FUNDING THEORIES FOR SOCIAL INSURANCE

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Within the social sciences the influence of Paul Samuelson, the MIT economist, is almost omnipresent. The majority of fledgling students of economics learn the rudiments of the subject from a textbook he has written. Technical papers he has penned have appeared in most of the major journals devoted to economics, political science, or statistics. Readers of the popular magazine Newsweek have grown accustomed to his periodic essays on current political and economic topics. One of these essays, "Social Security,"[1] was devoted to making the point that because of the growth of population and of real per capita income, the participants in a social insurance system which involves transfer payments from active to retired workers will receive more in benefits than they will contribute in social insurance taxes which are set on a pay-as-you-go level. In an earlier technical paper Samuelson had discussed the same idea.[2]

Henry Aaron has further formalized this idea in the form of a theorem.[3] Aaron's paper containing this theorem was reprinted in a compendium of papers on policy issues in public and private pension systems published for the use of the Joint Economic Committee of Congress.[3] Therefore, presumably, Aaron's theorem is better known at the center of political power than are most such formal propositions. To label a statement a theorem seems to sanctify it, to place it on a plane above controversy, and to thereby silence anyone who would question its eternal truth. Because the proof of Aaron's theorem is a simple exercise in actuarial mathematics, and because actuaries have a professional interest in social insurance, it seems appropriate to record the theorem and a modified proof in actuarial literature. A few of the developments in the proof are closely related to some found in a paper by Nowlin.[4]

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[1] Samuelson, Paul "Social Security," *Newsweek*, Feb. 13, 1967.

[2] Samuelson, Paul, "An Exact Consumption-Loan Model of Interest With or Without a Social Contrivance of Money," *Journal of Political Economy*. Vol. 66 (1958).

[3] Aaron, Henry, "The Social Insurance Paradox," *Canadian Journal of Economics and Political Science*, Vol. 32 (1966). Reprinted in Old Age Assurance, A Compendium of Papers on Problems and Policy Issues in the Public and Private Pension System, Part V: Financial Aspects of Pension Plans, submitted to the Subcommittee on Fiscal Policy of the Joint Economic Committee, Congress of the United States.

[4] Nowlin, Paul, "Insufficient Premiums," *Transactions, Society of Actuaries*, Vol. 11 (1959).

Theorem. If the annual rate of growth in the number of entrants into the working population plus the annual rate of growth of real average wages exceeds the annual rate of interest, which is assumed equal to the marginal rate of time preferences and the marginal rate of transformation of present into future goods, then the introduction of social insurance pensions equal to current average real wages on a pay-as-you-go funding basis will improve the expected welfare position of persons in the population who receive average real wages.

This statement of the theorem indicates that the conclusion is concerned with *expected* results for a person with *average* income. Aaron did not place this emphasis in his original statement. Those with above average incomes may not individually fare so well because of the average benefits paid retired workers.

Proof. We will adopt a continuous model and the following notation and assumptions.

1. The symbol $s(x)$ will denote the survival function for the population under consideration. We will assume that $s(x) = 0$ when x is greater than some finite limiting age.
2. The symbol h will denote the annual rate of increase of the average of the real wages paid those in the working population. This rate is assumed to be constant.
3. The symbol g will denote the annual rate of increase in the number entering the working population each year. This rate is assumed to be constant.
4. The average age of entry into the working population will be denoted by a and the average age of retirement will be denoted by r.
5. The annual rate of interest (force of interest) will be denoted by δ and it will be assumed that this rate is equal to the marginal rate of time preference and the marginal rate of transformation of present into future goods. This rate is also assumed to be constant.

The expected number in the working and retired populations at time t, where t is greater than the limiting age less the average age of entry, is denoted by $P(t)$ and is given by

$$P(t) = k\int_a^r e^{-(x-a)g+gt}[s(x)/s(a)]dx + k\int_r^\infty e^{-(x-a)g+gt}[s(x)/s(a)]dx,$$

where k is the annual rate of entry into the working population at an arbitrary starting time designated at time zero. The first integral in this expression represents the expected number in the working population and the second integral represents the expected retired population. Note that the same expected population would be achieved at time t by a constant increase in the survival function for each entering cohort at a constant annual rate g, as would be obtained by assuming a corresponding annual increase in the rate at which entrants come into the working population.

The expected total amount of wages to be received by a person who enters the work force at time t is denoted by $W(t)$ and is given by

$$W(t)=W(O)\int_a^r e^{h(x-a)+ht}[s(x)/s(a)]dx,$$

where $W(O)$ is the average annual wage rate at time zero.

The expected amount of benefits to be received by a person who enters the work force at time t, if benefits are paid at a rate equal to current average real wages, will be denoted by $R(t)$ and will be given by

$$R(t)=W(O)\int_r^\infty e^{h(x-a)+ht}[s(x)/s(a)]dx.$$

The constant payroll tax rate needed to fund, on a pay-as-you-go funding basis, the benefit payments is denoted by the symbol f and is given by

$$f=\frac{\int_r^\infty e^{-(x-a)g+gt+ht}[s(x)/s(a)]dx}{\int_a^r e^{-(x-a)g+gt+ht}[s(x)/s(a)]dx}$$

$$=\frac{\int_r^\infty e^{-gx}s(x)dx}{\int_a^r e^{-gx}s(x)dx}=\frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a}|}}$$

where the life annuity symbols are valued at force of interest g.

The expected accumulated value of taxes, at age r, paid by a worker who enters at time t is denoted by $C(t)$ and is given by

$$C(t)=fW(O)\int_a^r e^{(x-a)h+ht+(r-x)\delta}[s(x)/s(r)]dx.$$

The phrase "expected accumulated value" used in the definition of $C(t)$ does not imply the accumulation of a fund in this pay-as-you-go scheme. Rather the phrase uses the word "value" in a more subjective sense. The symbol $C(t)$ denotes the expected value, at age r, for decision making purposes that a person with marginal rate of time preference δ would attach to the taxes he is required to pay for a social insurance program.

The present expected value, valued at age r, of benefits paid to a worker who enters the program at time t is denoted by $B(t)$ and is given by

$$B(t) = W(O)\int_r^\infty e^{(x-a)h+ht-\delta(x-r)}[s(x)/s(r)]dx.$$

For an average participant we seek to determine which of the relationships $B(t) > C(t)$, $B(t) = C(t)$ or $B(t) < C(t)$ holds. The relationships $B(t) \gtreqless C(t)$ are equivalent to

$$\frac{\displaystyle\int_r^\infty e^{(x-a)h-(x-r)\delta}s(x)dx}{\displaystyle\int_r^\infty e^{-gx}s(x)dx} \gtreqless \frac{\displaystyle\int_a^r e^{(x-a)h-(x-r)\delta}s(x)dx}{\displaystyle\int_a^r e^{-gx}s(x)dx}.$$

We denote the right hand term of this expression by $R_1(\delta)$ and the left hand term by $L_1(\delta)$ and we note that the relationships indicated by $B(t) \gtreqless C(t)$ are equivalent to $L_1(\delta) \gtreqless R_1(\delta)$. We observe that $L_1(g+h) = R_1(g+h)$, therefore, if $\delta = g+h$ then $B(t) = C(t)$. Because $d\ L_1(\delta)/d\delta < O$ and $d\ R_1(\delta)/d\delta > O$, we have that if $\delta < g+h$, then $L_1(\delta) > R_1(\delta)$ and $B(t) > C(t)$, and if $\delta > g+h$, then $L_1(\delta) < R_1(\delta)$ and $B(t) < C(t)$. We may verbalize this result by saying that for a person with average real wage level, if $\delta < g+h$, then the present expected value of his social insurance benefits exceeds the expected accumulated value of the required social insurance taxes. On the other hand, if $\delta > g+h$, then this pay-as-you-go social insurance system is a poor bargain for him.

This result is intuitively obvious to most actuaries and they would probably accept the conclusion without a mathematical development. The theorem simply states the technical conditions for the success of an assessment system. At the Seventeenth International Actuarial Congress, several papers discussed "assessmentism" as a funding method for pension programs.[5,6,7] The direct transferring of income from the current working generation, with the benefits tied to current living standards, is related to the "repartition" system developed for private pensions in France.[8]

In commenting on his theorem, Aaron makes the following acknowledgment: "If savings and, hence, investment and, hence, the rate of growth of income are reduced as the level of social insurance increases, this conclusion does not necessarily follow." This possibility is, in fact, a very critical factor to consider in drawing any public policy conclusions from Aaron's theorem. Nevertheless, the tone of Aaron's paper is such that it is natural to infer that the conventional economic assumption is that, in fact, $\delta < g + h$ and therefore $B(t) > C(t)$. However, it would be wise to point out that on our finite planet we cannot tolerate, for any extended period of time, a rate of increase in the working population (g) other than zero. Hopefully the rate of increase in real income (h) will remain positive, although historically it has tended to average out at only around three per cent. On the other hand δ, the force of interest, which is assumed in this theorem to be the marginal rate of time preference may be, for at least certain members of the working population, relatively high. For example, the economic behavior of many young people in not taking advantage of potentially valuable educational opportunities and in acquiring current goods through expensive installment plans indicates that their preference for current goods may be very high.

The thrust of these remarks is not to refute Aaron's theorem, for it is quite valid. Rather the remarks are intended to indicate the limited scope of the theorem and to stress that it is seldom possible to justify a broad and long-term public program by a strictly formal chain of reasoning.

Robert J. Myers, Chief Actuary, Social Security Administration, has written a penetrating review of Aaron's paper.[9] Myers has provided a guide to some technical errors in the original paper and raises some interesting points concerning the economic reasoning that Aaron followed.

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[5] Hagstroem, K. G., "National Pension Schemes: Necessity of Investment," *Transactions, 17th International Congress of Actuaries,* Vol. 3.

[6] Kaikkonen, M., "Pensions and the Cost of Living in Finland," *Transactions, 17th International Congress of Actuaries,* Vol. 3.

[7] Mazoué, L., "Variations in Retirement Pension Schemes in France under the Influence of Monetary Instability," *Transactions, 17th International Congress of Actuaries,* Vol. 3.

[8] Dyer, J. K., "Variable Pensions: An International Survey," *Proceedings of the Conference of Actuaries in Public Practice,* Vol. 16 (1966–67).

[9] Myers, Robert. J., "Review of 'The Social Insurance Paradox,'" *Transactions, Society of Actuaries,* Vol. 20 (1968).

It is instructive at this point to examine a similar development in which a distinction is made between δ, the marginal rate of time preference, and δ', the marginal rate of transforming present goods into future goods. We shall determine a social insurance payroll tax rate by the principle of equivalence.

That is, we will set the present expected value of benefits at the average wage level, equal to the present expected value of payroll taxes for each individual. This tax rate will be analogous to an entry age normal rate in the momenclature of pension funding. In this case we are dealing with a financial system which may generate a fund which will earn interest at a continuous annual rate δ'. The tax rate, denoted by n, applicable to those who enter the working population at time t turns out to be independent of t and given by

$$n = \frac{\int_r^\infty (e^{gt+[t+(r-a)]h+(x-r)h-(x-r)\delta'})s(x)dx}{\int_a^r (e^{gt+[t+(x-a)]h+(r-x)\delta'})s(x)dx}$$

$$= \frac{\int_r^\infty (e^{-(\delta'-h)x})s(x)dx}{\int_a^r (e^{-(\delta'-h)x})s(x)dx} = \frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a|}}},$$

where the life annuities are evaluated at force of interest $\delta' - \mathrm{h}$. The accumulated expected value at age r of contributions at rate n of real wages for a person entering the working population at time t will be denoted by $Y(t)$ and is given by

$$Y(t) = nW(O)\int_a^r (e^{(x-a)h+th+\delta(r-x)})s(x)/s(r))dx.$$

Once again the amount $Y(t)$ does not represent an expected individual reserve fund; rather it is the value at age r, for decision making purposes, that a person with marginal rate of time preferences δ would attach to the taxes that he has paid for the social insurance plan. If δ' replaces δ in the integral which defines $Y(t)$, the result would be the expected fund at age r for a life which entered at time t and survived until age r. The factor $e^{[(x-a)+t]h}$ plays the role of a salary scale in conventional pension mathematics.

It remains to determine if $B(t) > Y(t)$, $B(t) = Y(t)$ or $B(t) < Y(t)$. The relationships $B(t) \gtreqless Y(t)$ are equivalent to

$$\frac{\int_r^\infty (e^{(x-a)h-(x-r)\delta})s(x)dx}{\int_r^\infty (e^{-(\delta'-h)x})s(x)dx} \gtreqless \frac{\int_a^r (e^{(x-a)h-(r-r)\delta})s(x)dx}{\int_a^r (e^{-(\delta'-h)x})s(x)dx}$$

We denote the right hand term of this expression by $R_2(\delta)$ and the left hand term by $L_2(\delta)$ and observe that if $\delta = \delta'$, then $R_2(\delta) = L_2(\delta)$ and $B(t) = Y(t)$. Because $d\ L_2(\delta)/d\delta < O$ and $d\ R_2(\delta)/d\delta > O$, we have that if $\delta < \delta'$, then $L_2(\delta) > R_2(\delta)$ and $B(t) > Y(t)$, and if $\delta < \delta'$, then $L_2(\delta) > R_2(\delta)$ and $B(t) > Y(t)$. Once again this result conforms to what our actuarial intuition would indicate; if the marginal rate of time preference is less than the marginal rate at which present goods may be transformed into future goods, the expected value of social insurance benefits exceeds the expected value of the associated taxes when these taxes are determined by the principle of equivalence. If the marginal time preference rate is greater, the converse value judgment would hold.

A final interesting comparison is between the expected accumulated value of taxes under the pay-as-you-go funding plan $(C(t))$ and under what is essentially an entry age normal funding plan. We seek to determine whether $C(t) > Y(t)$, $C(t) = Y(t)$ or $C(t) < Y(t)$ holds. The relationships $C(t) \gtreqless Y(t)$ are equivalent to the relationships $f \gtreqless n$ which in turn are equivalent to

$$\left.\frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a|}}}\right|_g \gtreqless \left.\frac{{}_{r-a|}\bar{a}_a}{\bar{a}_{a:\overline{r-a|}}}\right|_{\delta'-h}$$

where the bar symbol is intended to indicate that the left hand member of the relationships is valued at force of interest g and the right hand member is valued at force of interest $\delta' - h$.

It comes as no surprise that if $g = \delta' - h$, then $f = n$ and $C(t) = Y(t)$. Because the derivative of ${}_{r-a|}\bar{a}_a/\bar{a}_{a:\overline{r-a|}}$, with respect to the force of interest is negative, we may conclude that if $g > \delta' - h$, then $C(t) < Y(t)$ and if $g < \delta' - h$, $C(t) > Y(t)$. That is, if the population growth rate exceeds the marginal rate of transformation of present goods into future goods less the rate of increase in average real wages, then the expected value of taxes required on an individual under pay-as-you-go funding is less than that required by entry age normal funding. If the rate of population growth is less than the marginal rate of transformation of present goods into future goods minus the rate of growth of real wages, the entry age normal type of funding appears to be more favorable when judged by the size of the expected accumulated amount of social insurance taxes.

These developments are summarized in the following table:

	If	Then
(1)	$\delta < \delta' < g + h$	$C < Y < B$
(2)	$\delta < g + h < \delta'$	$Y < C < B$
(3)	$g + h < \delta < \delta'$	$Y < B < C$
(4)	$g + h < \delta' < \delta$	$B < Y < C$
(5)	$\delta' < g + h < \delta$	$B < C < Y$
(6)	$\delta' < \delta < g + h$	$C < B < Y$

Symbols

δ = Annual marginal rate of time preference

δ' = Annual marginal rate of transformation between present and future goods

g = Annual rate of increase in the rate at which new entrants come into the working population

h = Annual rate of increase in average real wage rate

C = Accumulated expected value at age r of payroll taxes on a pay-as-you-go funding method

Y = Accumulated expected value at age r of payroll taxes on an entry age normal funding method

B = Present expected value of social insurance benefits at age r

Inequality (1), in which the marginal time preference rate is less than the marginal rate of transformation between present and future goods, tends to support a social insurance system funded on a pay-as-you go basis. Inequality (2) on the other hand, in which marginal rate of transformation of present into future goods is high, tends to support a social insurance system with entry age normal funding. Inequality (3) supports entry-age normal funding as the only economic alternative. Inequalities (4) and (5), in which the time preference rate is relatively high, imply that a social insurance system would be an uneconomic innovation. Inequality (6), in which there is a low marginal rate of transformation, indicates that pay-as-you go funding would conform to the value judgment of the average worker.

The situations described in these inequalities, in which the marginal rate of time preferences is not equal to the marginal rate of transformation of present into future goods, are not in economic equilibrium. Classical economic theory describes the market forces which tend to push these two rates together. The practical answer to these possible objections to the results exhibited in the foregoing table, based on the disequilibrium of the interest rates, is that even in a free market economy there are forces at work which tend to disturb perfect equilibrium. In fact, part of the explanation for the driving force in a competitive economy may come from the fact that the marginal rate of time preferences for many persons exceeds the marginal rate at which present goods may be transformed into future goods. Even in an economy which is at approximate equilibrium position, there will probably be groups within the economy for which each of the inequalities in the table is a reality.

Of course these formal results simply reinforce conventional actuarial wisdom about the characteristics of various funding methods for social insurance systems. However, before becoming smug about this reinforcement, we should recall the rather artificial nature of the static assumptions made in this demonstration. In the real world probably no particular order relation among the rates under study would remain unchanged over a number of years. Indeed, it is practically impossible for some of the rates to remain positive indefinitely. The results exhibited in the table were obtained by averaging. In fact very different inequalities might be obtained for subpopulations whose real wages are not average. All that these results can do is to provide an analytic machine which may be helpful in examining proposals for social insurance programs. Social insurance programs evolve as a result of practical political compromises rather than abstract reasoning. However, it is our professional actuarial responsibility to examine by analytic methods the economic implications of proposed social insurance programs.