UNDERWRITING INDIVIDUAL DRIVERS: A SEQUENTIAL APPROACH

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Abstract

Adaptive decision models have been used with great success in many fields. This paper shows the value of the adaptive approach in underwriting individual automobile risks. Dropkin's model of the accident process serves as the basis by which adaptive and non-adaptive decisions are compared. The expected value of information about past or future driving experience is explained and developed to illustrate why adaptive and non-adaptive decisions may differ. Further insight into the adaptive model and the underlying accident process is developed by evaluating the value of information from stage *m* and the "true" value of stage *m*. The paper concludes by studying the adaptive model with discounting for the probability that a policy may lapse prematurely.

The insurance underwriter's basic task is risk selection—deciding which risks should be given insurance and which should not. To differentiate between risks, the underwriter must project the future accident experience of a driver and compare these costs against premium revenue. If expected revenues are at least as large as expected costs, the risk is acceptable.

The accuracy of the analysis is extremely important to the insurance company. If the underwriting policy is too restrictive, desirable risks will be overlooked. However, underwriters who are too liberal cost the company money by accepting undesirable business.

For the purposes of this paper, premiums are considered to be the sole source of revenue. On the cost side, expenses and insurer profits are ignored. Only the cost of accidents is considered. A simple decision rule then follows: accept the applicant if

$$\text{Premiums} \geq \text{EV (accident costs)}$$

These assumptions are made to simplify explanation of the model. Adjusting the model to include other sources of revenue and cost is simple and will not hamper its implementation.

The purpose of this paper is to propose a model which can assist the underwriter in selecting risks. The model in no way supplants the need for underwriting expertise, and it requires substantial input from the underwriter to operate properly. Although the paper considers accident frequency in determining the quality of business, other factors are clearly relevant. The effect of lapses, for example, is indicated in the final section of the paper.

It is possible to draw parallels between the underwriting decision and the decision of whether or not to grant credit. Like the underwriter, the credit manager of a company must distinguish between profitable and unprofitable risks. The credit manager must attempt to determine which risks will repay their loans and which will not. By establishing an unnecessarily restrictive policy, good risks are again overlooked. Too liberal a policy incurs unnecessary bad debts expenses.

Bierman and Hausman[1] have studied the problem of granting credit. Their results indicate that the most realistic decisions about extending credit are made when the decision-maker determines his optimal action from a multi-period analysis of the problem. The multi-period framework permits the credit manager to consider both the current and future benefits of granting credit. The model requires that the credit manager make an initial subjective estimate of the customer's probability of repayment. The decision to grant credit is made for one period, but it depends upon the expected value from current and future periods. After one period, the decision is revaluated, based now upon a revised probability of collection. The revised probability value is determined by modifying the prior estimate by the individual's repayment experience in the first period. If the expected monetary value (including costs) is still positive, credit will again be extended. The authors use Bayesian analysis to revise the probability of collection and dynamic programming to permit consideration of the expected returns from current and future periods.

The model presented by Bierman and Hausman is entirely compatible

[1] Harold Bierman, Jr., and Warren H. Hausman, "The Credit Granting Decision", *Management Science*, Vol. 16, No. 8 (1970), pp. B-519—B-532.

with the underwriting problem. Since the insurance company wishes to retain good risks and eliminate bad ones, it is quite natural to consider the underwriting decision in a sequential framework. The underwriter will determine his optimal action, based upon current information, for one period. After observing the accident experience of this period, the underwriter must incorporate the information with the original data and determine his optimal course of action for the next period. Bayesian analysis can be used to revise the underwriter's prior predictions of accident experience.

This paper presents an adaptive model similar to Bierman and Hausman's for use in underwriting individual drivers. Analysis will show that better underwriting decisions are made when the underwriter uses a sequential decision model which considers individual driving records.

A very strong case can be developed for incorporating information about driving records in underwriting decisions. A survey of the insurance literature reveals that individual driving records are often utilized in ratemaking. Studies by Wittick[2] of Canadian driving experience and by Harwayne[3] with California drivers indicate that there is significant and consistent variation in claims experience amongst drivers with different accident and traffic violation histories. Bailey and Simon[4] have established that the accident experience of an individual driver can be given a credibility weight for the purposes of determining his appropriate premium. Dropkin[5] shows that the distribution of the number of accidents for a group of individuals is most accurately described by a negative binomial function. In a subsequent paper, Dropkin[6] describes a method whereby the parameters of the group's negative binomial distribution can be updated by the accident records of individual drivers. The updated distribution serves to indicate future accident experience for each individual.

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[2] Wittick, Herbert E., "The Canadian Merit Rating Plan for Individual Automobile Risks," *PCAS* XLV, pp. 214–220.

[3] Harwayne, Frank, "Merit Rating in Private Passenger Automobile Liability Insurance and the California Driver Record Study," *PCAS* XLVI, pp. 189–195.

[4] Bailey, Robert A. and Simon, Leroy J., "An Actuarial Note on the Credibility of Experience of a Single Private Passenger Car," *PCAS* XLVI, pp 159–164.

[5] Dropkin, Lester B., "Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records," *PCAS* XLVI, pp. 165–176.

[6] Dropkin, Lester B., "Automobile Merit Rating and Inverse Probabilities," *PCAS* XLVII, pp. 37–40.

1. A Model of the Accident Process

The negative binomial model is used here as a description of the automobile accident process. The model assumes that:

(1) Each driver generates accident events according to a Poisson process in time with constant rate λ accidents per year. For any time interval of length t, the number of accidents generated by one driver is a random variable having Poisson probability function with parameter λt.

$$P(n|\lambda, t) = \frac{(\lambda t)^n \, e^{-\lambda t}}{n!}, \, n = 0, 1, 2, \ldots \qquad (1)$$

The expected value of n is

$$E(n|\lambda, t) = \lambda t$$

(2) The population is heterogeneous; each driver has a different λ value. Prior to observing the experience of a specific driver, the probability density of his λ value is given by

$$g(\lambda|a, b) = \frac{a^b \lambda^{b-1} \, e^{-a\lambda}}{\Gamma(b)}, \, \lambda > 0 \qquad (2)$$

The expected value of λ is

$$E(\lambda|a, b) = b/a$$

The marginal distribution of the number of accidents in a time interval of length t, for one driver, may be determined from assumptions (1) and (2).

$$P(n|a, b, t) = \int_0^\infty P(n|\lambda, t) g(\lambda|a, b) \, d\lambda$$

$$P(n|a, b, t) = \binom{n + b - l}{b - l} \left(\frac{a}{t + a}\right)^b \left(\frac{t}{t + a}\right)^n, \, n = 0, 1, 2, \ldots \qquad (3)$$

The expected value of n in a time period of length t is bt/a and the variance is $tb(a + t) \,/\, a^2$.

In Bierman and Hausman's model the original probabilities of collection (the prior information) are often made subjectively with little or no past information available. People may feel that this fact weakens the results. This criticism can safely be ignored when using the model in underwriting, since in this case more information is available. For any individual, prior information is the probability density of the applicant's value, characterized by a and b. When the underwriter has little or no information about an individual, other than his designated rate class, prior information can be developed without subjective input from the past accident record of the rate class. The accident experience of the class is translated into usable form by fitting a negative binominal distribution to the actual data, and calculating the values of a and b.

2. *The Economic Structure*

Let the underwriting profit from one individual for one year be represented by

$$\pi_1 = P - Cn_1$$

where P represents the pure premium and n_1 represents the actual number of accidents generated by that individual within the time period. The cost of a claim is represented here by its expected value, C. Although claim size is really a random variable its expected value is all that is needed if two conditions are met.

a.) The frequency and severity of accidents are independent random variables. This is commonly held to be true.

b.) The decision criterion utilized is based upon expected value, as is done in this paper. This is reasonable since we are dealing with decisions whose consequences are individually small relative to the overall size of the firm. The more general criterion, maximization of expected utility, would give approximately the same results in this case.

When considering the profit from one individual policy it is necessary to consider the lifetime of the policy. The number of years a policy will remain in force, before it is terminated by the insured, is unknown. Future policy life is of great importance for the decisions to be considered here. It seems most reasonable to express all results as a function of the future duration of the policy. Let m be the number of future years which the individual would renew the policy, if the choice is entirely left to him.

Thus, at the beginning, m is the life of the policy if the company does not terminate it. At any later time, m represents the future life.

Another important economic consideration is the time-value of money. Let

$$\beta = \frac{1}{1 + r}$$

be a discount factor for one year, based upon interest rate r representing the best alternative use of money. Thus, consider one dollar to be received one year hence to have present value β.

The present value of profit from one individual policy of m future years duration can be expressed as:

$$\pi_m = \sum_{i=1}^{m} (P - Cn_i) \beta^{i-1} \tag{4}$$

Here, n_i represents the number of accidents in year i and all accident costs are treated as if paid at the start of the year.

3. *Non-Adaptive Decision Problem*

Consider the following individual, applying for insurance. The underwriter, considering past records for the applicant's rate class and other factors, suggests that the prior distribution of λ for this person is gamma (equation (2)) with $a = 13.5$ and $b = 1.37$. The underwriting profit function for one year is $\pi_1 = 100 - 1000n$, for the purpose of this example. The expected profit for one year is:

$$E(\pi_1) = \sum_{n=o}^{\infty} (100 - 1000n)\, P\, (n|13.5, 1.37, 1)$$
$$= 100 - 1000\, E(n|13.5, 1.37, 1)$$

By equation (3), $E(n|13.5, 1.37, 1)$ is b/a or .10148 accidents. Therefore,

$$E(\pi_1) = -\$1.48$$

The expected underwriting profit is negative and this applicant is rejected.

Similar results are obtained from considering multi-year policies. Suppose the company can offer a three-year policy which it could not terminate during the three-year period. The expected underwriting profit is

$$E(\pi_3) = E \left\{ \sum_{i-1}^{3} (100 - 1000n_i)\, \beta^{i-1} \right.$$

At the time of the decision, n_1, n_2, and n_3 all have the same expected value, .10148 accidents. The result is

$$E(\pi_3) = (-\$1.48)(1 + \beta + \beta^2)$$

This is also negative. If $\beta = 1$, it is just a factor of three times the result for the one-year policy.

Non-adaptive decision-making will result in the rejection of this applicant, regardless of the duration considered for the policy.

It will be shown that an adaptive plan, one which anticipates the utilization of accident experience for decision-making, will give very different results.

4. *Information and Its Expected Value*

The one piece of information which the underwriter would really like to know is the number of accidents which the applicant will have. Although this information is unattainable, the analysis of its value will still be useful. Suppose, for simplicity, that the policy will last only one year. If future loss experience were known in time to be used in making the underwriting decision, the underwriter would accept the applicant only if the number of accidents, n, is zero. This is easily seen from the profit expression which is negative for all $n > 0$. The future profit from the one-year policy, as a function of n, is 100 if the information is that $n = 0$, and zero if the information is $n > 0$. However, the underwriter must evaluate this information prior to its receipt. The expected value with the information is found by weighting the two possible outcomes by their probabilities. For the applicant having $a = 13.5$ and $b = 1.37$, the probability of no accident in the next year is

$$P(0|13.5, 1.37, 1) = .9067$$

Therefore, the expected value of the process with the information is

$$100(.9067) + 0(.0933) = \$90.67$$

The expected value of perfect information about n, denoted $EVPI_n$, is the above quantity less the expected profit without having the information.

Since it was best to reject the applicant, the latter quantity is zero, and the result is,

$$EVPI_n = \$90.67$$

This quantity is useful as an upper bound on the value of any information to be used in a one-year policy decision. It also serves here to illustrate the concept of expected value of information, which was developed by Raiffa and Schlaifer.[7]

Although the information of the previous type is not available at any price, it is possible that the underwriter can obtain some additional information about the individual's prior distribution from outside sources and use that information in making his decision. The extra information usually involves some cost and it is important to know how much can be paid for it.

The model of the accident process implies that the best information, short of knowing actual accident experience, would be the λ value of the individual applicant. The only parameter of a Poisson probability function is λ, the expected number of claims per year. Knowing λ exactly makes possible exact prediction of the probability of x ($x = 0, 1, 2, \ldots$) accidents.

The best information which is actually available is information about the individual's actual past experience. If many years of past experience are available this information will be almost as valuable as information about λ. However, it is important to realize that even a small amount of this information is quite useful in the decision process.

Suppose that the underwriter can purchase the actual past T years of experience of the applicant and can use this information in his decision of whether to approve a one-year policy. The information will tell him k, the number of accidents the applicant had in the T year period.

Without the information, the expected value of the one-year underwriting profit was found to be

$$E(\pi_1) = E(100 - 1000n) = 100 - 1000\frac{b}{a}$$
$$= -1.48$$

[7] Raiffa and Schlaifer, *Applied Statistical Decision Theory,* The MIT Press, 1961.

However, with the information, the underwriter will modify his prior gamma probability distribution of λ to obtain a posterior gamma probability distribution[8] having parameters

$$\begin{aligned} a' &= a + T \\ b' &= b + k \end{aligned} \tag{5}$$

which is a blending of the prior knowledge with the sample information. The expected underwriting profit becomes

$$E(\pi_1) = 100 - 1000 \left(\frac{b + k}{a + T}\right)$$

This expression depends upon k. The value of the best decision, as a function of k, is

$$\text{Max} \left\{ 0, 100 - 1000 \left(\frac{b + k}{a + T}\right) \right\}$$

Before k is known, the value of the decision process with the information can be found by taking the expectation of the above expression using the probability function

$$P(k|a, b, T) = \left(\frac{a}{a + T}\right)^{b} \left(\frac{T}{a + T}\right)^{k} \binom{b + k - l}{k}$$

This expectation can be represented by

$$\sum_{k=o}^{k^*} \left[100 - 1000 \left(\frac{b + k}{a + T}\right)\right] P(k|a, b, T)$$

where k^* is the largest value of k such that

$$\left[100 - 1000 \left(\frac{b + k}{a + T}\right)\right] > 0$$

For illustration, let $T = 1$, $a = 13.5$, $b = 1.37$. Then $k^* = 0$, and

$$\left[100 - 1000 \left(\frac{b}{a + 1}\right)\right] P(0|a, b, 1) = 5.00$$

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[8] See Mayerson, Allen L., "A Bayesian View of Credibility," *PCAS* LI, pp. 85–104.

As already shown, the expected value without the information is zero. Therefore, the expected value of sample information from T years of actual past experience is

$$EVSI(a, b, 1) = \$5.00$$

Summarizing the previous development,

$$EVSI(a, b, T) = \sum_{k=o}^{k^*} \left[100 - 1000 \left(\frac{b + k}{a + T} \right) \right] P(k|a, b, T)$$
$$- \operatorname{Max} \left\{ 0, 100 - 1000 \left(\frac{b}{a} \right) \right\}$$

Information about driving history will appear even more valuable if a multi-period policy is considered. The underwriting profit from a policy which, if granted, will last m years is

$$\pi_m = \sum_{i=1}^{m} (100 - 1000 n_i) \, \beta^{i-1}$$

At the time of the decision all the random variables $n_1, n_2, \ldots, n_m$ have the same expectation; $E(n|a, b, 1)$. Therefore, the expected value of underwriting profit is

$$E(\pi_m) = \left[100 - 1000 \left(\frac{b}{a} \right) \right] \left(\frac{1 - \beta^m}{1 - \beta} \right)$$

Now it can be seen that the expected profit from m years is just a multiple of the expected profit for one year. Therefore, the expected value of sample information to be used in the m year decision is

$$EVSI(a, b, T, m) = \left(\frac{1 - \beta^m}{1 - \beta} \right) EVSI(a, b, T)^{9}$$

It can be proven that $EVSI(a, b, T)$ is a non-negative quantity and is an increasing function of T, increasing at a decreasing rate and approaching an asymptotic value, which would be the expected value of perfect information about the individual's λ value.

The foregoing development establishes the value of information about individual driving experience. Sometimes this experience will be available

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[9] As β approaches one, the value of $(1-\beta^m) \,/\, (1-\beta)$ approaches m.

from policy records, and the question is whether the information is worth the cost of data processing required to extract it. Even when the information is not available from past records, it is clear that future experience information will be available and the decision-making structure ought to anticipate using the information as it becomes available.

5. *The Multi-Period Adaptive Decision*

To obtain information about driving experience for use in future periods, the applicant must first be accepted. The underwriter will then receive information which can be used to determine the best future underwriting decisions. A single-decision model ignores the value of such information by basing the underwriting decision solely upon the prior distribution for λ.

A multi-stage dynamic programming model does not have this weakness; it utilizes the information as it becomes available.

Let $V_m(a, b)$ = optimal expected present value of the next m periods when the prior distribution of λ has parameters a, b at the start of the first period.

$\beta = 1/1 + r$ = a factor discounting future returns to consider the time value of money.

The underwriter chooses that action (accept or reject) which results in the optimal expected profit at each stage. The expected profit for rejection at any stage is zero. If the applicant is accepted, the expected profit is the expected first year return plus the discounted returns from future periods. The expected profit from one year, as a function of the parameters a, b of the prior distribution at the start of the year is

$$R(a, b) = E(\pi_1) = P - C\,E(n|a, b, 1)$$

If the experience during the first period is n accidents, and the period is one year, then the prior parameters will be transformed into the posterior parameters $a + 1$ and $b + n$. The value of the remaining $m - 1$ periods will be represented by

$$V_{m-1}(a + 1, b + n)$$

However, n is unknown at the start of the first period and thus the expected value of $V_{m-1}(a+1, b+n)$ must be used. It is[10]

$$E_{n|a,b}\{V_{m-1}(a+1, b+n)\} = \sum_{n=0}^{\infty} P(n|a, b, 1)\, V_{m-1}(a+1, b+n)$$

The dynamic programming equation for V_m, in terms of V_{m-1}, is

$$V_m(a, b) = \text{Max}\,\{0, \overbrace{R(a, b)}^{\substack{\text{1st year}\\ \text{expected}\\ \text{return}}} + \beta\, \overbrace{E_{n|a,b}\{V_{m-1}(a+1, b+n)\}}^{\substack{\text{expected returns}\\ \text{from future periods}}}\} \tag{6}$$

$V_0(a, b)$ is defined as 0 for all a, b.

$V_1(a, b) = \text{Max}\,\{0, P - CE(n|a, b, 1)\}$ and corresponds to a single stage decision.

Example

A simple example has been chosen to illustrate the procedure. The optimal decision for the next three years, $m = 3$, will be determined, assuming a policy period of $t = 1$ year[11] and no discounting ($\beta = 1$). The profit function is $100 - 1000n$ and the prior distribution for λ is $g(\lambda|13.5, 1.37)$. The optimal decision for single stage models, under these conditions, is rejection, as was previously shown.

To calculate $V_3(a, b)$, various values of $V_2(a+1, b+n_1)$ are needed. $V_2(a+1, b+n_1)$ in turn depend upon the values for one stage problems, $V_1(a+2, b+n_1+n_2)$.

The one stage process has the following form for any a, b.

$$\begin{aligned} V_1(a, b) &= \text{Max}\,\{0, 100 - 1000E(n|a, b, 1)\} \\ &= \text{Max}\,\{0, 100 - 1000(b/a)\} \end{aligned}$$

The optimal decision will be to accept if $a > 10b$ and "receive" $100 - 1000(b/a)$; otherwise reject and receive 0.

[10] The notation $E_{n|a,b}\{\cdot\}$ denotes the same expectation operation previously written $E(\cdot\,|a, b, t)$. It will be used only when $t = 1$ and so the t value can be suppressed.

[11] The optimal policy period is one year. If the company issued a multi-year, non-cancellable contract, the underwriter would be forced to ignore the experience information until the next renewal date.

In this situation $a + 2 = 15.5$, $b = 1.37 + k$, $k = 0, 1, 2, \ldots$

$$\begin{aligned}
V_1(15.5, 1.37) &= \text{Max}\,\{0, 100 - 1000(1.37/15.5)\} = 11.61\\
V_1(15.5, 2.37) &= \text{Max}\,\{0, 100 - 1000(2.37/15.5)\} = 0\\
V_1(15.5, 3.37) &= \text{Max}\,\{0, 100 - 1000(3.37/15.5)\} = 0\\
V_1(15.5, 1.37 + k) &= 0, \text{ for } k = 1, 2, 3, \ldots
\end{aligned}$$

Returning to Stage 2,

$$\begin{aligned}
V_2(a, b) = \text{Max}\,\{0, 100 - 1000(b/a)\\
+ E_{n|a,b}\{V_1(a + 1, b + n)\}\}
\end{aligned}$$

For $a = 14.5$, $b = 1.37$,

$$\begin{aligned}
V_2(14.5, 1.37) = \text{Max}\,\{&0, 5.52 + V_1(15.5, 1.37)\; P\,(0|14.5, 1.37, 1)\\
&+ V_1(15.5, 2.37)\; P\,(1|14.5, 1.37, 1) + \cdots\}
\end{aligned}$$

All terms involving $V_1(15.5, 1.37 + k)$ for $k = 1$ or higher are zero. Therefore

$$V_2(14.5, 1.37) = \text{Max}\,\{0, 5.52 + (11.61)(.91268)\} = \$16.12$$

For $a = 14.5$, $b = 2.37$

$$\begin{aligned}
V_2(14.5, 2.37) = \text{Max}\,\{&0, 100 - 1000(2.37/14.5)\\
&+ V_1(15.5, 2.37)\; P(0|14.5, 2.37, 1)\\
&+ V_1(15.5, 3.37)\; P\,(1|14.5, 2.37, 1) + \cdots\}
\end{aligned}$$

All of the terms for the continued decision are negative or zero. Therefore, $V_2(14.5, 2.37) = 0$. Similarly,

$$V_2(14.5, 1.37 + k) = 0, \text{ for } k = 2, 3, \cdots.$$

Finally, for stage 3,

$$\begin{aligned}
V_3(a, b) = \text{Max}\,\{0, 100 - 1000(b/a)\\
+ E_{n|a,b}\{V_2(a + 1, b + n)\}\}
\end{aligned}$$

After exclusion of terms which are zero, this is

$$\begin{aligned}
V_3(13.5, 1.37) &= \text{Max}\,\{0, 100 - 1000(1.37/13.5)\\
&\qquad + V_2(14.5, 1.37)\; P\,(0|13.5, 1.37, 1)\}\\
&= \text{Max}\,\{0, -1.48 + (16.12)(.90674)\}\\
&= \$13.14
\end{aligned}$$

The optimal underwriting decision from a sequential analysis is to insure the applicant for the first year. Whether or not further insurance will be granted depends upon the individual's accident experience, but a decision rule has been found. The expected profit is \$13.14, as opposed to an expected return of \$0 from a single stage analysis (where the optimal action is rejecting the applicant). The increase in expected profit occurs because the underwriter receives and utilizes additional information. The information about individual driving records allows the underwriter to retain good risks and eliminate bad ones. The value of this information is \$13.14, the expected profit with the information about driving records less the expected profit without information about driving records.

6. *The Value of Information From Stage m*

At each stage in the sequential process, the underwriter obtains information about the insured's accident rate λ from the individual's accident experience. Since knowledge about driving records enables the underwriter to make better decisions, the information has value.

Define $CVSI_m(n|a, b)$ as the conditional value of the sample information gathered in stage m, which is the first year of an m year process. If the individual is insured during stage m, the underwriter observes the accident experience—n accidents in one year—and has an optimal expected value at stage $m - 1$ of $V_{m-1}(a+1, b+n)$. If the individual is not insured during stage m, the optimal expected value at stage $m - 1$ is $V_{m-1}(a, b)$.

$$CVSI_m(n|a, b) = \beta\{V_{m-1}(a+1, b+n) - V_{m-1}(a, b)\}$$

The discounting factor is applied, because the information obtained in stage m can first be used in stage $m - 1$, one year later.

Define $EVSI_m(a, b)$ as the expected value of information from stage m.

$$\begin{aligned} EVSI_m(a, b) &= E_{n|a,b}\{CVSI_m(n|a, b)\} \qquad (7) \\ &= \beta\, E_{n|a,b}\{V_{m-1}(a+1, b+n)\} - \beta V_{m-1}(a, b) \end{aligned}$$

The expected value of the information gained during the first year of the three-year period is calculated as an example.

$$\begin{aligned} EVSI_3(13.5, 1.37) &= E_{n|13.5,\ 1.37}\{V_2(14.5, 1.37 + n)\} \\ &\quad - V_2(13.5, 1.37) \\ &= P(0|13.5, 1.37, 1)V_2(14.5, 1.37) \\ &\quad - V_2(13.5, 1.37)^{12} \end{aligned}$$

after zero terms are omitted. This reduces to

$$EVSI_3(13.5, 1.37) = 14.62 - 3.52 = 11.10$$

The definition of $EVSI_m(a, b)$ has the important property that it is non-negative. This agrees with the intuitive notion that information is always expected to be beneficial, although on an after-the-fact basis it can have negative value. This is shown in the appendix.

7. *The True Expected Value of Stage m*

It has been shown that an adaptive policy often leads to the acceptance of an individual whose application would be rejected on the basis of a non-adaptive decision. This is the situation when the expected one-year profit, $R(a, b)$, is negative but at the same time, for a horizon of m years, $V_m(a, b)$ is positive. The difference is due to the value of the information to be gained in stage m and utilized thereafter in the $m - 1$ remaining decisions.

It is useful to define $R_m^*(a, b)$ to be the "true" expected value which can be attributed to stage m. If the applicant were not to appear until one year later, then period m would be idle and the present value would be $\beta V_{m-1}(a, b)$. Let the true value of stage m be defined by

$$R_m^*(a, b) = V_m(a, b) - \beta V_{m-1}(a, b),$$

the change in expected value between utilizing and not utilizing stage m. It is shown in the appendix that this is equivalent to

$$R_m^*(a, b) = \begin{cases} R(a, b) + EVSI_m(a, b) \text{ if optimal to continue} \\ 0 \text{ if optimal to terminate or not grant the policy.} \end{cases}$$

[12] $V_2(13.5, 1.37)$ was not calculated before. It requires $V_1(14.5, 1.37 + k)$ for $k = 0, 1, 2$. These also were not calculated previously. The results are $V_1(14.5, 1.37) = 5.52$ and $V_1(14.5, 1.37 + k)$ for $k = 1, 2, 3 \ldots$, are all zero. Finally, $V_2(13.5, 1.37) = 3.52$.

Thus, the true expected value of stage m is the sum of the expected immediate profit of stage m plus the expected value of the information to be received during stage m. Thus, it will be optimal to continue the policy even when $R(a, b)$ is negative, if the value of $EVSI_m(a, b)$ is large enough to make the sum positive.

To illustrate the concept of true expected value, return to the example with $a = 13.5$, $b = 1.37$, $m = 3$, $\beta = 1$. We have found that the

$$EVSI_3(13.5, 1.37) = 11.10$$
$$\text{and } R(13.5, 1.37) = -1.48$$

Therefore

$$R_3^*(a, b) = -1.48 + 11.10 = 9.62$$

and we conclude that the information value far exceeds the small expected loss during the first-year of the adaptive decision. Here the expected loss $R(13.5, 1.37)$ can be viewed as a cost of sampling for information.

8. The Administration of Adaptive Decision Rules

The administration of these decision rules begins with the assignment of prior parameter values (a, b) to each new applicant based upon his rate classification and other, possibly subjective, information of use in the underwriting function. The optimum decision rule is calculated to give the accept/reject decision and the decision rule for future periods.

On the anniversary of each policy the prior parameters are updated to include the experience of the past year. The decision to continue or terminate the policy is made according to the decision rule previously calculated.

A useful classification system can be utilized, based upon the calculations shown. All existing policies can be classified into the following states which characterize their current condition:

"Trial" or "Tentative" State—the current values of a, b and m are such that $R(a, b) < 0$ while $R_m^*(a, b) > 0$. Such a policy is being continued only as an experiment which may result in favorable information.

Secure State of Degree n—the current values of a and b are such that $R(a, b) > 0$. Such a policy has positive expected profit for the current year. However, if more than n accidents were to occur during this year, this policy would revert to the trial stage, or even be terminated. In other words, n is the largest number such that $R(a+1, b+n)$ is still positive. The number n is very easy to compute.

The classification of policies into these states emphasizes the various different degrees of security of policies.

Each policy may move, with experience, from one phase to another. Good experience will tend to move a policy into higher states of security while unfavorable experience will move a policy rapidly downward in security level.

9. *The Effect of Lapses*

The expected profit from an applicant depends upon m, the future life of the policy, or the policy horizon. Conditional upon a future life of m years, $V_m(a, b)$ gives the optimal expected profit. The optimal decisions depend strongly upon m; the longer the life of the policy, the more valuable is the adaptive ability. For policies which would always lapse after one year, the adaptive feature is useless. The feature becomes very valuable for small and moderate m. The sensitivity to m is decreased beyond that, however, because of the discounting.

Fortunately, the effect of policy lapse (termination by the insured) can easily be introduced into the dynamic program equations. Let α_i be the probability that the policy will lapse during the i^{th} year given that it has entered the i^{th} year. These "lapse rates" may be constant and equal for all years or they may depend upon the policy age or other policy characteristics. Given the conditional probabilities, the unconditional probability that the policy will remain in force at least n years is

$$\prod_{i=1}^{n} (1 - \alpha_i)$$

The unconditional probability that it will lapse first in year n is

$$\prod_{i=1}^{n-1} (1 - \alpha_i)\, \alpha_n \text{ [13]}$$

The expected future life is

$$\sum_{n=1}^{\infty} n \prod_{i=1}^{n-1} (1 - \alpha_i)\, \alpha_n \text{ which is } \frac{1}{\alpha} \text{ if all } \alpha_i = \alpha$$

[13] $\prod_{i=1}^{n-1}(1 - \alpha_i)$ is defined to be equal to one for $n = 1$.

The dynamic programming equation becomes

$$V_m(a, b) = \text{Max}\ \{0, R(a, b) + (1 - \alpha_1)\ \beta E_{n|a,\,b}\{V_{m-1}(a+1, b+n)\}\}$$

for the first year of a period of m years duration, and

$$V_{m-i}(a, b) = \text{Max}\ \{0, R(a, b) + (1 - \alpha_{i+1})\ \beta E_{n|a,\,b}\{V_{m-i-1}(a+1, b+n)\}\}$$

for the successive years.

If the lapse probability for each year is the same, this process amounts to using a larger discount rate $(1 - \alpha)\beta$ in place of β.

The significance of m now is different than previously used. Here it represents the planning horizon of the company rather than the policy horizon. The results will become insensitive to m as long as m is taken to be larger than the expected life of the policy.

To consider the effects of interest and the probability of lapse, the expected returns from each stage must be multiplied by the appropriate discount factor. This discounting factor increases over time. Discounted, the expected returns are very small from stages far in the future. Thus, as m gets large, the discounted value of $V_m(a, b)$ will converge to a constant amount.

Example

Suppose that an applicant having the same description as before is being considered. Again $\beta = 1$. The new feature is that the probability of lapse during the first year is $\alpha_1 = .5$, $\alpha_2 = .5$ for the second year and $\alpha_3 = 1$ for the third year.

Recalling the original results for $a = 13.5, b = 1.37$ without consideration of lapse;

$V_1(13.5, 1.37) = 0$ $\qquad$ $V_2(13.5, 1.37) = 3.52$

$V_1(14.5, 1.37) = 5.52$ $\qquad$ $V_2(14.5, 1.37) = 16.12$

$V_1(15.5, 1.37) = 11.61$ $\qquad$ $V_3(13.5, 1.37) = 13.14$

$R(13.5, 1.37) = -1.48$

With lapse

$$(1)\quad V_1^L(13.5, 1.37) = \text{Max}\,\{0, R(13.5, 1.37) + (1 - \alpha_3)\beta E_{n|13.5,\,1.37}\{V_0(14.5, 1.37 + n\}\}$$
$$= 0$$

Similarly, $V_1^L(14.5, 1.37) = 5.52$ and $V_1^L(15.5, 1.37) = 11.61$

$$\begin{aligned}(2)\quad V_2^L(13.5, 1.37) &= \text{Max}\,\{0, R(13.5, 1.37) \\ &\quad + (1 - \alpha_2)\beta E_{n|13.5,\,1.37}\{V_1^L(14.5, 1.37 + n\}\} \\ &= \text{Max}\,\{0, \text{R}(13.5, 1.37) \\ &\quad + [\tfrac{1}{2}\, \text{P}(0|13.5, 1.37)V_1^L(14.5, 1.37) \\ &\quad + \tfrac{1}{2}\, \text{P}(1|13.5, 1.37)V_1^L(14.5, 2.37) + \cdots]\} \\ &= \text{Max}\,\{0, -1.48 + \tfrac{1}{2}[(.90674)(5.52) \\ &\quad + (.08326)(0)]\} \\ &= \$1.02\end{aligned}$$

A similar calculation will reveal that $V_2^I\,(14.5, 1.37) = 10.82$

$$\begin{aligned}(3)\quad V_3^L(13.5, 1.37) &= \text{Max}\,\{0, R(13.5, 1.37) \\ &\quad + (1 - \alpha_1)\beta E_{n|13.5,\,1.37}\{V_2^L(14.5, 1.37 + n\}\} \\ &= \text{Max}\,\{0, R(13.5, 1.37) \\ &\quad + [\tfrac{1}{2}P(0|13.5, 1.37)V_2^L(14.5, 1.37) \\ &\quad + \tfrac{1}{2}P(1|13.5, 1.37)V_2^L(14.5, 2.37) + \cdots] \\ &= \text{Max}\,\{0, -1.48 + (.90674)(10.82) \\ &\quad + (.08326)(0)\} \\ &= \$3.43\end{aligned}$$

The results of this calculation illustrate the importance of adaptive decision-making even when the lapse rate is high. Although the policy has an expected life of less than two years, by making adaptive decisions the underwriter expects to realize a profit for a policy horizon of more than one year.

10. Applications

Applications of the model are not limited to underwriting. With appropriate modifications, the model can also be used as a ratemaking tool.

For risk selection decisions, the decision-maker must be able to formulate a profit function similar to the one shown in section 2. This function can be specified on a net basis by ignoring all sources of revenue and cost except premiums and claims costs. Or, if the appropriate information is available, a profit function including these costs may be derived. The model then requires the decision-maker to specify the applicant's rate class and the loss characteristics of that class. At this point additional information about the applicant will be recognized and appropriate adjustments to the parameters of the prior probability distribution should be made. The best decision follows directly from equation (6) as illustrated in the paper.

For ratemaking applications the model is used somewhat differently. In this case the decision variable is the level of premiums rather than whether to accept or reject the applicant. The decision-maker wishes to find the premium for which $V_m(a, b)$ is just equal to zero. This rate represents the minimum amount the company should charge to insure the individual. At any rate lower than the minimum the company would expect to lose money on each person it insures. To determine the minimum rate an underwriting profit function must again be specified and the parameters of the individual's loss distribution must be developed. The model can be a tool for pricing both existing and proposed contracts. One interesting possibility would be to price a non-cancellable, multi-year policy where the premium rate is held constant between renewal dates, irrespective of the insured's accident experience.

Appendix

The Non-negativity of the Expected Value of Information from Stage m

It is to be proved that $EVSI_m(a, b) \geq 0$ for all m. This is equivalent to

$$E_{n|a,b}\{V_{m-1}(a+1, b+n)\} \geq V_{m-1}(a, b) \text{ for all } m.$$

This will be done by induction. Letting $m = 2$, by definition,

$$V_1(a+1, b+n) = \text{Max}\,\{0, R(a+1, b+n)\} \text{ for all } n$$

So

$$E_{n|a,b}\{V_1(a+1, b+n)\} = E_{n|a,b}\{\text{Max}\,\{0, R(a+1, b+n)\}\}$$

Reversing the operations of expectation and maximization, the inequality is found to be

$$E_{n|a,b}\{\text{Max}\,\{0, R(a+1, b+n)\}\} \geq \text{Max}\,\{0, E_{n|a,b}\{R(a+1, b+n)\}\}$$

This right hand side is, by definition, $V_1(a, b)$.

Hence,

$E_{n|a, b}\{V_{m-1}(a+1, b+n)\} \geq V_{m-1}(a, b)$ has been shown for the special case $m = 2$. Now assume the inequality holds for $m-2$ and consider whether it is true for $m-1$. A similar argument is used. By definition,

$$E_{n|a, b}\{V_{m-1}(a+1, b+n)\} = E_{n|a, b}\{\text{Max}\ \{0, R(a+1, b+n) + E_{n'|a+1, b+n}\{V_{m-2}(a+2, b+n+n')\}\}\}$$

Now, reversing the operations, the inequality is obtained

$$E_{n|a, b}\{V_{m-1}(a+1, b+n)\} \geq \text{Max}\ \{0, E_{n|a, b}\{R(a+1, b+n)\} + E_{n|a, b}\{E_{n'|a+1, b+n}\{V_{m-2}(a+2, b+n+n')\}\}\}$$

Now it can be shown that

$$E_{n|a, b}\{R(a+1, b+n)\} = R(a, b).$$

The iterated expectation over n and n' can be reversed, using the fact than n and n' are conditionally independent given λ. This reversal gives

$$E_{n|a, b}\{E_{n'|a+1, b+n}\{V_{m-2}(a+2, b+n+n')\}\} = E_{n'|a, b}\{E_{n|a+1, b+n'}\{V_{m-2}(a+2, b+n'+n)\}\}$$

By assumption, the inner term obeys

$$E_{n|a+1, b+n'}\{V_{m-2}(a+2, b+n'+n)\} \geq V_{m-2}(a+1, b+n')$$

Hence, the inequality is obtained

$$E_{n|a, b}\{V_{m-1}(a+1, b+n)\} \geq \text{Max}\ \{0, R(a, b) + E_{n'|a, b}\{V_{m-2}(a+1, b+n')\}\}$$

the right hand side is just $V_{m-1}(a, b)$ and so the result is true for $m-1$ and, by induction, true for any m.

Appendix

The True Expected Value of Stage m

The true expected value of stage m has been defined as the optimal expected value of an m stage process less that of the $m-1$ stage process starting from the same state of information but discounted by one period.

$$R^*_m(a, b) = V_m(a, b) - \beta V_{m-1}(a, b)$$

Suppose momentarily that the decision is between continuation of the policy or delaying the renewal one period. This is represented by the dynamic program

$$V_m(a, b) = \text{Max}\ \{\beta V_{m-1}(a, b), \mathrm{R}(a, b) + \beta E_{n|a, b}\{V_{m-1}(a+1, b+n)\}\}$$

The first term in the bracket represents the choice of delaying the decision one period. It will later be shown that this implies that $\beta V_{m-1}(a, b) = 0$ and hence that this dynamic program is equivalent to the one originally discussed.

Equation (7) can be written as

$$\beta E_{n|a, b}\{V_{m-1}(a+1, b+n)\} = EVSI_m(a, b) + \beta V_{m-1}(a, b)$$

Substitution into the above dynamic program gives

$$V_m(a, b) = \text{Max}\ \{\beta V_{m-1}(a, b), R(a, b) + EVSI_m(a, b) + \beta V_{m-1}(a, b)\}$$

subtraction of $\beta V_{m-1}(a, b)$ from all terms gives

$$R_m^*(a, b) = \text{Max}\ \{0, R(a, b) + EVSI_m(a, b)\}$$

Thus the true expected value of stage m is $R(a, b) + EVSI_m(a, b)$ if if it is optimal to continue the policy and zero otherwise.

It remains to show that the above dynamic program implies the original dynamic program. It can be shown from

$$V_m(a, b) = \text{Max}\ \{\beta V_{m-1}(a, b), R(a, b) + \beta E_{n|a, b}\{V_{m-1}(a+1, b+n)\}\}$$

that $V_m(a, b) = \beta V_{m-1}(a, b)$ which implies that

$$V_m(a, b) = V_{m-1}(a, b) = \cdots = V_0(a, b) = 0.$$

This will not be shown here in detail but is based upon the inductive argument that if delay is optimal with m stages remaining it will also be optimal with $m - 1$ stages remaining. By induction it is optimal with zero stages remaining also, and thus has expected value zero.