Finally, in an attached appendix Mr. Valerius includes a very convenient tabulation of the coefficients for the iteration equations, corresponding to the more useful cases of Formula A.

DISCUSSION BY RICHARD H. SNADER

Mr. Valerius' notes on Whittaker-Henderson Formula A have provided casualty actuaries with an opportunity to improve one of the most powerful tools at their disposal. The problem of examining a series of data, detecting a trend, and projecting that trend is one with which we are all vitally concerned. To fully appreciate the value of his contribution, a brief synopsis of the basic concepts of graduation might be helpful.[1]

Graduation may be defined as the process of securing from an irregular series of observed values a smooth, regular series of values consistent in a general way with the observed series. The smooth series is then taken as a representation of the underlying law that gave rise to the observed values. The set of observed values is usually donated by $\{u_x''\}$ and the graduated values by $\{u_x\}$.

Graduation is characterized by two essential qualities, smoothness and fit. These qualities are not independent. An increase in smoothing results in a reduction in fit; conversely, when fit is improved, smoothness usually suffers. Whittaker-Henderson formulas are the product of the difference equation method of graduation. In this method, the graduated series is determined by a difference equation derived from an analytic measure of the relative emphasis placed on smoothness and fit.

The combination of smoothness and fit may be expressed by $F + hS$, where h is a positive number fixing the relative weight assigned to smoothness and fit. Smoothness is measured by the smallness of the sum of the squares of the z^{th} order of differences of the graduated values:

$S = \sum(\Delta^z u_x)^2$, where Δ is the difference operator.

Closeness of fit is measured by the smallness of

$$F = \sum(u_x - u_x'')^2.$$

[1] The description of the graduation process is based almost entirely on Morton D. Miller's monograph *Elements of Graduation* published by the Society of Actuaries.

The best graduation, according to these assumptions, will result from requiring $F + hS$ to be as small as possible.

If each value of u can be considered an independent variable and $\{u_x''\}$ is considered to be a set of given constants, the expression $F + hS$ can be minimized by

$$\frac{\partial}{\partial u_x} F + h \frac{\partial}{\partial u_x} S = 0;$$

$$\frac{\partial}{\partial u_x} \sum (u_x - u_x'')^2 + h \frac{\partial}{\partial u_x} \sum (\Delta^z u_x)^2 = 0.$$

The conditions for a minimum are

$$(u_x - u_x'') + (-1)^z h \delta^{2z} u_x = 0;$$

$$u_x'' = u_x + (-1)^z h \delta^{2z} u_x,$$

where δ is the central difference operator. When second differences are minimized for smoothness $(z = 2)$, a fourth order difference equation results:

$$u_x'' = u_x + h\delta^4 u_x.$$

When third differences are minimized, $(z = 3)$, a sixth order difference equation results:

$$u_x'' = u_x - h\delta^6 u_x.$$

The difference equation can be factored into two lower order difference equations. For values of z equal to one, two, or three, the lower order equations are:

$$Eu''_{x+a} = Au'_x - Bu'_{x-1} + Cu'_{x-2} - Du'_{x+3}$$

$$Eu'_{x-a} = Au_x - Bu_{x-1} + Cu_{x+2} - Du_{x+3}$$

where $\{u_x' \pm a\}$ is an intermediate series. A new parameter, a, replaces h. The parameter, h, can be expressed in terms of a.

$$Z = 1, h = a(a + 1);$$

$$Z = 2, h = \frac{1}{4} a(a + 1)^2 (a + 2);$$

$$Z = 3, h = \frac{a(a + 1)^3 (a + 2)^3 (a + 3)}{16(2a + 3)^2}$$

The coefficients A, B, C, D, and E are all expressible in terms of a. The purpose of a is to fix the relative emphasis to be placed on smoothness and fit. The factoring process results in the Whittaker-Henderson Type A formulas, and a practical method of utilizing the difference equation for graduation is obtained.

Practical Application

The usefulness of the Whittaker-Henderson Type A formulas in constructing mortality tables is well known. Until now, however, practical applications in casualty actuarial work have been virtually non-existent. The author points out that the graduation of a time series can be used for predicting the future. Graduation by mathematical formula is one method that can be employed; the difference equation method is another.

It is difficult to find material in the *Proceedings* concerning the problem of extrapolation of an observed series of data. Mr. Paul Benbrook discusses the need for trend and projection factors and describes an early method.[2] The method currently employed in automobile ratemaking is described by Mr. Philipp K. Stern.[3] It consists of finding the line of best fit, by the method of least squares, for several observations of average paid claim costs and extending the line to determine trend and projection factors. The same method is employed in the rate level calculations of the Multi-Line Insurance Rating Bureau and the Fire Insurance Research and Actuarial Association, except the data are observations of the Composite Current Cost Index.[4]

The procedure of extending the line of best fit is almost universally accepted. Although Charles F. Cook has given us a new method for fitting the line, no alternatives have been offered to the basic concept that trends must be determined from linear relationships.[5] It is not possible, however, that the line of best fit may not fit the observed data very well? A trend line applied to spiraling hospital costs, for example, may produce projections which are hopelessly inadequate.

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[2] Benbrook, Paul, "The Advantages of Calendar-Accident Year Experience and the Need for Appropriate Trend and Projection Factors in the Determination of Automobile Liability Rates," *PCAS* Vol. XLV, p. 20.

[3] Stern, Philipp K., "Current Rate Making Procedures in Automobile Liability Insurance," *PCAS* Vol. LII, p. 139.

[4] The Composite Current Cost Index is a weighted average of the Consumer Price Index and the Composite Cost Index. The Composite Cost Index is published by the Department of Commerce and is a composite of several indexes representative of the major types of construction.

[5] Cook, Charles F. "The Minimum Absolute Deviation Trend Line," *PCAS* Vol. LIV, p. 200.

The author's work with the Whittaker-Henderson formulas has given us an opportunity to examine what may prove to be a practical alternative to determining trends from the line of best fit. If the observed series is graduated by the difference equation method, the graduated series is a curve of predetermined complexity that fits the observed data with a predetermined degree of fidelity. When second differences are minimized for smoothness, for example, the graduated series is an approximation of a linear function. The extensions of the graduated series are linear. When third differences are minimized for smoothness, the resulting graduated series is an approximation of a second degree curve; and the extensions are points that lie on a second degree curve.

Because it is not applicable when the value of z exceeds three, Formula A is seriously limited. Higher order difference equations, however, can be solved by direct algebraic methods. For any values of z or h, the difference equation will lead to a series of n linear equations in n unknowns, n being the number of terms to be graduated. The graduated values are uniquely determined from these equations. The direct algebraic solution was once thought to be impractical, but with the advent of modern computers the degree of impracticality has been greatly diminished and should no longer be considered a deterring factor.

The following table is based on data taken from Stern's paper.[6] The raw data consists of automobile bodily injury liability average paid claim costs for twelve month periods ending in successive calendar quarters. Using the method described for Exhibit I in the appendix of Valerius' paper, two graduations have been performed and are compared with the line of best fit. The first graduation was made with $z = 2$ and $a = 2$. The second graduation was made with $z = 3$ and $a = 2$. The graduated values were extended for 18 months and projection factors calculated. The comparison indicates that the projection factor based on the line of best fit may have been inadequate.

[6] Stern, op. cit., p. 174–175.

Comparison of Values

Year Ended	Average Paid Claim Cost	Line of Best Fit	Graduation No. 1 $z=2, a=2$	Graduation No. 2 $z=3, a=2$	
3/31/60	624	600.00	604.84	611.56	
6/30/60	602	609.56	610.58	610.53	
9/30/60	603	619.12	617.39	614.29	
12/31/60	620	628.68	625.85	622.30	
3/31/61	624	638.24	635.74	633.25	
6/30/61	661	647.80	646.54	645.42	
9/30/61	669	657.36	657.05	657.14	
12/31/61	672	666.92	666.90	667.61	
3/31/62	678	676.48	676.34	677.09	
6/30/62	670	686.04	685.96	686.50	
9/30/62	690	695.60	696.39	696.82	
12/31/62	718	705.16	707.42	708.48	
3/31/63		714.72	718.44	721.49	
6/30/63		724.28	729.46	735.84	
9/30/63		733.84	740.49	751.53	Extrapolated Values
12/31/63		743.40	751.51	768.56	
3/30/64		752.96	762.53	786.93	
6/30/64		762.52	773.56	806.65	
Projection Factor		1.081	1.093	1.139	

AUTHOR'S REVIEW OF DISCUSSIONS

Mostly I have only to thank Messrs. Nelson and Snader for their kind reviews.

Mr. Nelson recalls reading my remarks of twenty-five years ago on the subject of Whittaker-Henderson formulas, incidental to a paper on tables of risks inferred from the then rather new "excess ratio" tables. He says excess ratios were his main concern and the passing remarks on Hendersonian graduations got but passing attention from him. That was the emphasis intended. I wonder if he missed, as I find others have, the graphs which were for some reason printed on pages preceding the paper.

Graphical representations are so useful. I have read that the great Karl Pearson stressed graphical treatment. Therefore I appreciate Mr. Nelson's