each counted four times while the remaining three quarters receive counts of three, two, and one. Thus the earliest and latest quarters receive smaller counts than those in the middle and tend to offset any reverse weighting of the formula.

The minimum absolute deviation method of fitting a line will eliminate the deficiencies of the least squares method and, in addition, is many times easier to use. However, the absolute deviation procedure itself has a very serious drawback that was recognized by the author in his paper. This deficiency is not always present but only comes into effect when $Z_{k^*}' = MX$. In this instance the slope of the fitted line is not unique and any slope within a given range will satisfy the basic criterion of minimizing the sum of the absolute values of the differences. Mr. Cook suggests that in this case we use the average value in the range. While this suggestion is reasonable, the deficiency in the method still remains in that we are forced to enter a judgment factor into what ideally should be a completely objective method. It should be noted that the condition $Z_{k^*}' = MX$ is not highly unlikely since it occurs in fitting the latest countrywide automobile trend line both for bodily injury and for property damage.

The author describes the method that he has developed as an "algorithm of the operations analysis type" which perhaps could be stated as a method based upon a constructive proof. However, no matter what you call it, it is not an easy proof to read. In an effort to be concise, the author has left many gaps in the proof for the reader to fill in for himself, making it difficult for the casual reader to follow. Those with the spare time will find the exercise rewarding.

Mr. Cook has again demonstrated his unique talent for mathematics and we hope that he will come forward soon with more work in this area.

DISCUSSION BY KENNETH L. McINTOSH

This paper most certainly demonstrates, should such demonstration be necessary, that "an algorithm of the operations analysis type" need not involve complex and interminable arithmetical detail.* A word of arithmetical caution may be in order, however. Since $a_i = (y_i - \bar{y})/x_i;\ x_i \neq 0;$

* The distinction between traditional "mathematics" and "Operations Analysis" may be a matter more of semantics than of substance. *Cf.*, *e.g.*: Newton's algorithm to obtain the roots of polynomials; also the Gauss, Gauss-Jordan, and Crout algorithms for solving simultaneous linear equations. Linear Programming is directly related to Combinatorial Analysis, and Dynamic Programming seems to have an impact upon the theory of the Calculus of Variation. Where is the line to be drawn?

the difference $y_i - \overline{y}$ must contain at least as many significant digits as are desired in a_i. In many cases this will require retention in the original data of more decimal places than necessary to obtain equal precision in the results of the least squares calculation.

The application of the algorithm is not so restricted as the author states it to be. "Equal intervals between measurements" are *not* required; it is required only that $\overline{x} = 0$, and even that constraint may be by-passed. Nowhere does the proof of the method rest upon the value of any individual x_i, nor upon the value of any interval $x_{i+1} - x_i$. It follows that the algorithm is valid for arbitrary spacing of the measurements. The necessity for the constraint $\overline{x} = 0$ is not apparent from the analysis presented in the paper, but becomes apparent upon further analysis of certain mathematical detail which is totally unnecessary to Mr. Cook's rigorous and beautifully concise proof. To avoid the constraint, suppose that $X \neq 0$; where $\overline{X}$ is the mean of the original abscissae. Let $x_i = X_i - \overline{X}$, and minimize $\sum^n{}_i \mid ax_i + \overline{y} - y_i \mid$ by the algorithm. If the solution is $a = a^*$, then the desired line on the original X_i's will be:

$$y = a^* + X + b^*; \quad (b^* = \overline{y} - a^*\overline{X})$$

On the necessary assumption that some simple linear model (as opposed to a curvilinear model) will be an acceptable approximation of the true but unknown trend function, projection of the least squares line to a future time point, x_{n+p}, yields an unbiased estimate of y_{n+p}, even where a normal distribution cannot be hypothecated. If $\overline{x} = 0$, of course $\overline{y}$ will be an unbiased estimate of the y-intercept no matter how the line may be fitted. An unbiased estimate of the y-intercept is not of itself sufficient, however, to guarantee that the projection of the minimum absolute deviation line to x_{n+p} will yield an unbiased estimate of y_{n+p}. This matter requires further investigation.

Mathematically, Mr. Cook has neatly sliced a Gordian knot, but the mathematical structure of the algorithm becomes apparent only when the knot is untied in laborious fashion. The difficulty of minimizing $E(a) = \sum^n{}_i \mid ax_i + \overline{y} - y_i \mid$ stems, of course, from the fact that the derivative, $E'(a) = dE(a)/da$, is not continuous on the entire line $-\infty < a < \infty$. However, on any open interval, $a_i < a^* < {}_{i+1}$, the derivative $E'(a^*)$ can be shown to exist, and on that interval $E'(a^*) \equiv$ constant. Further, the one-sided derivatives exist at the end points of the interval, and:

$$E'(a_i + 0) = E'(a^*) = E'(a_{i+1} - 0)\,; \; i = 1, 2, \ldots, n - 1$$

It follows that there will exist some unique interval, say $a_\mu < a^* < a_{\mu+1}$, such that:

$$E'(a_\mu - 0) < 0; \; and: \; E'(a_{\mu+1} + 0) > 0$$

and either:

$$E'(_\mu + 0) = E'(a^*) = E'\ (a_{\mu+1} - 0) < 0 \qquad \text{CASE I.}$$

or else:

$$E'(a_\mu + 0) = E'(a^x) = E'\ (a_{\mu+1} - 0) \equiv 0 \qquad \text{CASE II.}$$

In Case I it follows that $E(a_{\mu+1}) = Min\ E(a)$. In Case II, $E(a)$ is minimized by any a^* in the closed interval, $a_\mu \leqq a^* \leqq a_{\mu+1}$. The algorithm is neither more nor less than a simple, by no means obvious, and, to say the least, ingenious technique which immediately locates the interval $a_\mu < a^* < a_{\mu+1}$ among the n–1 intervals $a_i < a < a_{i+1}$, and which in the process discriminates between Case I and Case II without need to calculate the derivatives in either case. Once the algorithm is given, it is not too difficult to prove its validity by direct reference to the intervals and derivatives noted above, though the development is longer and more involved than the proof given in the paper. Developing the algorithm in the first place is a very different matter indeed.

Mr. Cook has broken a trail into very interesting territory. Though not yet persuaded that minimum absolute deviation should supplant least squares in the trend calculation, I suggest that the matter is well worth further study.