ACTUARIAL APPLICATIONS IN CATASTROPHE REINSURANCE

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DISCUSSION BY DAVID G. HALMSTAD

Actuarial literature is filled with technical studies of varying complexity, but few of these emphasize the importance of consistency in the actuary's work. On the life insurance side, few actuaries would directly recognize that consistency in the results is perhaps the most important reason for using the traditional life insurance actuarial model. All of them should agree that the process of system control afforded by that model—the multi-faceted analysis of expense, mortality, lapse, investment and the role of contractual provision in both the estimation of earnings and the setting of dividends—relies heavily on the use of studies of these variables defined consistently with their use in the model itself. Most life actuaries might even agree that such consistency is a prerequisite for "equity." But I doubt that they would take one additional step with me to hold consistency to be more important than the precise level of any of the individual parameters. Yet I think that this is the case, and that it is exemplified by the case that Mr. Simon has chosen for this paper.

Mr. Simon's case is given by simple illustrations, the sign of a classic paper. In the illustrations used, an insurance is analyzed for which one acceptable price (dependent on a precise definition of the coverage being provided) is given. The problem considered is simply one of finding the consistent costs of alternative definitions of coverage.

To complement Mr. Simon's artful exposition of the relationships between catastrophe coverages of various design, I would like to examine an additional form which I have had to analyze recently. I hope to reinforce Mr. Simon's plea that the actuary can help discern the relative differences amongst alternative designs of such products.

In direct analogy to Mr. Simon's cases, let us consider a catastrophe cover that will cost the insured a pure premium of $\underline{P}$, a lower bound on the premium, if no event occurs, but a maximum premium during the period of $\overline{P}$ ($> \underline{P}$) if the cover has to be paid. This would be a trivial extension if we restricted ourselves to a single possible covered event [equivalent to providing a cover without reinstatement of $L - (\overline{P} - \underline{P})$ when $L > \overline{P}$], but we shall extend the cover "drastically" by providing the cover of L on every such event during the insured period for the maximum total premium of $\overline{P}$.

By making this change, I may well be violating every usual definition of "catastrophe reinsurance," but I do plan to offer my own such definition in due course. Dropping the limitation on the number of events covered may be unsound underwriting in some cases, but there are frequently sound reasons for the actuary to consider the case even if the underwriter imposes such limitations. And in specific cases it will be infeasible to impose such limits on either the number of events or the total possible amount of claims that may be paid.

Allowance for all possible numbers of events during a period involves relatively small changes in premiums for small numbers of expected events. For example, if Mr. Simon had added, as another example, the determination of the premium for an unrestricted cover in his example A, he would have proceeded, with $m = .09553$, to get the pure premium for an unrestricted number of events, $m \times 9 = .85979$, and then to a gross premium (using the same 18% of gross for margins) of $.85979 \div .82 = 1.0485$ (of $1,000,000.) The relative size of making the assumption of coverage for all events in this example, under 5%, is probably less than the uncertainty the underwriter has in the underlying probability of having such an event.

The case I am suggesting for analysis in this review, allowing an unlimited number of events to be covered, with a fixed amount L payable on each event, develops quite naturally from the large claim portion of liability forms in which premiums are defined on a European "swing rate" basis.

When used for non-catastrophe forms, the swing rate structure simply specifies a minimum and maximum premium with a self-insurance role played in those periods when total claims lie between the two extreme premiums. In an American version of swing rates, an expense allowance is paid to the insurer/reinsurer through a guaranteed loss ratio which is less than 1 in the central "self-insurance" band.[1] I shall consider only the simpler and more drastic European form.

[1] In effect, if the year's claims are C and the guaranteed loss ratio is R, the cost of the cover is $C \div R$, subject to minimum and maximum limits of $\underline{P}$ and $\overline{P}$, respectively.

Most coverages offered on such a swing rate basis are on contingencies where the expected losses are believed to be reasonably manageable with the possible exception of rare but spectacular disasters. Thus in the analysis of such coverages, one is faced with the mixture of two components in the risk: one reasonably stable and amenable to standard statistical and actuarial analysis, the second one which may properly be treated as a catastrophe cover. I shall assume hereafter that the premium analysis for the stable portion has been completed and that analysis of the catastrophe portion of the swing rate premium remains.

In such a catastrophe portion, when we limit the claim amount to a single level L, we have (continuing to use Mr. Simon's notation):

$$\text{Expected Losses} = \sum_{c=0}^{\infty} P_c \times cL, \text{ and}$$

$$\text{Expected Premiums} = \sum_{c=0}^{\infty} P_c \times \max(\underline{P}, \min\{\overline{P}, cL\}).$$

After equating these expressions, and dropping the similar central terms from both sides, we find that

$$\sum_{c=0}^{[P \div L]} p_c(\underline{P} - cL) = \sum_{c=1+[\overline{P} \div L]}^{\infty} p_c(cL - \overline{P}), \tag{1}$$

where [•] denotes the lower integer operator (i.e., [x] is the greatest integer less than or equal to x). Expression (1) states the obvious: those cases in which apparent profits occur (left side) must balance those where losses occur (right side).

The question that we are asked in this situation is quite similar to those of consistency explored by Mr. Simon. Since we generally know what probability (and pure premium) should be assigned to the catastrophe element of the cover, we are interested in determining, for a given $\underline{P}$ and $\overline{P}$ given to us, the level equivalent P (or Lm). That is, we wish to find a simple relation between the given set $\underline{P}$, $\overline{P}$ and L, and the logically equivalent P.

The answer is surprisingly simple when $L > \overline{P}$. In this case, which I feel should be made part of the definition of a catastrophe coverage, equation (1) becomes

$$p_0\underline{P} = \sum_{1}^{\infty} p_c(cL - P) = p_0\overline{P} + \sum_{0}^{\infty} p_c(cL - \overline{P})$$
$$= p_0\overline{P} + Lm - \overline{P} = p_0\overline{P} + P - \overline{P}, \text{ or}$$
$$p_0 = \frac{\overline{P} - P}{\overline{P} - \underline{P}} \tag{2}$$

In the specific case when claims are Poisson, this becomes $e^{-m} = (\overline{P} - P) \div (\overline{P} - \underline{P})$ and, appealing once more to the level claim assumption, we get

$$\text{(Poisson)} \quad e^{-P \div L} = \frac{\overline{P} - P}{\overline{P} - \underline{P}}. \tag{3}$$

It should be reemphasized that equations (2) and (3) are valid only when $L > \overline{P}$. In the general unit claim swing rate case represented by equation (1), the numerical solution is a bit trickier, but not impossible.

In Appendix A, we illustrate the use of an APL system designed to solve equation (1). Lines beginning in column 7 were typed by the system user; computer responses usually start in column 1. The system responds to inquiries about its status regarding three variables: *TYPE,* the distribution for numbers of events, *MEAN,* the expected number of events, and *SIZE,* the (fixed) claim size for each event, as well as resetting these variables when the user desires and it solves four forms of queries of it:

UPPER FOR	gets $\overline{P}$ for a given $\underline{P}$ (and a global[2] L and $P = mL$),
LOWER FOR	gets $\underline{P}$ for a given $\overline{P}$ (and a global L and $P = mL$),
LEVEL FOR	gets P for a set $\underline{P}$ and $\overline{P}$ (and a global L), and
N FOR	which is essentially the same as *LEVEL FOR.*

A word should be given about the algorithms used to solve equation (1). The applicable programs are given in Appendix B, along with the routines used to handle the Poisson distribution. Lines 7 and 8 (for *LOWER*) and 9 and 10 (for *UPPER*) in the program *<FOR>* solve (1)

[2] A global parameter is one which is essential to the numerical calculation of a function but which is not specifically set in the function definition. Good examples are the specific values for the interest rates and mortality rates in life insurance and annuity functions such as A_x and $\ddot{a}_x$.

for a $\underline{P}$ (or $\overline{P}$) for each possible integer of the limit of c, and take that one which is consistent with $[\underline{P} \div L]$ (or $[\overline{P} \div L]$) for the needed limit.

Solution of (1) for a level P, given $\overline{P}$ and $\underline{P}$ is not as simple in the general case. The expected number of events, m, is not known (the program *<FOR>* ignores *MEAN* in this case) and the solution is accomplished by a Newton approximation (to the root of the line 6 of *<FUNC>*). An equivalent method for expressing the level P is given by *N FOR* which returns the N in

$$P = \underline{P} + (\overline{P} - \underline{P}) \div N. \tag{4}$$

This form is an easier one to communicate to underwriters, partially since, for swing rate coverages, they frequently use the "One Third" rule expressable in (4) by setting $N = 3$. The last request shown in Appendix A is for an N of this form when the swing rate limits are 8 and 14 on a cover of 10 per event, assuming a negative binomial claim number distribution with $k = 1$ (i.e. a geometric distribution).[3] The "One Third" rule is quite accurate in this case.

We have studied the "One Third" rule as it applies to catastrophe coverage on a swing rate basis in the Poisson case. As I indicated previously, I would suggest that a catastrophe coverage be (at least in part) defined as one on which the insurer cannot, from his entire portfolio of business, expect sufficient premium to cover a single claim during the period of insurance. This at least serves as a definition of the conditions under which the use of the "One Third" rule is questionable.

In Appendix C, we present the APL algorithm used to solve the simultaneous equations (3) and (4) for N, given $\underline{P}$ and $\overline{P}$. The algorithm includes the function *<WEW>* which solves the transcendental equation

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[3] The APL function name for the negative binomial case is NEGBINOMIAL and it is followed, in the Appendix A demonstration by the parameter k which is left free after the mean m has been set. That is, the probability of exactly c claims is, in the NEGBINOMIAL k case,

$$\mathrm{Pc} = \binom{k + c - 1}{k - 1} \left(\frac{m}{k}\right)^{c} \left(1 + \frac{m}{k}\right)^{-(k + c)}$$

$$c = 0, 1, 2, \ldots$$

The system provides for assumption of any of the following distributions for the number of claims: Poisson, binomial, negative binomial, geometric, logarithmic, discrete Pareto, single point and harmonic.

$x = w \times e^w$ for w given a value $x > -e^{-1}$. Using these routines, we have, finally, constructed a contour map for finding a conservative N (giving an understated P) in the Poisson catastrophe reinsurance case. It is presented in Appendix D. The ratio $\underline{P} \div \overline{P}$ is the horizontal axis of this map, and $(\overline{P} \div L) < 1$ the vertical axis. The next integer higher than the true solution for N is shown by odd integers where applicable, with blank bands for even values of N. No values are shown in the area of $N \geq 10$, and it is suggested that $\underline{P}$ be considered as the equivalent value of P in such a case.

I would like to make one final addition that bears directly on the examples presented both by Mr. Simon and myself. All of the "consistency" calculations should be made on a pure (or net) premium basis. Mr. Simon has made this much more explicit than I, but he has neglected to include a marginal ingredient which I believe to be as important as the tax, commission and profit margins: the risk theory fluctuation loading. For catastrophe coverages, this element is both logically includeable and reasonably simple to calculate, especially when the number of possible claims is limited.

APPENDIX A

```
      ⍝ A SAMPLE SESSION WITH THE SWING RATE PACKAGE
      )LOAD SWING
SAVED 22.43.56 06/22/73
          AVIATION REINSURANCE SWING RATES
      STATE
TYPE IS UNDEFINED
MEAN IS  1
SIZE IS  1
PURE EXPECTED CLAIMS = MEAN×SIZE =  1
      UPPER FOR .9
WHAT TYPE OF DISTRIBUTION FOR NUMBERS OF CLAIMS?
⎕:
      POISSON
1.139221119117733
      )DIGITS 6
WAS 16
      LEVEL FOR .9 1.13922
1
      LOWER FOR 1.25
0.82043
      NEGBINOMIAL 2
WAS POISSON
      LEVEL FOR .9 1.13922
0.991011
      MEAN IS .5
WAS  1
      STATE
TYPE IS NEGBINOMIAL  2
MEAN IS  0.5
SIZE IS  1
PURE EXPECTED CLAIMS = MEAN×SIZE =  0.5
      UPPER FOR .3
0.855556
      LOWER FOR .75
0.359375
      NEGBINOMIAL 3
WAS NEGBINOMIAL  2
      LOWER FOR .75
0.353009
      GEOMETRIC
WAS NEGBINOMIAL  3
      LOWER FOR .75
0.375
      NEGBINOMIAL 1
WAS GEOMETRIC
      LOWER FOR .75
0.375
      MEAN IS 10
WAS  0.5
      UPPER FOR 8
13.8794
      LEVEL FOR 8 14
10.04
      N FOR 8 14
2.94111
```

```
      )WSID
SWING
      )GRP SWING
UPPER   LOWER   LEVEL   N       FOR     TYPE    MEAN    SIZE    IS      M       L       T̲YPE
PTCALC  MLRESET STATE   STATUS  RESET   SWINGAUX        NEWTON  FUNC    POISSON BINOMIAL
NEGBINOMIAL     GEOMETRIC       LOG     DPARETO SINGLE  HARMONIC        ZETA    LNGAM   POIST
BINOMT  NEGBINT GEOMT   LOGT    DPARETOT        HARMONICT       EXPECTED
```

```
      ∇FOR[⎕]∇
    ∇ Z←X FOR Y;PU
[1]    →(0=ρρT̲YPE)↑OK
[2]    0 0 ρ⎕,0ρ⎕←'WHAT TYPE OF DISTRIBUTION FOR NUMBERS OF
       CLAIMS?'
[3]   OK:→((⌈/,Y÷L)<ρP)/Z←X↓UPPER,LOWER,LEVEL,LEVEL
[4]    T←, 1 0 ↓P←M PTCALC⌈⌈/,Y÷L
[5]    P←, ¯1 0 ↓P
[6]    →Z
[7]   LOWER:Z←(L×+\P×T)+(L×M-+/P×T×Z)-Y×1-+/P×Z←Y≥T×L
[8]    →0,ρZ←(Y≥L×M)×(T=⌊Z÷L)/Z←Z÷+\P
[9]   UPPER:Z←(L×M-+\P×T)-+/P×0⌈Y-T×L
[10]   →(0≠ρZ←(Y≤M×L)×(T=⌊Z÷L)/Z←Z÷(Z≤0)+1-+\P)↑0
[11]   →(M=+/P×T)↑0×ρZ←'IMPOSSIBLE'
[12]   T←, 1 0 ↓P←M PTCALC 2×ρP
[13]   →UPPER,ρP←, ¯1 0 ↓P
[14] LEVEL:→(1=ρZ←,Y)↑F̲UNC←0
[15]   Z←L× 50 1E¯5 1E¯5 NEWTON PU←¯1↑Y←(>/Y)⌽Y÷L
[16]   →(X≠3)↑0
[17]   Z←(-/Y)÷(1↑Y)-Z÷L
    ∇
```

```
      ∇FUNC[⎕]∇
    ∇ Z←FUNC X;P;T
[1]    →F̲UNC↓FA,FB
[2]   FB:→((X>0)∧X<1)↓0×Z←1+|X
[3]    →0,ρZ←M+X÷(1-X)×⍟1-X
[4]   FA:T←(P←X PTCALC⌊PU)[1+Z←''ρι1;]
[5]    P←P[Z;]
[6]    Z←X-(PU×1-+/P)++/P×T⌈1↑Y
    ∇
```

```
      ∇POISSON[⎕]∇
    ∇ Z←POISSON
[1]    Z←'WAS',2↓TYPE
[2]    T̲YPE←0
[3]    M MLRESET L
    ∇
```

```
∇PTCALC[⎕]∇
    ∇ Z←M PTCALC MAX;P;T
[1]    T←(-ι1)+ι1+MAX
[2]    →T̲YPE↓POISSON,BINOMIAL,NEGBIN,GEOM,LOG,
       DPARETO,SINGLE,HARMONIC
[3]   POISSON:→E,ρP←T POIST M
[4]   BINOMIAL:→E,ρP←T BINOMT(M÷P̲),P̲
[5]   NEGBIN:→E,ρP←T NEGBINT K̲,M
[6]   GEOM:→E,ρP←T GEOMT M
[7]   LOG:→E,ρP←T LOGT M
[8]   DPARETO:→E,ρP←T DPARETOT B̲
[9]   SINGLE:→E,P←(ρT)↑| 1 0 -P̲
[10] HARMONIC:→E,P←T HARMONICT B̲
[11] E:Z←(Z/P),[¯0.5+ι1](Z←P≠0)/T
    ∇
```

```
      ∇POIST[⎕]∇

    ∇ Z←K POIST MEAN;KT
[1]    K←K+8×KT←K<8
[2]    →(Z←~1∊KT)↑OK
[3]    Z←(KT←(ρ,K)ρKT)/,K←(ρK=MEAN)ρK
[4]    Z←(ρK)ρ(~KT)+KT\×⌿(7 6 5 4 3 2 1
       0 ∘.-Z)÷(8,ρZ)ρKT/,MEAN
[5]   OK:KT←(K∘.*,¯2)⊥ 13860 ¯4146 1820
       ¯1287 1716 ¯6006 180180 ÷2162160
[6]    Z←Z×*(K×(1-(MEAN÷K)-⍟MEAN÷K))-(KT÷K)+
       0.5×⍟○2×K
    ∇
```

APPENDIX C

```
      )WSID
SWING
      )GRP POISSONCATASTROPHES
NN      WEW
      ∇NN[⎕]∇
    ∇ Z←PU∆L NN PL∆PU
[1]   Z←÷PU∆L
[2]   Z←1+Z×WEW(PL∆PU-1)÷Z×*÷Z
[3]   Z←(PL∆PU-1)÷PL∆PU-Z
[4]  ⍝ PU∆L IS PU÷L * PL∆PU IS PL÷PU
    ∇

      ∇WEW[⎕]∇
    ∇ W←WEW X;Z;E;B
[1]   B←(2+1∊X<0)⍴4
[2]   W←(Z←X+(X≤0)÷*1)∘.+,0
[3]   W←(1000⌊X)+(X>0)×(⍟Z)⌈(W⊥ 8 6 0)÷W⊥ 5 14 6
[4]   Z←(⍟X÷W)-W
[5]   W←W×1+E←Z×E÷(1+W)×Z+E←Z-2×(1+W)×1+W+2×Z÷3
[6]   →B←1↓B
[7]   W←W×X≠0
[8]  ⍝ SOLVES X=W×*W FOR W, GIVEN X>-*-1 [¯1<W<0 FOR X<0]
[9]  ⍝ X-W×*W←WEW X IS ¯3E¯17 AT ¯.367<X;
[10] ⍝                   ¯3.5E¯7 AT -*-1.
[11] ⍝ SEE CACM ALGORITHM 443 (1973 FEBRUARY)
    ∇

      .5 NN .1 .2 .3 .4 .5
11.8913  6.81447  5.08915  4.20882  3.66992
      .3 .5 .8 NN .6
5.46871  3.3036  2.14072
      .9 NN .9
1.73106
      .99 NN .99
1.59477
      .999999 NN .999999
1.58198
      .1 NN .1 .05 .02 .01
91.5502  181.553  451.554  901.555
      .01 .02 .05 .1 NN .1
991.505  491.509  191.524  91.5502
      .25 .5 .75 NN .25 .5 .75
13.6179  3.66992  2.08691
```

WEIGHTING 'N' FOR EQUIVALENT LEVEL PREMIUM

APPENDIX D

GIVEN AS A FUNCTION OF P[LOWER]÷P[UPPER] AND P[UPPER]÷CLAIM∆SIZE

UPPER PREMIUM ÷ CLAIM SIZE

```
    0.1       0.2       0.3       0.4       0.5       0.6       0.7       0.8       0.9       1.
     v         v         v         v         v         v         v         v         v         v
1.0>  3333333333333333333333333333           |                                                  <1.0
        333333333333333333333333333333       |
          3333333333333333333333333333333    |
           3333333333333333333333333333333333|                N<2 - - BUT USE 3
0.9>         3333333333333333333333333333333333333                                              <0.9
     5         333333333333333333333333333333333333333
     55          333333333333333333333333333333333333333333
     5555          333333333333333333333333333333333333333333333
0.8> 55555            33333333333333333333333333333333333333333333333                           <0.8
      555555            33333333333333333333333333333333333333333333333333
        55555             33333333333333333333333333333333333333333333333333333
     7   555555             3333333333333333333333333333333333333333333333333333333333
0.7> 77   5555555              33333333333333333333333333333333333333333333333333333333333333  <0.7
     777    5555555                33333333333333333333333333333333333333333333333333333333333
       77    55555555                 33333333333333333333333333333333333333333333333333333333
     9  777    555555555                 3333333333333333333333333333333333333333333333333333
0.6> 99  777     555555555                   | 333333333333333333333333333333333333333333333333 <0.6
      99  7777     5555555555                |     33333333333333333333333333333333333333333333
       99   777      55555555555             |          333333333333333333333333333333333333333
        99   7777       55555555555          |               3333333333333333333333333333333333
0.5> -----99---7777-------5555555555555------+---------------------3333333333333333333333333333 <0.5
           99    77777       55555555555555  |                            333333333333333333333
            999    77777        555555555555555                                  33333333333333
              999    777777         5555555555555555                                     333333
0.4>            999     777777          555555555555555555                                      <0.4
                  9999     7777777           5555555555555555555
                     9999     77777777       |    5555555555555555555555
                        9999      777777777  |          555555555555555555555555
0.3>                       99999       777777777               555555555555555555555555555      <0.3
                               999999       77777777777                 55555555555555555555555
       N≥10                         999999   |     777777777777                   5555555555555
                                          9999999          77777777777777                     5
0.2>                                         |   999999999           7777777777777777           <0.2
                                             |             999999999              7777777777777
                                             |                         999999999999
                                             |                                           999999
0.1>                                         |                                                  <0.1
     ^         ^         ^         ^         ^         ^         ^         ^         ^         ^
    0.1       0.2       0.3       0.4       0.5       0.6       0.7       0.8       0.9       1.
```

← ← ← ← ← ← P[LOWER]÷P[UPPER] → → → → → →

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