NOTES ON WHITTAKER-HENDERSON FORMULA A

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VOLUME LIV, PAGE 218

DISCUSSION BY DALE NELSON

Mr. Valerius has now contributed two papers to the *Proceedings* dealing with the Whittaker-Henderson (or difference equation) method of graduation. I recall reading his earlier paper on "Risk Distributions Underlying Insurance Charges in the Retrospective Rating Plan" (*PCAS* Vol. XXIX, p. 96) while studying for Part 7 of the exams. But since "excess ratios" were my main area of concern and bewilderment at the time, little attention was paid to his remarks on graduation. Later, a long paper by LeRoy Simon on "The 1965 Table M" (*PCAS* Vol. LII, p. 1) touched briefly on Formula A graduation; but, again, I — and undoubtedly many others — took little note of the passing remarks concerning graduation.

Graduation techniques are very common in life insurance ratemaking. They are also widely used in the non-life lines, although it is probably safe to say that the techniques normally used by the casualty actuary are much less refined than the Whittaker-Henderson process — and, also, much less arduous. Now, thanks to Mr. Valerius, we have another opportunity to study this process. And, in the belief that a non-technical exposition might be in order — rather than a detailed critique of the method's fine points — most of my remarks will be toward that endeavor.

Basically, a graduation process is any technique applied to a set of ordered data to smooth out these data and to aid in uncovering any patterns or laws underlying the observed values. These data may represent a time series (such as the auto BI claim frequency for several consecutive periods of time) or, perhaps, some cross-sectional, functional relationship (such as an expense study by size of risk). But regardless of the data or the specific problem — which generally is one of prediction — the process is designed to eliminate the random (and sometimes, non-random) irregularities existent in the observed data.

We are all familiar with several methods of graduation:

(1) Graphing — where a convenient plot of the data is made and a

"smooth" curve is drawn among the data points. While done easily and quickly, graphing lacks an important quality: that of consistency.

(2) Moving averages — where each data point is replaced by a weighted average of itself and the points surrounding it. This method is used extensively, and has probably reached its highest level of sophistication in the techniques developed and used by the National Bureau of Economic Research.

(3) Mathematical models — where an "appropriate" formula is fitted to the raw data, using a standard technique such as the method of least squares. An important subcategory to this general approach is that of interpolation, where an $(n\text{-}l)$st degree polynomial is fitted (exactly) to the n data points.

The Whittaker-Henderson process falls under a general technique which is the inverse of the method of moving averages. Thus, under it, each of the original data points turns out to be a weighted average of the adjusted data points. Specifically, in the example used by Mr. Valerius,

$$\text{(A)}\qquad \begin{aligned} u_0'' &= 19u_0 - 36u_1 + 18u_2 \\ u_1'' &= -36u_0 + 91u_1 - 72u_2 + 18u_3 \\ u_x'' &= 18u_{x-2} - 72u_{x-1} + 109u_x - 72u_{x+1} + 18u_{x+2}\ (2 \leq x \leq 16) \\ u_{17}'' &= 18u_{15} - 72u_{16} + 91u_{17} - 36u_{18} \\ u_{18}'' &= 18u_{16} - 36u_{17} + 19u_{18} \end{aligned}$$

Note that this system can be interpretated in two distinct ways. Regarding the u_x as the raw data, it defines a moving average process. On the other hand, if the u_x'' denote the raw data, then it is a Whittaker-Henderson Formula A process. Strictly speaking, however, a given system would not be used interchangeably in this fashion. In fact, system (A) defines a very poor moving average process; and, conversely, a good moving average system will usually produce a bad difference equation process.

Although (A) is nothing more than a system of 18 linear equations in the same number of unknowns, from a computational point of view, the actual Whittaker-Henderson process is easier than directly solving these equations. It involves "factoring" this system, using the methods of difference equations, into two smaller, simpler systems which for all practical purposes are (but actually, are not) moving average processes. The main problem lies in determining the values at the end points. [Note the asymmetry of the first two and the last two equations in (A).] This problem plagues all graduation processes, but it is particularly serious in the Whittaker-Henderson process.

A new approach to this particular problem is presented in the first of the author's Notes. When one reconciles himself to the fact that graduation is a tedious task (at least when done by hand), the iterative technique suggested by Mr. Valerius seems quite satisfactory. There appears to be an important theoretical flaw in the development, though, in that no proof is given to the implied convergence of the iteration process. (However, on this point, it should be noted that I have not seen the original Henderson or Spoerl papers — and the necessary convergence properties may be developed therein.)

The second Note in the paper is the observation that the Whittaker-Henderson process is additive — a point of considerable practical value.

The third Note is concerned with an extremely important aspect of graduation: that of projection. It would be foolish to state categorically that a particular method for projection is good or bad. It suffices to say that the difference equation approach provides us with another tool. For example, in one of the other papers presented in November, 1967, on "The Minimum Absolute Deviation Trend Line" by Charles Cook (*PCAS* Vol. LIV, p. 200) a simple illustration was presented, involving the projection of the series: 110, 109, 112, 111, 115, 112, 113, 114, 112, 116, 114, 117, 119. Mr. Cook's procedure yields estimates for the next two points of 118.4 and 119.2. Fitting a straight line, via the method of least squares, yields 117.7 and 118.4. The Formula A process, using the author's specific case: $z = 2$, $a = 2$, gives estimates of 118.9 and 120.0. (The interpolation method, fitting a 12th degree polynominal to the 13 points, yields -1723 as the next point, thus illustrating an extreme case where the measure of closeness of fit, by itself, is not adequate.) If one plots the actual data and these alternative estimates, it is easily "seen" that the Formula A estimates, in this particular case, are more realistic. Unfortunately, the procedure is extremely time-consuming in comparison to other methods. It also suffers the same fault as other methods — it is only as good as the assumed model fits the data. For example, it will not track a cyclical movement unless the latter has been programmed into the model.

Finally, in an attached appendix Mr. Valerius includes a very convenient tabulation of the coefficients for the iteration equations, corresponding to the more useful cases of Formula A.

DISCUSSION BY RICHARD H. SNADER

Mr. Valerius' notes on Whittaker-Henderson Formula A have provided casualty actuaries with an opportunity to improve one of the most powerful tools at their disposal. The problem of examining a series of data, detecting a trend, and projecting that trend is one with which we are all vitally concerned. To fully appreciate the value of his contribution, a brief synopsis of the basic concepts of graduation might be helpful.[1]

Graduation may be defined as the process of securing from an irregular series of observed values a smooth, regular series of values consistent in a general way with the observed series. The smooth series is then taken as a representation of the underlying law that gave rise to the observed values. The set of observed values is usually donated by $\{u_x''\}$ and the graduated values by $\{u_x\}$.

Graduation is characterized by two essential qualities, smoothness and fit. These qualities are not independent. An increase in smoothing results in a reduction in fit; conversely, when fit is improved, smoothness usually suffers. Whittaker-Henderson formulas are the product of the difference equation method of graduation. In this method, the graduated series is determined by a difference equation derived from an analytic measure of the relative emphasis placed on smoothness and fit.

The combination of smoothness and fit may be expressed by $F + hS$, where h is a positive number fixing the relative weight assigned to smoothness and fit. Smoothness is measured by the smallness of the sum of the squares of the z^{th} order of differences of the graduated values:

$$S = \sum (\Delta^z u_x)^2, \text{ where } \Delta \text{ is the difference operator.}$$

Closeness of fit is measured by the smallness of

$$F = \sum (u_x - u_x'')^2.$$

[1] The description of the graduation process is based almost entirely on Morton D. Miller's monograph *Elements of Graduation* published by the Society of Actuaries.