DISCUSSION BY CHARLES A. HACHEMEISTER

The only way in which a working analytical model for retrospective rating will ever be produced is by initially proposing a model which can be criticized and improved. It is extremely difficult to produce a finished working model without having a jumping off place for one's thoughts. We should be grateful indeed to Mr. Hewitt for having given us this paper.

This review will be divided into two main parts. The first is a general discussion of models for loss ratio distributions with particular reference to Table M and the gamma distribution. The second comments on some of the technical aspects of the paper.

MODELS FOR LOSS RATIO DISTRIBUTIONS

We are all aware of the deficiency of assuming that all insureds developing the same expected losses or premium should be subject to the same insurance charges. A large clerical risk and a small oil well drilling risk can produce the same expected losses, but the loss ratio of the clerical risk is much more stable than that of the oil well drilling risk. However, it is not difficult to understand why different tables of insurance charges do not now exist.

If one were to take the time to read through the recent paper "The 1965 Table M"[1] the reason would be eminently clear. It was in his review of this paper that Mr. Hewitt first commented on the difficulties surrounding the use and generation of Table M type statistics. At that time he proposed a program of constructive steps to be taken to do away with these difficulties. The essence of the program was the construction of a mathematical model of the family of loss-ratio distributions. More particularly, the model would contain parameters which would vary by different types of insureds (i.e. sub-line of insurance, class, geographical location, time and size).

The current paper under review expands upon a model first mentioned by the author in his review of "The 1965 Table M." The model was developed by fitting gamma distributions to data in a California Inspection Rating Bureau report containing loss ratio distributions for California experience-rated insureds grouped by premium size. This procedure implies that the composite distribution of loss ratios for all insureds developing the same premium is a gamma distribution. This idea is comparable to the current use of a common Table M for all classes of business being retro-rated.

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[1] Simon, LeRoy J., *PCAS*, Vol. LII

A comparison of the parameters of the distributions for different premium sizes showed an apparent theoretical discrepancy. In particular, the ratio of the parameter r to premium size should remain constant. This ratio did not remain constant; it decreased as premium size increased. The reason was correctly assessed to be the different mixes of frequency and severity in different premium size groups.

In the light of this result, the author challenges "the present method of equating insurance charges for three-year retrospective rating plans with insurance charges for a one-year plan on a risk three times as large," in addition to 50% and 200% quotes for Plan D. Let us carry this one step further. If we take any insured and assume that he has just enough exposure to be eligible for retrorating, then his insurance charges will be calculated from the smallest premium group. If then we calculate his insurance charge from the premium group indicated by any increase in exposure, the same problem arises as in the case of the one to three year comparison. If we have assumed that the variability of loss ratios implicit within the smallest premium group is appropriate for this insured, then the variability within the large premium group is too large because of the greater predominance of high severity insureds in this group. In spite of the fact that the larger premium group may exhibit a smaller variability than the smaller group, that variability is larger than would be expected. This problem is independent of whether the gamma distribution is the model or not. Whenever insurance charges are calculated from a model (Table M, gamma, or otherwise) wherein all insureds with the same premium are grouped together in spite of their severity, this problem will arise. If a different criterion, say, perhaps, a combination of class severity and premium were used, more consistent insurance charges could be calculated.

There is some question whether the gamma distribution is an admissible model for loss-ratio distributions, even without considering the fit. Loss ratios are defined continuously over all positive numbers. In addition, zero loss ratios are not only possible, but occur frequently for small premium sizes. Hence the probability of a zero loss ratio must be greater than zero in a realistic model. A model defined strictly in terms of a continuous probability density function, such as the gamma distribution, cannot supply this greater than zero probability mass since $\int_o^o f(x)\,dx = o$, by definition,[2] for any probability density function. At the time Mr. Hewitt

[2] Riemann integration.

first discussed his gamma distribution model, he mentioned that a good fit was obtained for large premium groups. However, for premium groups below $5,000, the fit was unsatisfactory. He ascribed this to the presence of zero loss ratios. The inability of the gamma distribution to properly handle zero loss ratios is certainly a major contributor to the unsatisfactory fit.

Perhaps we need to look one step deeper into the process to overcome this difficulty. Maybe the model for loss ratios should be composed of frequency and severity elements appropriately mixed. One such possibilty would be Poisson frequency, $g(i)$, and gamma severity, $h(x)$, yielding a composite distribution of loss ratios,[3] $f(y)$:

$$f(y) = \begin{cases} e^{-\lambda} & , y = o \\ \sum_{i=1}^{\infty} \dfrac{\lambda^{(r+1)i}\, r^{ri}\, y^{ri-1}\, e^{-\lambda(ry+1)}}{i\,!\,\Gamma(ri)} & , y > o \end{cases}$$

FITTING THE GAMMA DISTRIBUTION

The procedure outined and used in the paper purports to use a more stable procedure than moments by taking advantage of the less skewed moment distribution, $r'g(r')$. Unfortunately, it reduces to one of fitting by moments, since by definition $E_{g(r')}(r'^2) = E_{g(r')}(r') \cdot E_{r'g(r')}(r')$

It is preferable to use maximum likelihood estimates where possible since they are asymptotically minimum variance unbiased estimators. On first blush the maximum likelihood estimates of the gamma parameters look intractable. However, with the aid of a tabulated function of r the solution is straightforward. The maximum likelihood estimate of a is $\dfrac{\hat{r}}{x}$ even if r is not known. The maximum likelihood estimate of r is the value of $\hat{r}$ such that $\Psi(\hat{r}) - Ln(\hat{r}) = \overline{Lnx} - Ln\bar{x}$

where $\Psi(Z) = \dfrac{d \log \Gamma(Z)}{dZ}$

Note that the maximum likelihood equation for r contains the average of the logarithms of the sample observations, $\overline{Lnx}$. This is only defined for sample values greater than zero. In other words, maximum likelihood estimation cannot be used when the data contains zero values. The gamma

[3] Wadsworth, George P., and Bryan, Joseph G., *Introduction to Probability and Random Variables,* p. 139, Example 5-17, McGraw-Hill, 1960.

distribution can be modified slightly to allow for a zero case by simply introducing a probability, p, of the event zero:

$$f(x) = \begin{cases} p & , x = o \\ \dfrac{(1-p)\, a^r\, x^{r-1}\, e^{-ax}}{\Gamma(r)} & , x > o \end{cases}$$

The maximum likelihood estimates of r and a do not change when this function is fitted. The maximum likelihood estimate, $\hat{p}$, of p is the ratio of the number of zero observations to the total number of observations.

$\Psi(Z)$ is a well tabulated function.[4] For maximum likelihood fitting the gamma distribution this reviewer has tabulated $\Psi(Z) - LnZ$.

I fitted the modified gamma distribution by maximum likelihood to the same C.I.R.B. data. The fit was not improved for the smaller premium size groups. For the larger premium size groups sometimes the fit was better, sometimes worse. However, the estimate for r was consistently lower than that calculated by moments. This leads to the conclusion that perhaps the moment estimator for r is biased on the high size.

A few comments are in order concerning the calculation of the gamma probabilities. Pearson's tables of the incomplete gamma function are used by the author. There is nothing wrong with using these tables; however, the following relationships allow the complete gamma, and the incomplete gamma probabilities can be calculated directly by computer:

$$GAM(r) = \Gamma(r) = \begin{cases} \prod_{i=1}^{[r]+1} (r-i)\, \Gamma(r-[r])/(r-[r]-1), & r > o \\ \Gamma(r-[r])/\prod_{i=1}^{[r]} (r-i+1), & r < o \end{cases}$$

where $[r]$ is the greatest integer less than r.

$$\frac{1}{\Gamma(Z)} = \sum_{k=1}^{\infty} C_k Z^k$$

where C_k are constants.[5]

$$GAMIN(p, a, r) = \int_o^p \frac{a^r x^{r-1} e^{-ax}}{\Gamma(r)}\, dx = \int_o^{ap} \frac{x^{r-1} e^{-x}}{\Gamma(r)}\, dx = \sum_{i=1}^{\infty} \frac{(ap)^{r-1+i} e^{-ap}}{\Gamma(r+i)}$$

[4] *Handbook of Mathematical Functions,* AMS55 — U.S. Department of Commerce, pp. 267-273.

[5] Ibid p. 256.

Note that when r is an integer this sum reduces to the probability of at least r successes for a Poisson frequency function.

$$\frac{(ap)^{r+i}\, e^{-ap}}{\Gamma(r+i+1)} \bigg/ \frac{ap^{r+i-1}\, e^{-ap}}{\Gamma(r+i)} = \frac{ap}{r+i}$$

Errata in Author's Paper

Section 2.21(h)	p. 39 should read p. 391
Section 3.6(d)	replace r′ by E(r′) replace r″ by E(r″)
Appendix Table 2	Degrees of freedom for Chi-Square is 5 not 7, since one d.f. must be deducted for each parameter estimated.