AN ACTUARIAL NOTE ON ACTUARIAL NOTATION

JEFFREY T. LANGE

"When I use a word," Humpty-Dumpty said, "it means just what I choose it to mean — neither more nor less."

— Lewis Carroll

One of the invariants in the Syllabus of Examinations of our Society has been the study of life insurance mathematics and the associated actuarial notation. As a result probably every practicing actuary has, at one time or another, worked with this unique notation. Multiple indices appear both prefixed and suffixed to a symbol along with exponents resulting in a halo of characters about the basic symbol. Special print characters, e.g. horizontal bars, ⌉, °, ¨, are used to modify both indices and basic symbols in order to alter the meaning of an expression. Sometimes the actuarial usage is quite different from ordinary practice; for example, exponents are not to be interpreted as exponents. The end result is that to the non-actuary the notation of the actuary may resemble the jottings of the astrologer or alchemist.

The complexity of actuarial notation results in an unusual succinctness. An intricate insurance policy involving a number of benefit options and a complicated payment plan may be reduced to a single expression. Once a problem is translated into actuarial notation, it is frequently a relatively elementary task to manipulate the symbols and solve the equations. Having learned the notation, even the student, equipped only with tables, pencil, and paper, can evaluate rather involved policy forms. The concise, compact form of the notation allows the experienced actuary to elegantly express complicated insurance schemes in a limited number of equations, which can be of great aid in obtaining a solution.

The usefulness of the notation is evidenced by the fact that from life insurance work it was extended to pension work and to health insurance. It has achieved a universal status, the current version being settled on by the 14th International Congress of Actuaries in Madrid in 1954.[1]

[1] A brief history of the notation and a general, although dated, description of it is given by F. S. Perryman, "International Actuarial Notation," *PCAS* Vol. XXXVI, pp. 123–131.

Standard actuarial notation has never been adopted for property and liability work. One of the first tasks undertaken in 1914 by the newly formed Casualty Actuarial Society was an attempt to develop such notation. A committee was formed, but it decided that casualty actuarial science was too young to permit the establishment of a stable notation.[2] Later, in 1920, an individual proposed a system of workmen's compensation notation, which was not generally accepted because of its complexity. A plea was made for a simple and universal system of notation embracing all non-life lines.[3] Still later, in 1932, another individual proposed a standard system of notation for casualty work,[4] but it was not generally used by other authors. (Even its author developed an entirely different system of notation in one of his later papers.[5])

This lack of a standard notation implies that each author of a technical paper must develop his own notation which his reader must learn. This is time consuming for both writer and reader, and can make technical papers more difficult to comprehend, thus leading to unnecessary confusion. Tracing a concept through several papers can be particularly troublesome since the same idea may appear in substantially different form in each author's notation. As a result, it is difficult to make comparisons, to recognize parallelisms, and to extend work from one area to another since the variation in notation tends to obscure similarities and impede pattern recognition. Finally, the value of the *Proceedings* as a reference work is reduced since the reader must restudy the author's notation whenever he consults a paper. Research and communication are made more difficult by the lack of any degree of standardization in non-life notation.

While life actuarial notation has achieved standardization, the notation is less than perfect. Any actuary who has attempted a typed report including some sophisticated equations in the notation probably realizes the difficulties in accurately portraying the halo of indices and the special characters. It is often hard with a typewriter to differentiate first (or last) symbols in

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[2] Committee on Terms, Definitions and Symbols, *PCAS* Vol. I, p. 76, *PCAS* Vol. II, pp. 163, 317, and 497.

[3] Perkins, Sanford, "A Suggested System of Standard Notations for Actuarial Work in Workmen's Compensation Insurance," *PCAS* Vol. VII, pp. 36–56.
and
Michelbacher, G. F., Discussion of Perkins' paper, *PCAS* Vol. VII, pp. 405–407.

[4] Carlson, T. O., "Suggestions for a Standard System of Notation in Casualty Actuarial Work," *PCAS* Vol. XX, pp. 264–274.

[5] Carlson, T. O., "An Actuarial Analysis of Retrospective Rating," *PCAS* Vol. XXVII pp. 283, 317, and 318.

multiterm indices from individual basic symbols; hence, the result can be confusing. The notation is clearest in its published form, but translation into print may be both difficult and expensive due to the extraordinary nature of the notation.

Actuarial notation evolved over a long period, having first been considered by an International Congress in the last years of the nineteenth century. That it developed independently from the mainstream of mathematical thought may have resulted from two circumstances: first, the applications of probability during the late nineteenth and early twentieth centuries were limited to insurance and gambling and the actuary could consider his application of probability to be unique;[6] second, probability theory itself was rather independent from mathematical analysis prior to Kolmogorov. Hence, the actuary of fifty or seventy-five years ago would not have concerned himself with the development of a notational system consistent with the rest of mathematics.

Today's actuarial student has a strong background in mathematical analysis and views probability as a branch of function theory. He finds actuarial notation inconsistent with the mathematical notation to which he is accustomed. During his career he expects to borrow techniques from other mathematical disciplines, which will necessitate his using two different notations: one for the actuarial fraternity and one for the remainder of the scientific community. Existing actuarial notation may appear to be an anachronism and may prove to be a handicap.

Increasing use of electronic computers in actuarial work has added to the confusion in that the notation is not readily adaptable to computer programming. While each computer langauge is different, the higher level languages (ALGOL, COBOL, FORTRAN and PL/1) generally require that variables be expressed in a linear form: variable name followed by indices (in parentheses) separated by commas. Special characters cannot be used and capital letters must be used exclusively, except that the variable name may include numerics.

There is no obvious way to mechanically translate existing notation into a form which could be included in a programming language. For example, either a, $\ddot{a}$, A or $\bar{A}$ in current notation could be rendered as A. Should D be used to denote d, δ, or D of the current notation? How should the indices

[6] Borch, Karl, "The Theory of Risk," *Journal of the Royal Statistical Society*, Series B (Methodological), Vol. 29, p. 433.

which prefix the symbol be handled? What alphabetics will be used to replace the various special characters? Currently, each actuary (or programmer) must make these decisions independently. If his program needs to be revised, then someone else will have to learn his notation to work with the program. Aside from leading to wasted effort, this situation also makes the exchange of completed programs among actuaries more difficult. The discussions of the influence of computers on actuarial problem solving at the 18th International Congress of Actuaries in Munich illustrate this point. Two actuarial programming languages were presented, each designed for essentially the same type of work; however, they were notationally quite different and apparently incompatible.[7] Several other actuaries doing similar work stated that they could not incorporate either language in their own work because the languages were too different from what they had developed thus far. Many of the differences cited in the discussion seemed to be notational.

In recognition of these problems, a group of German, Austrian, and Swiss actuaries presented a revision of actuarial notation to the 18th Congress,[8] which did not take final action on the suggestion. A review of their notation serves to illustrate the degree of change which would be necessary if actuarial notation is to conform to current mathematical usage and if it is to be adaptable to a computer language format.

Actually, their paper gives two new sets of notation; a publication language designed with a view toward consistency with mathematical function theory, and a computer notation developed from a translation of the publication language into a computer acceptable format. In the former, exponents are employed only to raise a variable to a power, and indices prefixed and suffixed to a variable have been eliminated in almost all cases. The only exception arises in cases where the index was itself indexed; in the new notation the second index would be retained.

$$ {}_{n_1|n_2}a_x^{(2)} \quad \text{becomes} \quad a(x\,;n_1 : n_2\,;2) $$

These changes result in the removal of the cluster of indices about the central character. The use of both upper and lower case characters has been

[7] Benjamin, Sidney, "A Language for Routine Actuarial Programming," *Transactions of the 18th International Congress of Actuaries* (*TICA*), Subject 5, pp. 771–782, and Kunz, Peter, "Die Programmierung AKTUARIAT," *TICA*, Subject 5, pp. 931–947.

[8] Boehm, Carl; Reichel, Georg and others, "Vorschlage für eine internationale versicherungsmathematische Veröffentlichungssprache und ihre computerverträgliche Darstellung," *TICA*, Subject 5, pp. 815–842.

retained as have been Greek letters, but special characters (other than the umlaut) have been deleted.

In most cases, the translation into the publication language was as follows;

$${}_{prefix}VARIABLE^{exponent}_{suffix} \quad \text{becomes} \quad VARIABLE\,(suffix; prefix; exponent)$$

where exponent denotes an index of some kind (not a power of the variable) and where multiple characters in prefix or suffix are separated by colons or commas, while the semicolon is used only to separate suffix from prefix from exponent. A few examples are given below:

a_x becomes $a(x)$

$a_{\overline{n}|}$ becomes $a(;n)$

$\overline{A}_x$ becomes $Am(x)$

$A_{x:\overline{n}|}$ becomes $AE(x;n)$

While life insurance actuaries are more qualified to comment on the notation and will probably suggest revisions in it, a superficial review indicates that the notation is no more difficult than the existing and no less meaningful once one has become accustomed to it. It has the advantage of being more readily understandable since it resembles normal mathematical notation, and the disadvantages of requiring (if adopted) a rewrite of actuarial texts and of still not being in a computer format.

This later difficulty is overcome by the development of computer notation in which only upper case letters are used and in which all special characters and Greek letters are translated into alphabetics. In some cases, the resulting computer notation is more meaningful than the original; permanent and temporary annuities are denoted by *AP* and *AT* instead of *a* and *ä* respectively. While A_x can be simply rendered as *AM(x),* more complicated expressions are not quite as obvious, for example:

${}_{n_1|n_2}a^{(8)}_x$ becomes $AT5A2\,(X, N1, N2)$

$A_{x:n}$ becomes $AE7\,(X, N)$

$a^{(m)}_n$ becomes $APN0A0\,(N, K)$

While the numerics included in the variable names have been assigned in a systematic way, the resulting expressions appear more complex and less meaningful than the existing notation.

Only a few examples of the two new languages have been given, but the examples were chosen to be representative, being neither more nor less complicated that the renderings of the many other symbols translated. The examples fail to illustrate the great amount of work on the part of European actuaries in developing the extensive and intricate set of rules for translating in a consistent manner the existing notation into the two new notations.

The possibility of the development of new actuarial notation for life, health, and pension work raises several questions for casualty actuaries:

1. Should casualty actuaries, either independently or through the Society, have any role in the development of the new notation? Some casualty actuaries do work in health insurance, which would be affected, and all have studied the notation, thus giving casualty actuaries some interest.

2. Is standard notation needed for casualty and property actuarial work? Such notation might improve communication among actuaries, aid in the solution of technical problems, make the *Proceedings* a more valuable reference tool, and generally enhance the Society's scientific work; on the other hand, these arguments have not been compelling in the past.

3. If developed, should the casualty-property actuarial notation be a derivative of life, health, and pension notation? Past attempts at casualty notation never followed this avenue, but a reformulation of life insurance notation would provide an ideal time to develop a more ecumenical actuarial notation embracing all lines of business.

4. If the first three questions are answered positively, how might the problem of notation be studied further? As noted above, individuals have developed standard notations not generally accepted by other actuaries; however, group efforts have been no more successful: on May 19, 1898 the International Congress of Actuaries voted unanimously "That a Universal Notation be adopted, not only for Life Assurance, but for all other branches of assurance."[9]

[9] As quoted by Valerius, N. M., Discussion of Carlson's paper on notation, *PCAS* Vol. XXI, p. 163.