A MATHEMATICAL MODEL FOR LOSS RESERVE ANALYSIS

CHARLES L. McCLENAHAN

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

— *Lewis Carroll*

It has long been recognized that loss reserving is, or should be, within the domain of the Casualty Actuary; but in no other area have we applied our expertise with as little success. We have devised classification systems which generate unique automobile insurance rates for single female farmers living in Manhattan and we have developed so many formulae for partial credibility that we are in danger of losing ours. In our sixty year history we have truly put the "science" in "actuarial science." But, as a review of the experience of the past few years points out, we still have difficulty establishing accurate loss reserves.

One reason for this difficulty is the dearth of analytical tools with which to quantify the effects of changes in payment patterns, inflation, frequency and other factors upon reserve adequacy. Where a line of business has a "long tail" we must go back several years in order to examine a relatively complete development pattern — and the intervening years may have brought changes which should be taken into account in establishing current loss reserves.

Over the years, actuaries and others have developed several mathematical models to deal with the projection problem. These models range from the rather simple deterministic model underlying the calculation of loss development factors to the sometimes quite complex models of incurred losses which have been built into probabilistic planning models.

More recently, attention has been turned to the use of mathematical models in the analysis of loss reserves.[1] Not only in the area of adequacy determination but also in the area of financial planning it is becoming more

[1] See, for example, Simon, "Distortion in IBNR Factors", *P.C.A.S. LVII, p. 64.*

and more important that paid losses and loss reserves be treated separately rather than being dealt with, on a combined basis, as incurred losses. As cash flow begins to rival profitability as the key area for analysis by management, investors and regulators, the need for reserving models based upon paid losses becomes more intense. This paper presents one such model.

Any loss payment model which is proposed for use in the analysis of loss reserves must meet certain minimum requirements. First, the cumulative paid losses for a given incurred period must obviously converge to the ultimate incurred losses. Second, the model should allow for the varying of frequency and severity assumptions separately. Finally, the model should provide a reasonable approximation of reality. Where the model is designed to serve as a component of a larger corporate model it is also desirable that the model be simple — especially if the macro model is probabilistic.

The model described herein represents the results, to date, of the formulation and testing (mostly on a trial-and-error basis) of several paid loss development models.

THE MODEL

Assume that, where severity — that is the pure loss cost resulting from the average claim — is constant over time, losses of 1 incurred during a given (accident) month m are paid during subsequent months $m+n$ in amounts equal to pq^{n-d} where $0 < p < 1$, $q = 1-p$, $n \geq d$, and where d is the average delay in months between loss occurrence and loss reporting. In other words, assume that no payments are made for the first d months and then monthly payments are made at the rate of 100p% of the unpaid losses at the beginning of each subsequent month.[2]

If we let x represent the uniform monthly rate of change in severity, and y the uniform monthly rate of change in accident month incurred losses due to claim frequency and exposure volume increases or decreases, we are able to develop certain relationships between paid losses, incurred losses and loss reserves. It is necessary that assumptions x and y be treated separately because, while x impacts the amount of loss through the date of payment, the effect of y is felt only through the incurred date.

[2] This, of course, assumes that all losses occur on the first day of the month and represents an average delay of d-½ months assuming a uniform distribution of loss occurrence.

Let ${}_nP_m$ represent the amount paid during month $m+n$ $(n \geq 0)$ on losses incurred during accident month m. If $n < d$ *then*, ${}_nP_m = 0$. If $n \geq d$ *then*, ${}_nP_m$ can be expressed as:

$$ {}_nP_m = cpq^{n-d}\,(1+x)^{m+n}\,(1+y)^m \tag{1}$$

where c represents the constant-dollar losses (i.e. incurred losses where $x=0$) for some base accident month $(m=0)$.

It will be helpful at this point to define three additional values:

$$z = x + y + xy$$

$$r = q(1+x)$$

$$b = cp(1+x)^d$$

Note that z represents the combined effects of x and y (i.e. $1+z = (1+x)\ (1+y)$); r is a combination of the effects of severity increases $(1+x)$ during a month and the unpaid loss factor (q); and b represents the payments during month d on losses of c incurred during month $m=0$. Substituting into (1):

$$ {}_nP_m = br^{n-d}(1+z)^m \quad , \qquad n \geqslant d \geqslant 0 \tag{2}$$

Formula (2) is the basis for the model described in this paper. All of the subsequent formulae and relationships follow directly from (2).

Defining I_m as the losses incurred in month m:[3, 4]

$$I_m = \sum_{k=d}^{\infty} {}_kP_m = \frac{b(1+z)^m}{1 - r} \tag{3}$$

And, defining ${}_nU_m$ as the losses incurred in month m which remain unpaid at the end of month $m+n$:

$$ {}_nU_m = \sum_{k=n+1}^{\infty} {}_kP_m = \frac{br^{n-d+1}(1+z)^m}{1-r}\ , n \geqslant d \geqslant 0 \tag{4}$$

[3] The derivations of the formulae in this section will be found in Technical Appendix 1.

[4] In this and all subsequent formulae it is assumed that $-1 < x < p/q$ (i.e. $0 < r < 1$). Note that if $\geq p/q$ ultimate incurred losses are infinite as severity is increasing the value of unpaid losses faster than the losses are being settled.

Now, if we let R_m represent the total required reserve at the end of month m:

$$R_m = \sum_{k=0}^{d-1} I_{m-k} + \sum_{k=d}^{\infty} {}_kU_{m-k}$$

$$= \frac{b(1+z)^{m-d+1}}{1-r}\left[\frac{(1+z)^d-1}{z} + \frac{r}{1+z-r}\right], d \geqslant 0 \quad (5)$$

One final definition is necessary. Let ${}_mP_{tot}$ be the total losses paid during month m. Then:

$${}_mP_{tot} = \sum_{k=d}^{\infty} {}_kP_{m-k} = \frac{b(1+z)^{m-d+1}}{1+z-r}, d \geqslant 0 \quad (6)$$

ACCIDENT YEAR MODEL

We can now examine the paid loss model in the accident year mode.

Let: AI_t = incurred losses for accident year t;

${}_nAP_t$ = accident year t losses paid during the year $t+n$ $(n \geqslant 0)$;

${}_nAR_t$ = required reserve for accident year t at the end of the year $t+n$ $(n \geqslant 0)$.

The accident year incurred formula is fairly straightforward. Since the payment model is predicated upon losses incurred in a given month, the accident year incurred losses are simply the sum of twelve months of incurred losses:[5]

$$AI_t = \sum_{k=12t}^{12t+11} I_k = \frac{b(1+z)^{12t}}{1-r}\left[\frac{(1+z)^{12}-1}{z}\right] \quad (7)$$

The accident year payment formulae present a more difficult computational task. Even if there were no delay between incurred date and reported date, a separate formula would be required for the payments made during the accident year. Where d exceeds 1 additional formulae are re-

[5] The derivations of the formulae in this section will be found in Technical Appendix 2.

quired. In developing the following it has been assumed that d is at least 1 month but does not exceed 12 months. It is obvious that d can never be zero as that would require the average loss to be reported 1/2 month prior to its occurrence.

${}_0AP_t$ represents the payments on accident year t incurred losses made during year t. Thus:

$$
\begin{aligned}
{}_0AP_t &= \sum_{k=12t}^{12t+11-d} \; \sum_{j=d}^{12t+11-k} {}_jP_k \\
&= \frac{b(1+z)^{12t}}{1-r}\left[\frac{(1+z)^{12-d}-1}{z} - r\left[\frac{(1+z)^{12-d}-r^{12-d}}{1+z-r}\right]\right], \\
&\qquad 1 \leqslant d \leqslant 12 \qquad (8)
\end{aligned}
$$

${}_1AP_t$ represents the payments on accident year t incurred losses during the first subsequent year. Where $2 \leqslant d \leqslant 12$ payments on the last $d-1$ months of the accident year will not be made until the full d month delay has elapsed. For this reason the formula for ${}_1AP_t$ requires two double summations as follows:

$$
\begin{aligned}
{}_1AP_t &= \left[\sum_{k=12t}^{12t+12-d} \; \sum_{j=12t+12-k}^{12t+23-k} {}_jP_k\right] \\
&\qquad + \left[\sum_{k=12t+13-d}^{12t+11} \; \sum_{j=d}^{12t+23-k} {}_jP_k\right] \\
&= \frac{b(1+z)^{12t}(1-r^{12})}{1-r}\left[\frac{(1+z)^{13-d}-r^{13-d}}{1+z-r}\right] \\
&\quad + \frac{b(1+z)^{12t+13-d}}{1-r}\left[\left[\frac{(1+z)^{d-1}-1}{z}\right]\right. \\
&\quad \left. - (r^{13-d})\left[\frac{(1+z)^{d-1}-r^{d-1}}{1+z-r}\right]\right], \; 2 \leqslant d \leqslant 12 \qquad (9)
\end{aligned}
$$

Where $n \geqslant \frac{d+11}{12}$, each accident month has payments during each of the twelve months of the year $t+n$. Therefore, a single formula will serve for all such years as follows:

$$_nAP_t = \sum_{k=12t}^{12t+11} \sum_{j=12(n+t)-k}^{12(n+t)-k+11} {}_jP_k$$

$$= \frac{b(1-r^{12})(1+z)^{12t}(r^{12n-11-d})}{1-r} \left[\frac{(1+z)^{12}-r^{12}}{1+z-r}\right], n \geqslant \frac{d+11}{12} \qquad (10)$$

The accident year reserve formulae present similar problems to those encountered with the accident year paid. Again the formulae assume $1 \leqslant d \leqslant 12$.

The reserve at the end of the accident year is the difference between the accident year incurred and the amount paid during the accident year:

$$_0AR_t = AI_t - {}_0AP_t$$

$$= \frac{b(1+z)^{12t}}{1-r} \left[\left[\frac{(1+z)^{12}-(1+z)^{12-d}}{z}\right] + r \left[\frac{(1+z)^{12-d}-r^{12-d}}{1+z-r}\right] \right], \qquad 1 \leqslant d \leqslant 12 \qquad (11)$$

The reserve for accident year t at the end of the year $t+n$ $(n \geqslant 1)$ can be expressed as follows:

$$_nAR_t = \sum_{k=12t}^{12t+11} {}_{12(t+n)-k+11}U_k$$

$$= \frac{br^{12n-d+1}(1+z)^{12t}}{1-r} \left[\frac{(1+z)^{12}-r^{12}}{1+z-r}\right], n \geqslant \frac{d}{12} \qquad (12)$$

COMPARISON OF MODEL WITH ACTUAL ACCIDENT YEAR DATA

In order to test the model, the automobile bodily injury loss data for accident year 1968 of five large writers of automobile insurance was compiled. This data was as follows (000 omitted):

Paid through 12/31/68	$112,528
Paid through 12/31/69	328,420
Paid through 12/31/70	453,371
Paid through 12/31/71	528,505
Paid through 12/31/72	575,449
Paid through 12/31/73	597,843
Paid through 12/31/74	608,204
Reserve at 12/31/74	9,931

The accident year model was then applied using the following values:

$p = .0498 \qquad x = .004 \qquad c = 44{,}414$

$d = 2 \qquad y = .007 \qquad t = 0$

The following tables detail the relationship between the actual and theoretical data. It should be noted that no attempt was made to obtain the best fit between theoretical and actual data. p was estimated from the actual data and d, x, and y were selected as representative of the line and the conditions extant in 1968.[6]

TABLE I
DISTRIBUTION OF ACCIDENT YEAR 1968 LOSSES

	Theoretical	Actual	Difference	Percent of Actual
Paid during 1968	110,947	112,528	− 1,581	− 1.4
Paid during 1969	217,453	215,892	+ 1,561	+ 0.7
Paid during 1970	125,078	124,951	+ 127	+ 0.1
Paid during 1971	71,082	75,134	− 4,052	− 5.4
Paid during 1972	40,396	46,944	− 6,548	− 13.9
Paid during 1973	22,957	22,394	+ 563	+ 2.5
Paid during 1974	13,047	10,361	+ 2,686	+ 25.9
Reserve 12/31/74	17,175	9,931	+ 7,244	+ 72.9
Total Incurred	618,135	618,135		

[6] See Technical Appendix 3 for an explanation of this application.

TABLE II
ACCIDENT YEAR 1968 REQUIRED RESERVES

Reserve Date	Theoretical	Actual	Difference	Percent of Actual
12/31/68	507,188	505,607	+ 1,581	+ 0.3
12/31/69	289,735	289,715	+ 20	+ 0.0
12/31/70	164,657	164,764	− 107	− 0.1
12/31/71	93,575	89,630	+ 3,945	+ 4.4
12/31/72	53,179	42,686	+ 10,493	+ 24.6
12/31/73	30,222	20,292	+ 9,930	+ 48.9
12/31/74	17,175	9,931	+ 7,244	+ 72.9
Average	165,104	160,375	+ 4,729	+ 2.9

TABLE III
ACCUMULATIVE PERCENTAGE OF 1968 ACCIDENT YEAR LOSSES PAID

Date	Theoretical	Actual
12/31/68	17.9%	18.2%
12/31/69	53.1	53.1
12/31/70	73.4	73.3
12/31/71	84.9	85.5
12/31/72	91.4	93.1
12/31/73	95.1	96.7
12/31/74	97.2	98.4

The above tables indicate that, while the model does not provide as close a fit to the sample data as might be desirable (especially in the later years of development), the fit is sufficiently close to allow us to use the model to advantage—particularly where total reserves (as opposed to reserves for a specific accident year) are the subject of study. For example, the model can be quite useful in the analysis of the effects upon reserve adequacy of changes in various exogenous variables and in the testing of the established loss reserves on a prospective basis.

APPLICATIONS TO LOSS RESERVE ANALYSIS

The remainder of this paper is devoted to the analysis of loss reserves through the use of certain theoretical relationships developed from the model.

Effect of "discounting" loss reserves

If, instead of reserving a full dollar for each dollar to be paid t months hence, we "discount" the t month-deferred dollar by v^t, where $v = (1+i)^{-1}$ and i represents the assumed monthly yield on our invested reserves, the result is the present value of the loss reserve.

The model may be used to approximate the effect of such "discounting" of the loss reserves. Defining DR_m as the present value of R_m:[7]

$$DR_m = \left[\sum_{k=0}^{d-1} \sum_{j=d}^{\infty} (v^{j-k})\ {}_jP_{m-k}\right] + \left[\sum_{k=d}^{\infty} \sum_{j=k+1}^{\infty} (v^{j-k})\ {}_jP_{m-k}\right]$$

$$= \frac{bv(1+z)^{m-d+1}}{1-vr}\left[\frac{v^d(1+z)^d - 1}{v(1+z)-1} + \frac{r}{1+z-r}\right]$$

And the ratio of DR_m to R_m can be expressed as:

$$\frac{DR_m}{R_m} = \frac{vz(v-r)}{(1-vr)\,[v(1+z)-1]}$$

$$\left[\frac{v^d(1+z)^{d-1}(1+z-r) - (1-vr)}{(1+z)^{d-1}(1+z-r) - (1-r)}\right]$$

Effect upon reserve adequacy of a change in x

The same approach may be taken in determining the effect of a change at the end of month m in the monthly severity increase rate x. If the "new" rate is denoted by x', and defininig $r' = q(1+x')$ and $z' = x' + y + x'y$, then the required reserve at m, adjusted for the change from x to x', can be expressed as follows simply by replacing v in the expression for DR_m by $\frac{1+x'}{1+x}$:

$$R'_m = \frac{b(1+x')\,(1+z)^{m-d+1}}{(1+x)\,(1-r')}\left[\frac{(1+z')^d - 1}{z'} + \frac{r}{1+z-r}\right]$$

And:

$$\frac{R'_m}{R_m} = \frac{z(1+x')\,(1-r)}{z'(1+x)\,(1-r')}$$

$$\left[\frac{(1+z')^d\,(1+z-r) - (1+z)\,(1-r')}{(1+z)^d\,(1+z-r) - (1+z)\,(1-r)}\right]$$

[7] The derivations of the relationships described in this section will be found in Technical Appendix 4.

Required reserve relative to current monthly payment rate

Where a line of business has been written for a sufficient period to have closed substantialy all of the losses incurred during the first year of writing, an approximation based upon the model may be used in testing the adequacy of current loss reserves.

The test consists of determining the ratio of R_m to ${}_mP_{tot}$ from the model, and multiplying that ratio by the current average monthly loss payment rate. The resultant product is an approximation of the current required reserve. The ratio of R_m to ${}_mP_{tot}$ can be expressed as:

$$\frac{R_m}{{}_mP_{tot}} = \frac{(1+z)^d - (1+z)}{z} + \frac{(1+z)^d}{1-r}$$

CONCLUSION

The model described in this paper is but one of many models of loss payment patterns which can be developed and successfully applied to reserving problems. The applications described herein likewise represent but a few of the potential applications of such a model. It is this author's hope and expectation that the next few years will see additional actuarial papers presented on this and related subjects.

TECHNICAL APPENDIX 1

Development of formula (3):

$$I_m = \sum_{k=d}^{\infty} {}_kP_m = \sum_{k=d}^{\infty} br^{k-d}(1+z)^m = br^{-d}(1+z)^m \sum_{k=d}^{\infty} r^k$$

$$= \frac{b(1+z)^m}{1-r}, \quad 0 < r < 1$$

Development of formula (4):

$$ {}_nU_m = \begin{cases} \displaystyle\sum_{k=n+1}^{\infty} {}_kP_m = br^{-d}(1+z)^m \sum_{k=n+1}^{\infty} r^k = \frac{br^{n-d+1}(1+z)^m}{1-r}, & n \geqslant d \\ I_m & , \quad n < d \end{cases}$$

Development of formula (5):

$$R_m = \sum_{k=0}^{d-1} I_{m-k} + \sum_{k=d}^{\infty} {}_kU_{m-k} = \sum_{k=0}^{d-1} \frac{b(1+z)^{m-k}}{1-r}$$

$$+ \sum_{k=d}^{\infty} \frac{br^{k-d+1}(1+z)^{m-k}}{1-r}$$

$$= \frac{b(1+z)^m}{1-r} \left\{ \sum_{k=0}^{d-1} (1+z)^{-k} + r^{1-d} \sum_{k=d}^{\infty} \left(\frac{r}{1+z}\right)^k \right\}$$

$$= \frac{b(1+z)^m}{1-r} \left\{ \frac{(1+z)^d - 1}{z(1+z)^{d-1}} + r^{1-d} \left[\frac{r^d}{(1+z)^{d-1}(1+z-r)} \right] \right\}$$

$$= \frac{b(1+z)^{m-d+1}}{1-r} \left[\frac{(1+z)^d - 1}{z} + \frac{r}{1+z-r} \right]$$

Development of formula (6):

$$
{}_{m}P_{tot} = \sum_{k=d}^{\infty} {}_{k}P_{m-k} = \sum_{k=d}^{\infty} br^{k-d}(1+z)^{m-k}
$$

$$
= br^{-d}(1+z)^{m} \sum_{k=d}^{\infty} \left(\frac{r}{1+z}\right)^{k}
$$

$$
= \frac{b(1+z)^{m-d+1}}{1+z-r}
$$

TECHNICAL APPENDIX 2

Development of formula (7):

$$AI_t = \sum_{k=12t}^{12t+11} I_k = \frac{b}{1-r} \sum_{k=12t}^{12t+11} (1+z)^k$$

$$= \frac{b(1+z)^{12t}}{1-r} \left[\frac{(1+z)^{12}-1}{z} \right]$$

Development of formula (8):

$$_0AP_t = \sum_{k=12t}^{12t+11-d} \; \sum_{j=d}^{12t+11-k} {}_jP_k = b \sum_{k=12t}^{12t+11-d} (1+z)^k \sum_{j=d}^{12t+11-k} r^{j-d}$$

$$= b \sum_{k=12t}^{12t+11-d} (1+z)^k \left(\frac{1-r^{12t+12-d-k}}{1-r} \right)$$

$$= \frac{b}{1-r} \left[\sum_{k=12t}^{12t+11-d} (1+z)^k - r^{12t+12-d} \sum_{k=12t}^{12t+11-d} \left(\frac{1+z}{r} \right)^k \right]$$

$$= \frac{b}{1-r} \left[(1+z)^{12t} \left[\frac{(1+z)^{12-d}-1}{z} \right] - r^{12-d}(1+z)^{12t} \left[\frac{(1+z)^{12-d}-r^{12-d}}{r^{11-d}(1+z-r)} \right] \right]$$

$$= \frac{b(1+z)^{12t}}{1-r} \left[\frac{(1+z)^{12-d}-1}{z} - r \left[\frac{(1+z)^{12-d}-r^{12-d}}{1+z-r} \right] \right],$$

$$1 \leqslant d \leqslant 12$$

Development of formula (9):

$$
{}_1AP_t = \sum_{k=12t}^{12t+12-d} \; \sum_{j=12t+12-k}^{12t+23-k} {}_jP_k
$$

$$
+ \sum_{k=12t+13-d}^{12t+11} \; \sum_{j=d}^{12t+23-k} {}_jP_k
$$

$$
= b \sum_{k=12t}^{12t+12-d} (1+z)^k \sum_{j=12t+12-k}^{12t+23-k} r^{j-d}
$$

$$
+ b \sum_{k=12t+13-d}^{12t+11} (1+z)^k \sum_{j=d}^{12t+23-k} r^{j-d}
$$

$$
= b \sum_{k=12t}^{12t+12-d} (1+z^k r^{12t+12-k-d} \left(\frac{1-r^{12}}{1-r}\right)
$$

$$
+ b \sum_{k=12t+13-d}^{12t+11} (1+z)^k \left(\frac{1-r^{12t+24-k-d}}{1-r}\right)
$$

$$
= \frac{b}{1-r}\Bigg[(1-r^{12})r^{12t+12-d} \sum_{k=12t}^{12t+12-d} \left(\frac{1+z}{r}\right)^k
$$

$$
+ \sum_{k=12t+13-d}^{12t+11} (1+z)^k - r^{12t+24-d} \sum_{k=12t+13-d}^{12t+11} \left(\frac{1+z}{r}\right)^k \Bigg]
$$

$$
= \frac{b}{1-r}\Bigg[(1-r^{12})r^{12t+12-d} \left(\frac{1+z}{r}\right)^{12t} \left[\frac{(1+z)^{13-d}-r^{13-d}}{r^{12-d}(1+z-r)}\right]
$$

$$
+ (1+z)^{12t+13-d} \left[\frac{(1+z)^{d-1}-1}{z}\right]
$$

$$
- r^{12t+24-d} \left(\frac{1+z}{r}\right)^{12t+13-d} \left[\frac{(1+z)^{d-1}-r^{d-1}}{r^{d-2}(1+z-r)}\right]\Bigg]
$$

$$= \frac{b(1+z)^{12t}(1-r^{12})}{1-r}\left[\frac{(1+z)^{13-d}-r^{13-d}}{1+z-r}\right]$$

$$+\frac{b(1+z)^{12t+13-d}}{1-r}\left[\left[\frac{(1+z)^{d-1}-1}{z}\right]-(r^{13-d})\right.$$

$$\left.\left[\frac{(1+z)^{d-1}-r^{d-1}}{1+z-r}\right]\right], 2 \leqslant d \leqslant 12$$

Development of formula (10):

$${}_nAP_t = \sum_{k=12t}^{12t+11}\ \sum_{j=12(n+t)-k}^{12(n+t)-k+11} {}_jP_k = b\sum_{k=12t}^{12t+11}(1+z)^k$$

$$\sum_{j=12(n+t)-k}^{12(n+t)-k+11} r^{j-d}$$

$$= b\sum_{k=12t}^{12t+11}(1+z)^k[r^{12(n+t)-k-d}]\left(\frac{1-r^{12}}{1-r}\right)$$

$$= \frac{b(1-r^{12})r^{12(n+t)-d}}{1-r}\sum_{k=12t}^{12t+11}\left(\frac{1+z}{r}\right)^k$$

$$= \frac{b(1-r^{12})\ r^{12(n+t)-d}}{1-r}\left[\frac{(1+z)^{12t}}{r^{12t+11}}\right]\left[\frac{(1+z)^{12}-r^{12}}{1+z-r}\right]$$

$$= \frac{b(1-r^{12})(1+z)^{12t}(r^{12n-11-d})}{1-r}\left[\frac{(1+z)^{12}-r^{12}}{1+z-r}\right],$$

$$n \geqslant \frac{d+11}{12}$$

Development of formula (11):

$$
{}_0AR_t = AI_t - {}_0AP_t
$$

$$
= \frac{b(1+z)^{12t}}{1-r}\left[\left[\frac{(1+z)^{12}-1}{z}\right]-\left[\frac{(1+z)^{12-d}-1}{z}\right] + r\left[\frac{(1+z)^{12-d}-r^{12-d}}{1+z-r}\right]\right]
$$

$$
= \frac{b(1+z)^{12t}}{1-r}\left[\frac{(1+z)^{12}-(1+z)^{12-d}}{z} + r\left[\frac{(1+z)^{12-d}-r^{12-d}}{1+z-r}\right]\right],
$$

$$
1 \leqslant d \leqslant 12
$$

Development of formula (12):

$$
{}_nAR_t = \sum_{k=12t}^{12t+11} \quad 12(t+n)-k+11^{U}k
$$

$$
= \frac{br^{12(t+n)+12-d}}{1-r}\sum_{k=12t}^{12t+11}\left(\frac{1+z}{r}\right)^k
$$

$$
= \frac{br^{12n-d+1}(1+z)^{12t}}{1-r}\left[\frac{(1+z)^{12}-r^{12}}{1+z-r}\right], \qquad n \geqslant \frac{d}{12}
$$

TECHNICAL APPENDIX 3

3.1 Estimation of x and y

For the five companies and groups included in the study, automobile bodily injury net written premium for 1967 and 1968 was used to determine the total premium growth rate as follows:

Net written premium 1968	\$1,212,517,000
Net written premium 1967	\$1,062,432,000
Growth factor 1968/1967	1.1413

A review of automobile bodily injury rate filings made during 1968 and 1969 indicated that the average annual trend factors being used were approximately:

Severity	+ 5.0% per year
Frequency	− 1.0% per year

x and y were then determined as follows:

1) Premium growth factor	.1413
2) Severity growth factor	.0500
3) Frequency growth factor	− .0100
4) Pure premium growth factor	
(1.0500) (.9900) − 1	.0395
5) Volume growth rate	
(1.1413/1.0395) − 1	.0979
6) Frequency and volume combined	
(.9900) (1.0979) − 1	.0869
7) $x = (1.0500)^{1/12} - 1 =$	.004
8) $y = (1.0869)^{1/12} - 1 =$	.007

3.2 Selection of d

2 was selected as the value of d based upon the author's experience with automobile bodily injury loss reporting patterns.

3.3 Selection of r

.0498 was selected as the value for r by setting the theoretical ratio of losses paid through 12/31/69 to total incurred losses equal to the actual ratio and solving for r:

$$\frac{{}_1AP_0}{AI_0} = \frac{328{,}420 - 112{,}528}{618{,}135} = .3493$$

3.4 Determination of b and c

b was determined to be 2229.56 by solving $AI_0 = 618{,}135$ for b.

c was determined to be 44,414 by solving $b = cp(1 + x)^d$ for c.

TECHNICAL APPENDIX 4

$$DR_m = \sum_{k=0}^{d-1} \sum_{j=d}^{\infty} (v^{j-k})_j P_{m-k} + \sum_{k=d}^{\infty} \sum_{j=k+1}^{\infty} (v^{j-k})_j P_{m-k}$$

$$= \sum_{k=0}^{d-1} \sum_{j=d}^{\infty} (v^{j-k}) br^{j-d}(1+z)^{m-k}$$

$$+ \sum_{k=d}^{\infty} \sum_{j=k+1}^{\infty} (v^{j-k}) br^{j-d}(1+z)^{m-k}$$

$$= br^{-d}(1+z)^m \left[\sum_{k=0}^{d-1} v^{-k}(1+z)^{-k} \sum_{j=d}^{\infty} (vr)^j \right.$$

$$\left. + \sum_{k=d}^{\infty} v^{-k}(1+z)^{-k} \sum_{j=k+1}^{\infty} (vr)^j \right]$$

$$= \frac{br^{-d}(1+z)^m}{1-vr} \left[\sum_{k=0}^{d-1} v^{-k}(1+z)^{-k}(vr)^d \right.$$

$$\left. + \sum_{k=d}^{\infty} v^{-k}(1+z)^{-k}(vr)^{k+1} \right]$$

$$= \frac{br^{-d}(1+z)^m}{1-vr} \left[(vr)^d \sum_{k=0}^{d-1} v^{-k}(1+z)^{-k} + vr \sum_{k=d}^{\infty} \left(\frac{r}{1+z} \right)^k \right]$$

$$= \frac{br^{-d}(1+z)^m}{1-vr} \left[(vr)^d \left(\frac{v^d(1+z)^{d}-1}{v^{d-1}(1+z)^{d-1}[v(1+z)-1]} \right) \right.$$

$$\left. + \frac{vr(1+z)}{1+z-r} \left(\frac{r}{1+z} \right)^d \right]$$

$$= \frac{bv(1+z)^{m-d+1}}{1-vr}\left[\frac{v^d(1+z)^d-1}{v(1+z)-1} + \frac{r}{1+z-r}\right]$$

$$\frac{DR_m}{R_m} = \frac{\dfrac{bv(1+z)^{m-d+1}}{1-vr}\left[\dfrac{v^d(1+z)^d-1}{v(1+z)-1} + \dfrac{r}{1+z-r}\right]}{\dfrac{b(1+z)^{m-d+1}}{1-r}\left[\dfrac{(1+z)^d-1}{z} + \dfrac{r}{1+z-r}\right]}$$

$$= \frac{\dfrac{v}{1-vr}\left[\dfrac{v^d(1+z)^d-1}{v(1+z)-1} + \dfrac{r}{1+z-r}\right]}{\dfrac{1}{1-r}\left[\dfrac{(1+z)^d-1}{z} + \dfrac{r}{1+z-r}\right]}$$

$$= \frac{v(1-r)\left[\dfrac{v^d(1+z)^d-1}{v(1+z)-1} + \dfrac{r}{1+z-r}\right]}{(1-vr)\left[\dfrac{(1+z)^d-1}{z} + \dfrac{r}{1+z-r}\right]}$$

$$= \frac{vz(1-r)}{(1-vr)[v(1+z)-1]}\left[\frac{v^d(1+z)^{d-1}(1+z-r) - (1-vr)}{(1+z)^{d-1}(1+z-r) - (1-r)}\right]$$

$$\frac{R_m}{{}_mP_{tot}} = \frac{\dfrac{b(1+z)^{m-d+1}}{1-r}\left[\dfrac{(1+z)^d-1}{z} + \dfrac{r}{1+z-r}\right]}{\dfrac{b(1+z)^{m-d+1}}{1+z-r}}$$

$$= \frac{(1+z-r)\left[\dfrac{(1+z)^d-1}{z} + \dfrac{r}{1+z-r}\right]}{(1-r)}$$

$$= \frac{(1+z)^d-1}{z} + \frac{(1+z)^d-1+r}{(1-r)}$$

$$= \frac{(1+z)^d-(1+z)}{z} + \frac{(1+z)^d}{1-r}$$