

## BOOK NOTES

Hans Bühlmann, *Mathematical Methods in Risk Theory*, 210 pages, Springer-Verlag, 1970.

Reviewed by JAMES C. HICKMAN

Actuarial *theory* and the mathematical theory of risk are practically identical. There are, of course, many aspects of actuarial science and actuarial practice that are effectively independent of risk theory. Yet if one attempts to define the singular body of theory on which actuarial science is built, he comes up with a statement that satisfactorily describes risk theory. Unfortunately, despite the fundamental importance of the subject to actuarial science, twentieth century developments in risk theory have not, in any large measure, been incorporated into the mainstream of North American actuarial education. One can easily conjecture several explanations for this regrettable lag, but one which seems very plausible has been the lack of an English language textbook on the subject.

Since 1968 this impediment to the incorporation of new risk theory ideas into North American actuarial education and practice has been removed by the publication of four important books. In 1968 *The Economics of Uncertainty* by Karl Borch, Norwegian economist and actuary, was published by Princeton University Press. Borch provides his readers with a panoramic view of a collection of ideas for introducing probability components into economic models. This is not a book on risk theory; yet because so many of the examples are drawn from insurance, this book belongs on any list of volumes contributing to the propagation of risk theory ideas among English speaking actuaries.

In 1969 the book *Risk Theory* by Beard, Pentikäinen, and Pesonen was published by Metheun. It is fashionable for specialists to criticize this slim volume for oversights and misprints. Yet within this book the main ideas of collective risk theory are succinctly developed, several of the most promising methods for making approximate probability statements using the collective risk model are discussed, and a collection of miscellaneous topics including ruin probability approximations and the use of risk models in business plan-

ning are summarized. All of this is done at a mathematical level which is not inconsistent with that required of North American actuarial students.

Also in 1969 the book *Stochastic Theory of a Risk Business* by H. L. Seal was published by John Wiley. With his usual thoroughness Seal surveys and orders the scattered literature of risk theory and provides an invaluable guide to the relationship between risk theory and its mathematical parent, the theory of stochastic processes. Now in 1970, with the publication of Bühlmann's monograph, the diligent reader of English may work his way very close to the frontier of risk theory.

A reviewer must start by complimenting Bühlmann on his honesty. In the preface he makes it quite clear that he is concerned only with the development of mathematical models appropriate for risk processes. True enough; except for a very few examples that illustrate the application of some of the models, the practical problems involved in estimating the parameters of the models receive little attention in this book. Basically this volume is a research monograph, not a textbook. Yet Bühlmann claims that it may be read by a person "who is moderately familiar with [probability] theory on an intermediate level (without the use of measure theory)." Once again he is right. By prudently avoiding proofs that involve the direct use of measure theory, he has made it possible for a reader with the background he prescribes to appreciate his main ideas.

Perhaps the best way to indicate the scope of Bühlmann's book is to conduct a brief tour of the chapters. Chapter one contains a review of the fundamentals of probability theory required in the remainder of the book. Since most of the technical mathematics problems in probability relate to integration, Bühlmann has thoughtfully provided an appendix in which some of the principal definitions and theorems concerning the generalized Riemann-Stieltjes integral are summarized. Chapter one is amazingly self contained. However, it is practically impossible to encompass within 34 pages all of the basic probability ideas that one needs to advance to the frontier of modern risk theory. For example, chapter one says relatively little about techniques for establishing limiting distribution results. Yet the exercises for chapter one require the reader to employ the unstated and rather deep "continuity theorem" for characteristic functions to solve a limiting distribution problem.

In chapter two the fundamental risk model is developed. Those expecting a restatement of the classical collective risk model will be in for a

surprise. Bühlmann does not redevelop the Poisson distribution for the number of claims, but rather he makes the weaker assumption that the number of claims is a Markov process (that is, the probability of a claim in a period is permitted to depend on the number of past claims). Some of the interesting results when contagion is present in the form of a claims number intensity function that is a linear function of past claims, are developed. Bühlmann's general formulation of the distribution of total claims payments does not appear until page 55 and the classical collective risk model then turns up as a special case on page 58.

Although in chapter two the basic risk model is presented in greater generality than is customary, the ideas used in this chapter are in the mainstream of the study of general stochastic processes. However, in chapter three, *The Risk in the Collective*, Bühlmann commenced to introduce ideas that are directly motivated by practical insurance problems and have special actuarial interpretations. The key idea is that the parameters of a risk process, defined by claims number and claims amount distributions, are not known. These parameters may be thought of as belonging to a set of possible parameters called a collective. A distribution defined on the collective, or some subset of it, which provides information on the probability of the realization of a particular risk process is called a structure function. Bühlmann does not stress the estimation of the structure function. He admits that statistical investigations may cast some light on the structure distribution; but the possibility that it might be estimated by a blend of statistical and ancillary prior information, as could be permitted if one adopts a subjective view of probability, is not mentioned. Nor is the idea that the role of underwriting is, in this model, to reduce the dispersion of the structure distribution discussed.

The remainder of chapter three is taken up with embedding the previously developed general risk model into the still more general model that permits selection, according to the structure distribution, of the risk characteristics. Bühlmann devotes the final section of this chapter to discussing properties of portfolios of risks selected according to the structure distribution. The principal results pertain to the convergence in probability of the average claims payment in a portfolio of risk selected from the collective in the case where the number of risks increases (convergence obtains) and where the time period increases (convergence fails unless the risks are homogeneous).

Chapter four is spent on premium calculations and starts the second

part of the book which is devoted to the consequences of the theoretical model developed in the first part. Four alternative principles (expected claims loaded proportionately, expected claims loaded by a constant times the standard deviation of claims, expected claims loaded by a constant times the variance of claims, and expected claims plus a loading which will make the insurer's expected utility upon entering the insurance contract equal to its current utility) are discussed. The principal point in the chapter is that the risk premium, associated with a particular risk from the collective, remains essentially unknown because changing risk conditions prevent the collection of enough data to permit a precise estimate of the risk premium. Statistics are useful primarily for the estimation of the collective premium, the expected premium taken with respect to structure distribution over the set of risk premiums.

Bühlmann defines a credibility premium for a risk, using the variance loading principle, which he proves in Section 4.3.5 has the desired property that as the number of experience periods increase the credibility premium approaches the risk premium. In the remainder of chapter four the practical problem of estimating successive credibility premiums, starting with the collective premium, is discussed. In this project Bühlmann applies the ancient idea of approximating a function, in this case certain conditional expectations, with least squares lines. With some ingenious approximations and manipulations he develops a linearized approximate credibility premium which can be computed using claims statistics from the collective and which, as is shown in an exercise, retains the property that it converges to the risk premium.

Chapter five is spent on the important risk theory problem of making rational reinsurance decisions. Only individual risk reinsurance contracts of the proportional and excess of loss types are considered. Two problems are formulated. The first is to determine the optimal reinsurance agreement for each risk, in the sense of minimizing the variance of retained profits given a fixed profit expectation, for proportional and excess of loss reinsurance contracts. The second problem is to solve the absolute retention problem. In this problem the objective is no longer to spread the reinsurance coverage over individual risks, but it is that of determining the company's global reinsurance policy. The first problem is solved, but the second requires the risk carrier to formulate its overall business objective.

This requirement, to specify the firm's objective in order to solve the absolute retention problem, provides a bridge to chapter six in which

insurance carrier stability criteria are discussed. Thus chapter six provides a brief introduction to the intersection between risk theory and modern applied economics. The decision variables which are assumed to be under management control are the premium loadings, the retention level, and the amount of initial free reserve funds. It is management's objective to adjust these variables in order to achieve stability as measured by the probability of avoiding ruin, the present expected value of future dividends, or the achievement of the maximization of the expected utility associated with the future profit stream. The classical ruin probability results of collective risk theory are developed and alternative novel proofs are presented in some cases. The survival probability objective is used to solve the absolute retention problem which had been left in limbo in chapter five. In recent years, led by de Finetti and Borch, many actuaries have become interested in expected dividends as a business objective. Because new results due to Gerber (a doctoral student of Bühlmann's at Zurich), are presented for the classical collective risk model, the development of the stability criterion is of special interest. Finally, not only does Bühlmann discuss the maximization of expected utility as a business objective, but he makes his discussion almost self contained by providing a proof of the existence of a utility function over the class of risk processes by reasoning from basic axioms concerning preference orderings.

This is an important book. Most of the criticisms that a reviewer might make follow from the limitations on the scope of the book that Bühlmann imposed on himself. One yearns for more ideas on methods for solving the estimation problems in risk process, yet Bühlmann warned us that he would stick to model building. The outline that Bühlmann apparently set for himself is incomplete in places, such as in the case of dividend and utility stabilization criteria. However once again Bühlmann usually points out the missing steps in the outline and indicates the reason for the incompleteness. Several aspects of the models that he introduces are only partially exploited. For example, the idea of the collective and the structure distribution provide models for studying sales and underwriting operations. Although properties of posterior distributions play a vital role in proving the properties of Bühlmann's credibility premium, he avoids any direct Bayesian interpretation of the credibility premium. In summary, this is a challenging book. Those actuaries who read it will be carried to the working frontier of risk theory and will be rewarded with a stock of stimulating ideas which they can develop, expand and apply.