$$\int_{x_0}^{x_1} g(x, 1)\,dx = \int_{x_0}^{x_1} f(x - k)\,dx \tag{5}$$

The final formula presented will develop the value for unearned exposures at time x_t. We can view the unearned exposures as the amount of exposures earned following time x_t from all policies written prior to time x_t. Since the last policy contributing to this earned will expire at time $x_t + k$, we will be integrating between time x_t and time $x_t + k$. This will be similar to formula (4) except the lower limit for y will be the diagonal line connecting $(x_t, 0)$ and $(x_t + k, 1)$ instead of $y = 0$. The equation for this line is $y = (x - x_t)/k$, hence the integral for unearned exposures becomes:

$$\int_{x_t}^{x_t + k} \int_{\frac{x - x_t}{k}}^{1} g(x, y)\,dy\,dx =$$

$$\int_{x_t}^{x_t + k} \int_{\frac{x - x_t}{k}}^{1} f(x - ky)\,dy\,dx \tag{6}$$

APPLICATIONS

In developing rate indications for an established line it is unlikely that a bureau will see a significant change in written exposures in one year, but for new lines or declining lines there would be significant changes. On an individual company basis new lines, new program deviations and new market penetrations will in some cases cause changes in exposure large enough to have an impact on rate level indications. In these cases the change in exposure can have an effect on rate level indications which can be measured using the three dimensional model.

As an example of increasing exposures consider the following table of data which might represent the pattern for a new line of business.

Time	Written Exposures
1st Quarter Year	125
2nd Quarter Year	375
3rd Quarter Year	625
4th Quarter Year	875
5th Quarter Year	1125
6th Quarter Year	1375
7th Quarter Year	1625
8th Quarter Year	1875

With f(x) defined as the annual rate of written exposures it is necessary to convert the actual quarterly exposures to an equivalent annual rate in developing the equation for f(x).

Time	Written Exposures	Annual Rate
$0 \leq x \leq .25$	125	500
$.25 \leq x \leq .50$	375	1500
$.50 \leq x \leq .75$	625	2500
$.75 \leq x \leq 1.00$	875	3500
$1.00 \leq x \leq 1.25$	1125	4500
$1.25 \leq x \leq 1.50$	1375	5500
$1.50 \leq x \leq 1.75$	1625	6500
$1.75 \leq x \leq 2.00$	1875	7500

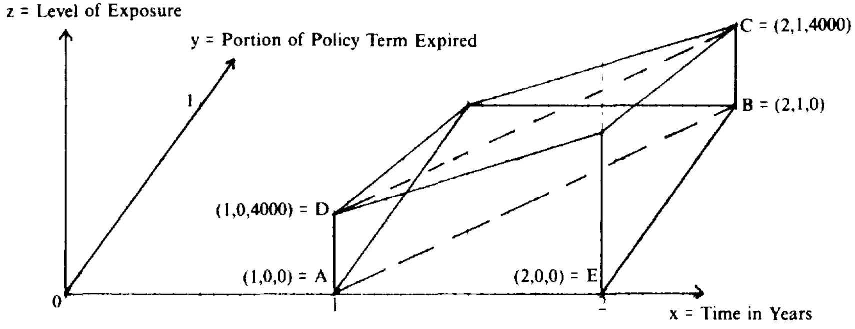

Figure 3

This data indicates a linear pattern since the growth between quarters is constant. By fitting each value to the midpoint of the time interval the equation f(x) = 4000x is developed. Figure 3 describes this situation for policies with one-year term. From our prior development we have g(x,y) = f(x-y) = 4000 (x-y).

We have assumed a rate level increase of 20% at the end of the first year. In figure 3 the diagonal plane ABCD separates the exposures at different rate levels. In our example the earned exposures at the higher rate level in year two are represented by the volume above triangle ABE. To evaluate this volume we apply formula (4) except that the upper limit for y will be the diagonal line AB whose equation is y = x-1. The expression to evaluate this volume is then:

$$\int_{1}^{2} \int_{0}^{x-1} 4000\,(x-y)dydx = 2667$$

The proportion this volume is of the full year's earned exposure is obtained from dividing the above result by the total earned exposure volume for the year. This latter value is developed directly from formula (4) as:

$$\int_{1}^{2} \int_{0}^{1} 4000\,(x-y)\,dydx = 4000$$

The following table compares the results of this method with those of the traditional method.

	Increasing Exposures	Level Exposures
Exposures at Base Rate Level	33%	50%
Exposures at 1.200 Rate Level	67%	50%
Average Rate Level	1.133	1.100
Premium Adjustment Factor	1.059	1.091

If the actual and expected loss ratios were 70% and 60% respectively the traditional method would yield a rate level indication of +6.9% whereas recognizing increasing exposures yields + 10.2%[4]. The above example shows that when rates have increased during a time when exposures are steadily increasing the traditional approach underestimates the average rate level for earned exposure. As a result the necessary premium adjustment is overestimated and the adjusted loss ratio is too low, leading to an inadequate rate indication.

In the case of declining exposures during a period of rising rate levels the traditional method overestimates rate level indications. This situation has been encountered recently in developing rates for a diminishing book of monoline fire business.

In addition to growth situations, irregular exposure patterns may also occur in a stable line where policy writing is heavily weighted towards specific effective dates (e.g., January 1, July 1). Whenever a non-level pattern of exposures is evident, it would be appropriate to look further for the actual exposure pattern. Certainly monthly exposure data would be optimal, but it may be more practical to rely on premium figures. This may be in the form of internal data such as quarterly production reports, monthly bureau transmittals, etc. This data will allow one to judge the value of a refined calculation in the particular instance. If a calculation is warranted, techniques from numerical analysis can be used to fit the data to an integrable function.

[4] $\frac{.70}{1.091} \div .60 = 1.069; \quad \frac{.70}{1.059} \div .60 = 1.102$