

PROCEEDINGS

MAY 23, 24, 25 and 26, 1965

THE 1965 TABLE M

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*An Argument Against the Empirical Method
Some haystacks don't even have any needle.*

... William Stafford

Ours did.
We found it.

The purpose of this paper is to set forth in an orderly fashion, a summary of certain aspects of the work done in compiling the 1965 version of Table M. It will be assumed that the reader has a basic knowledge of Table M and its use in obtaining insurance charges for retrospective rating plans.¹ This report will be presented in a fashion that will imply that we went straight as an arrow from the problem to the solution. The reason for doing this is to present a clear and easy to follow line of reasoning. I'm sure it is realized that the work could not and did not proceed in exactly this manner. Many false leads were pursued, and oftentimes decisions would be made near the end of the work which affected things done near the beginning. The earlier work would then be redone. Part of the reason for creating this written record is that the future researcher will have an easier time following the development. Through footnotes and appendices, I hope to indicate some of the areas where we investigated ideas and rejected them, where expediency prompted us to accept the results produced and where we could hope for improvement in the future.

On September 13, 1962, the subject of a revision for Table M was placed on the agenda of the Actuarial Committee of the National Council

¹ Basic reference papers on the compilation of excess pure premium ratios (that is, Table M) are Lewis H. Roberts, "Graduation of Excess Ratio Distributions by the Method of Moments," *PCAS Vol. XLIV*, pg. 45; Nels M. Valerius, "Risk Distributions Underlying Insurance Charges in the Retrospective Rating Plan," *PCAS Vol. XXIX*, pg. 96; and Thomas O. Carlson, "An Actuarial Analysis of Retrospective Rating," *PCAS Vol. XXVIII*, pg. 283. A technical definition of the excess pure premium ratio will be found in this paper as equation (C12) in Appendix C.

on Compensation Insurance. Aetna Casualty and Surety Company, American Mutual Liability Insurance Company, Insurance Company of North America and Liberty Mutual Insurance Company were appointed to a Subcommittee to Review Table M. Also attending all meetings and participating in the discussions was a representative of The Travelers Insurance Company.

Table M was last revised in 1954. Unfortunately, no paper was presented to the Society at the time of this revision, and the only convenient reference is the files of the National Council on Compensation Insurance. At times, it is difficult to differentiate between the proposals made and the ideas that were actually put into effect. However, it was clear that this revision was made to reflect the increased variance in the risk distributions since the 1943 table of the National Council was published. Basically, the table itself was not changed, but the column headings were revised so that the variance of risks having the expected losses shown at the top of a given column was equal to the variance of the underlying risk distribution for that column in Table M. Basic raw data, therefore, was only used to the extent of determining the variance for a given expected loss size, and was not used to actually calculate excess pure premium ratios.

It was decided that the 1965 revision of Table M would be based on actual workmen's compensation risk distributions. Individual risk information was gathered for policy year 1960-61 from all states where the National Council on Compensation Insurance is the filing agency plus the states of California, Hawaii, Massachusetts, Michigan, Montana, New York, North Carolina, Virginia and Wisconsin. The columns of charges² in the 1964 Table M were numbered from 1 through 37, and the expected loss ranges were converted to premium ranges by dividing by .596 (the permissible loss ratio most commonly used throughout the country). The risks were then sorted by standard premium and assigned "Old Premium Group" numbers on this basis (risks having a premium less than \$1,678 were excluded). Referring to Exhibit A, we see that experience was found in each of the first 36 premium groups of the 1954 Table M. All risks with standard premiums between \$1,678 and \$1,987 were included in the first group, and the last group included risks with standard premiums between \$608,221 and \$2,307,046. A total of 112,646 risks were processed with an aggregate premium of \$855,278,990.

Four interesting statistics are included on Exhibit A to help visualize

²I will use "excess pure premium ratios" and "charges" interchangeably. The European term "stop-loss premium" could also have been used.

the data better: (1) the loss ratio; (2) the charge at a ratio of 1.00, that is, at the mean; (3) the percent of risks falling below the mean and (4) the median value of the ratio. The last three statistics are based on the adjusted ratio of actual losses to expected losses, r , which will be described later.

Exhibit A also shows the raw data average loss per risk which is the quotient of the total actual unlimited losses in the premium group divided by the number of risks. A study of loss trends over the years has indicated that the average cost per case in workmen's compensation is steadily increasing. This is expected because of more costly medical bills, plus the fact that legislation has tended to increase the value of all the benefits. Exhibit B shows a six year trend of workmen's compensation average loss costs. This indicates a 33.7% increase in cost over a five year period. There is to be a five year gap between the period used in collecting the basic statistical information (1960) and the time when the new Table M will first go into effect (1965). Today's workmen's compensation risk, therefore, having a given number of expected losses, will be roughly 35% larger in terms of dollars of expected loss. It is proven in Appendix A that this means that average losses per risk should be modified 35% assuming the coefficient of variation of the claims distribution remains constant. Countering this argument is the slightly improved claim frequency experience resulting from improved safety measures and technological changes. However, it was felt that this was more than offset by other factors.

Another factor behind the modification was that the raw data was based on first reports under the unit statistical plan. It is well-known that the major developments from first to second reports occur in the area of large claims.³ This is almost certain to increase both the variance and the skewness of the distribution. Hence, we felt that the raw data risk distributions were undoubtedly more compact than the truth would show if it could be known. It was, unfortunately, not financially feasible to make a comparison of risk distributions between first and second reports under the unit statistical plan. Somewhat countering this argument again is the fact that unit reports split large interstate risks into smaller intrastate pieces. Because experience rating is on an interstate basis, the pieces are being adjusted in the direction of the mean of the entire risk. Therefore, we expect the basic data to have a little larger variance than if the smaller pieces had been subject to experience rating on their own. As a result of all these con-

³ For example, a special study of Serious losses for policy year 1960-61 totaling \$150,000,000, showed a development factor of 1.146 from first to second reports.

siderations, we arrived at E, the expected loss (current level) shown on Exhibit A, by modifying the raw data average loss per risk by a factor of 1.35. Three significant digits were retained here and were also used in the final column headings on Table M.

The work now progressed toward calculating and tabulating a column of insurance charges for each of the 36 premium groups. Working on one group at a time, the standard premium (P) for each risk was multiplied by .596, and the ratio of the actual losses (A) to the standard premium times .596 was calculated, rounded to two decimal places and designated R. The risks were then sorted on R, and each premium size group was tabulated as shown in Exhibit C. The standard premium and unlimited losses were shown for reference purposes and for checking. Two additional quantities, Sum 1 and Sum 2, were also calculated on the first pass of the cards. The first is simply a downward accumulation of the number of risks, and the second is the calculated number of "points" of excess over the ratio shown opposite R. This can be expressed as follows:

$$S_{2,i} = S_{2,i+1} + (R_{i+1} - R_i) S_{1,i+1}$$

where S_1 and S_2 are Sum 1 and Sum 2 respectively, and $i = 0, 1, 2, \dots, \omega$; that is, it is a sequential numbering of the R 's from the .00 end of the table up to the limiting value ω .⁴ Sum 2 continues until it reaches $R = .00$. (If a case did not occur with zero losses, such a card was put in the deck with a zero as the number of risks.) $S_{2,0}$ then contained the total number of points of excess over 0, that is, the sum of the frequencies times the ratios. The mean ratio is the quotient of $S_{2,0}$ divided by the total number of risks.

For Table M purposes this mean ratio should be exactly 1.00 and, therefore, a correction factor was applied to R which would adjust the mean to unity. Making this adjustment is equivalent to saying that we would accept the actual loss ratio of the entire group as being the best estimate of the expected loss ratio of the group. It also fitted in properly with theoretical considerations of Table M which will be referred to later. The adjusted ratios were called r and were used hereafter. On the second pass of the cards, two calculations were made at the same time. The first was to calculate r and the second to calculate the Table M charge, $\phi(r)$, by dividing Sum 2 by the entry opposite Sum 2 at $r = .00$.⁵

⁴ In actual practice the tabulation was made in a slightly different fashion which included Sum 1 as a part of Sum 2. However, in dealing with the material it was noted that the system outlined could have been used equally well and perhaps with a little advantage in calculating simplicity.

⁵ See Appendix B for a proof that the scale translation is permissible and for discussion of the interpolation problem when there are missing values.

A graphic plotting of some of this data across a common entry ratio indicated that there was a fair degree of regularity, but there were certain fluctuations. Exhibit D indicates how these occurred at an entry ratio of $r = 1.60$. It was decided that a preliminary smoothing would be accomplished by graduating the entries across a common entry ratio using the Whittaker-Henderson formula.⁶ Although the Whittaker-Henderson formula is based on the assumption of equally spaced data, it was applied to this material. In the lower premium group sizes this is not too disturbing since there are approximately even intervals between the values of E . However, as one goes from Old Premium Group 21 to Old Premium Group 22, the jump in E is significantly different than it had been up to that point. There is a similar sharp change at Old Premium Group 32. It will be developed later in this report that the smoothed data under the Whittaker-Henderson smoothing was only used up through Old Premium Group 31. The small circles on Exhibit D indicate the effect of this smoothing on the data. Calculations were made for $r = .20(.20)5.00$. We thus had a compilation of insurance charges smoothed under the Whittaker-Henderson formula which were arrayed in a matrix of 36 premium groups by 25 entry ratios.⁷

The actual production of an entire Table M is a mammoth job even under the best conditions. We expected to compute insurance charges from entry ratios of .00 through 3.00 for publication. We also expected to have in the neighborhood of 50 premium groups requiring the production of some 15,000 insurance charge values. An attempt to do this through any method such as graphing or interpolating between selected points seemed beyond the realms of possibility. We were particularly interested in producing a table as promptly as possible while exercising a minimum number of independent judgments. Constructing this Table M has certainly given all those who were involved in the work a great deal of appreciation and respect for those who constructed the original Table M values. It seemed likely that if we could use the experience data we had

⁶ Valerius, op. cit. pg. 107-110. With the advantage of hindsight, we can now question whether this preliminary graduation across premium groups was altogether necessary. Since we were successful in being able to obtain formulas to represent the columns of charges, we probably would have been just as successful in dealing with the raw data as we were in dealing with the Whittaker-Henderson smoothed data. However, at the time, it seemed that this procedure would give our other formulas a maximum opportunity for success; and at the same time, if they were unsuccessful, would provide the starting point from which the table could be constructed.

⁷ Early experience using desk calculators showed that the results using 25 points were as satisfactory as the results using 51 points on the curve fitting technique under test. We did not experiment to see if the 25 could have been reduced further.

accumulated and derive some formulas from it, we could then be in a position to reformulate the premium groups in some fashion that would be more to our liking. Finally, we could foresee the possibility of using high speed computing equipment to evaluate retrospective rating values if the Table M charges could be determined by the computer through the use of mathematical formulas. Because of these many advantages, a considerable amount of effort was spent in the search for "down-the-column" graduation formulas.

Perhaps as many as 25 different general equations were explored as possible mathematical formulas to describe the column of insurance charges, $\phi(r)$. Some were rejected very promptly because they did not appear to offer a sufficient amount of flexibility or because determining the constants necessary appeared to be an insurmountable task. Nine equations seemed to meet most of the subjective requirements, and these were thoroughly tested mathematically in accordance with the development in Appendix C. From a mathematical standpoint and from preliminary desk calculator tests which were made, two formulas were processed for electronic computer programming and thus considered eligible for the final selection. These two formulas were:

$$\phi(r) = \exp [- (r + a_2r^2 + a_3r^3 + a_4r^4 + a_5r^5 + a_6r^6 + a_7r^7)] \quad \dots\dots\dots (1)$$

$$\phi(r) = 1/(1 + r + b_2r^2 + b_3r^3 + b_4r^4 + b_5r^5 + b_6r^6 + b_7r^7) \quad \dots\dots\dots (2)$$

(Work was begun on Equation (C28), Appendix C, but was stopped when we decided to accept another equation as final.)

A computer program was written so that for a given premium group the computer would read in the 25 values of r and the corresponding smoothed data values for the insurance charge. When working with Equation (1), the computer would transform the smoothed charge, $\phi_8(r)$, as follows:

$$Y(r) = - [r + \ln \phi_8(r)]/r^2$$

We were thus able to rewrite Equation (1) as:

$$Y(r) = a_2 + a_3r + a_4r^2 + a_5r^3 + a_6r^4 + a_7r^5$$

The computer then solved the problem of obtaining the six coefficients a_2 through a_7 such that we would have a least squares best fit using r and $Y(r)$ as the two variables. Using these fitted coefficients, the computer then produced the 25 fitted values. By comparing the fitted values with the original smoothed values, the percentage error was calculated. This is all set out in Exhibit E. It should be noted that in calculating the per-

centage errors, the computer was working with eight significant digits, although the fitted values which were printed out were only shown to four decimal places. Therefore, the reader will not be able to exactly reproduce the percentage error figures shown in the exhibit.

On the same pass of the data, the information was transformed for Equation (2) as follows:

$$Z(r) = \left[\frac{1}{\phi_8(r)} - 1 - r \right] / r^8$$

We were thus able to rewrite Equation (2) as:

$$Z(r) = b_2 + b_3r + b_4r^2 + b_5r^3 + b_6r^4 + b_7r^5$$

Again, the coefficients were determined by a least squares best fit to the points r and $Z(r)$, the fitted values for the 25 points were produced by the computer and error calculations were made.

As we examine Exhibit E, there is little choice between the two formulas. Out to three decimal places, they produced the same result in nearly every case. However, the pattern shown on that exhibit was generally repeated throughout the first 22 premium groups and indicated that Equation (2) had a slight edge over Equation (1). From Old Premium Groups 23 through 28, Equation (1) was slightly better, and beyond Old Premium Group 28 neither one of the two equations produced satisfactory fits. It was decided that only one general form of equation would be used, and Equation (2) was selected.

In an attempt to extend the area over which the curves would satisfactorily fit, Equation (2) was revised to include an additional term and thus read.⁸

$$\phi(r) = 1/(1 + r + b_2r^2 + b_3r^3 + b_4r^4 + b_5r^5 + b_6r^6 + b_7r^7 + b_8r^8) \quad \dots \dots \dots (3)$$

An example of this output for Old Premium Groups 10 and 11 is shown on Exhibit F.

The computer run was completed through Old Premium Group 34

⁸ Originally, we were unsure whether either Equation (1) or Equation (2) would perform satisfactorily. The original computer programming which was based on orthogonal polynomials dealt with the input in a coded fashion such that the 25 values were sequentially numbered from -12 through +12. The coefficients which the computer actually determined were thus in this coded translation and were not directly usable. It was, therefore, necessary to do some additional programming to get the coefficients into a satisfactory form. We decided we would rewrite the entire job, and at the same time, omit any Equation (1) calculations and extend Equation (2) one more term.

and the results were examined for closeness of fit. In the first two premium groups it was observed that the curves were producing values which implied negative frequencies for the underlying distribution in the neighborhood of $r = 1.8$. This also occurred in Old Premium Group 3 in the neighborhood of $r = 3.8$. Other than this, the fit of the formula value to the smoothed value was quite good up through Old Premium Group 28. In Old Premium Groups 29, 30 and 31, the formulas were not fitting too closely at $r = 1.4$. The errors were in the direction such that the formulas were coming closer to the raw data on both Old Premium Groups 29 and 31, and we decided to retain the formula approach all the way through Old Premium Group 31. At Old Premium Group 32, it was felt that the results were wholly unsatisfactory and, therefore, nothing beyond this point would utilize the formulas produced by the least squares best fit of the smoothed data.⁹

A basic decision was made at this point about reformulating the entire format of Table M. It was decided that the premium groups would be reformulated on such a basis that the insurance charges at an entry ratio of $r = 1.00$ would be spaced at intervals of .010 between premium groups. For reference purposes the new columns of Table M were referred to as Premium Group .64 through Premium Group .01 where the premium group number identified the first two digits of the charge at an entry ratio of 1.00.¹⁰ The coefficients for the equation of a given premium group were determined by interpolation using the coefficients determined by the raw data. Exhibit G illustrates the method of computation¹¹ for premium Group .43, and Exhibit H sets forth the coefficients which were so determined for Premium Groups .64 through .21.

We next turned our attention to the problem of extending the tables beyond Premium Group .21. Referring to Exhibit A, we can see that the number of risks became quite small from Old Premium Group 32 onward, and we were not surprised to find random fluctuations playing a larger part. Appendix D gives the development of a technique to pro-

⁹ The formula approach was extended further on a different basis as will be discussed later.

¹⁰ The raw data furnished us with coefficients for Premium Groups .21 through .61. We further extrapolated the coefficients successfully to produce Premium Groups .62, .63 and .64.

¹¹ A graphic method of interpolation was attempted at one point in our studies, but was rejected when we felt a need for more significant digits than could be so obtained. The coefficients fell along surprisingly smooth curves when plotted against a $1/\phi(1.00)$ abscissa, and we were convinced that the interpolation technique of Exhibit G was quite satisfactory.

duce underlying risk distributions and, hence, charges using Pearson Type III curves.¹² The statistical foundation for this method is also given. In brief, the method consists of graduating the moments of the raw data distributions and then using these moments to produce Type III curves from which the charges are calculated.

In making some of the final checks, we found that the two approaches (reciprocal polynomials and Type III curves) met in such a fashion that the gap between the two could not be bridged in any rational fashion. We found that the cross-differences in the area of $r < 1.00$ were generally too large, and the results produced net insurance charges in retrospective rating which increased as the size of the risk increased. An occasional anomaly of this sort of .001 or .002 might be allowed, but these inversions were both large and frequent. Therefore, to eliminate this problem we used Premium Groups .21 and .11 as fixed end points and performed a linear interpolation between these two groups to produce the values for Premium Groups .20 through .12.¹³ The material in Appendix D is included partly for possible future use and partly because our findings closely parallel those reported by Bohman and Esscher.¹⁴

The final set of premium groups remaining were those from Premium Group .11 through .01. Using the Type III curve we found that the tabled values could only carry us as far as Premium Group .12. The limit of the Pearson tables is at $p = 50.0$ at which value we produced a $\phi(1.00) = .1119$. The curve was beginning to approach normality, and I felt we should swing over to a normal curve at Premium Group .11. This would be out in an area of about \$800,000 of expected loss which is beyond the size of the largest risk we had in the raw data. Appendix E sets forth the rationale and the technique used in utilizing the normal curve.

This, then, marked the end of the main effort on producing the columns of charges for the new Table M. Two peripheral areas remained.

¹² We also tried experimenting in the small expected loss sizes with (a) Pearson curves, (b) the lognormal curve and (c) transformations which would normalize the data. These met with varying degrees of success, but none seemed satisfactory in the final analysis.

¹³ We also tried to retain the Type III area and smooth out the irregularity by modifying the polynomials from Premium Groups .28 to .21. Time was very short at this point, and three quickly conceived methods each failed to produce the desired improvement. We did not experiment further.

¹⁴ Bohman, H. and Esscher F., "Studies in Risk Theory with Numerical Illustrations Concerning Distribution Functions and Stop Loss Premiums," *Skandinavisk Aktuarietidskrift*, 1963. A two page summary by Mr. Bohman of this 92 page paper is found in *The Astin Bulletin*, Vol. III, Part II, August, 1964, p. 185.

The first was the problem of extending every column out to a charge of zero. The use of these extended values is so rare that they were dealt with rather arbitrarily by using a straight line set of values of the form:

$$\phi(r) = c - mr \quad \dots \dots \dots (4)$$

For completeness, these formulas are listed in Exhibit I.

The second peripheral area was the desire to have two more premium groups calculated because of their special nature. Tabular retrospective plans have normally started at an expected loss of \$596 (i.e., premium = \$1,000). It will later be shown that Premium Group .85 is the one appropriate to $E = \$596$. To help guide our judgment in establishing this premium group, it was observed that if we wanted to place 20 risks at appropriate points along the r scale so they would closely reproduce Premium Group .64, they would be located at .00, .01, .02, .03, .04, .05, .06, .08, .09, .12, .14, .17, .21, .34, .84, 1.00, 1.27 and at two points which are beyond the maximum usable r of our Equation (3). To formulate Premium Group .85, this information was considered and it was decided to place eight risks at .00, five risks at .05, four risks at .10 and one risk at 5.00. The other two risks were to be at r_{18} and r_{20} . These two points are used to fix the conditions that the mean equals 1.00 and $\phi(1.00) = .850$; that is, $[8(.00) + 5(.05) + 4(.10) + 5.00 + r_{18} + r_{20}]/20 = 1.00$ and $[(5.00 - 1.00) + (r_{20} - 1.00)]/20 = .850$.¹⁵ Solving, $r_{18} = .35$ and $r_{20} = 14.00$.

The column of charges could then be produced from these values and is described in Exhibit J.

We also sought a more or less limiting set of values. A Premium Group .99 was constructed by assuming 99 of 100 risks had a zero entry ratio which meant the one was at $r = 100.00$. This simple column of charges is also described in Exhibit J. An expected loss of \$3 was attached to this premium group because one average Workmen's Compensation loss is about \$300.

Now that the premium groups have been reformulated from the 1954 premium group numbers to the new system, it becomes necessary to establish the expected loss ranges for the new groups. The problem breaks down into roughly three areas. In the first, we have formula (E1) from

¹⁵ This assumes that r_{20} will lie beyond 1.00 and r_{18} will not lie beyond 1.00 which is borne out as being true by the subsequent solution.

Appendix E which associates the *midpoint* of the premium group with the expected loss size by the formula

$$\bar{E} = 63200/\hat{\sigma}^*$$

where \bar{E} denotes the midpoint value. Let's define g as the premium group number and g' as the number which divides two premium groups, i.e., $g' = g + .005$ divides g from $(g + .01)$, thus establishing the lower limit of the expected loss range for g which will be called E'_g . Hence, we can write:

$$E'_g = 63200/\hat{\sigma}_g^{*2}$$

It was observed that, from Appendix E,

$$\hat{\sigma}_g = 2.5g$$

and it was inferred that we could validly write:

$$\hat{\sigma}_{g'} = 2.5g'$$

By substitution,

$$E'_g = 10112/(g + .005)^2$$

This formula applied to Premium Groups .01 through .10. The result for Premium Group .11 (765,000) was modified on a judgment basis (to 725,000) in order to make the transition run smoothly from this segment to the next.

The second area for establishing expected loss ranges was that in which Equation (D5) in Appendix D could be helpful. Although the Type III curves were not used to obtain the insurance charges, they were still the best guide to locating the expected loss ranges. A graph was drawn on semi-logarithmic paper of $1/\phi(1.00)$ vs. \bar{E}_g as calculated from:

$$\bar{E}_g = 53400(p_g + 1)$$

These points appear as small circles on Exhibit K and can be seen to fall almost perfectly along a straight line. The line was drawn on the graph and its equation, using the two point form, was found to be

$$\log E'_g = 3.5510 + .2730/g'$$

where "log" designates the common logarithm. It was used over the range from Premium Group .12 through .24.

The final area of consideration is shown on the graph as lying below $1/\phi(1.00) = 4.0$. The values of E and $\phi(1.00)$, as shown in Exhibit A, were used to plot the small crosses on Exhibit K.¹⁶ It appeared to the

¹⁶ Several other possible plottings were also considered, but this seemed the most satisfactory.

eye that the data fell into two sections where straight lines would fit the points rather well.¹⁷ A formula was preferred over simply drawing a line and reading out the values, because (a) we wanted values to three significant places, and (b) the line could best be drawn statistically. Therefore, Old Premium Groups 27 through 8 were used to determine a best fitting least squares line as

$$\log E'_g = 2.6651 + .4955/g'$$

which was used over the range from Premium Group .25 through .46.

Similarly, Old Premium Groups 7 through 1 were used to determine a best fitting least squares line as

$$\log E_g = 1.6363 + .9747/g'$$

which was used over the range from Premium Group .47 through .64.¹⁸ Carrying out this entire set of evaluations for expected loss ranges results in Exhibit L.

Very little space can be devoted to commenting on the gigantic task performed by the people who programmed the IBM 7080 in the offices of the Insurance Company of North America. They accomplished a quantity and quality of work which could not have been done by desk calculator methods "in a hundred years." The reader undoubtedly recognizes the scope of the work involved in solving the least squares fit of a sixth degree equation, and we solved 150 of these problems in the course of this project. The calculation of one charge using Equation (3) takes about five minutes with a desk calculator, and we calculated about 50,000 of them during this study. Joan Featherer did the majority of the programming using FORTRAN and programmed the final print-out of the table in such a well designed and executed manner that reviewing the results was made quite easy. The final running and testing of the table was done in a single program which accomplished a number of important steps. Using the polynomial formula, the computer calculated the insurance charge. If the entry ratio was less than 1.00, it also computed the saving from Equation (C23). If the saving was negative, zero was substituted for the calculated value and the charge was set equal to one minus the entry ratio. The values were rounded to three decimal places and were

¹⁷ A single concave parabola might also have fitted closely but was not tried. Looking back, this would have been better since we did run into a little "roughness" where the lines crossed. A single curve would have avoided this.

¹⁸ It was also extrapolated and used to show that $E = \$596$, $g' = .856$ and $g = .851$. Therefore, Premium Group .85 applies to $E = \$596$.

written out on tape. At the same time, the first and second differences down the column of charges were calculated so that we could readily examine the underlying risk distribution. One of the critical tests was to make certain that the charges produced did not imply the existence of negative frequencies. For example, the reason that Premium Group .64 can only use the polynomial equation out to $r = 1.74$ is that beyond that point the polynomial would imply that there was a negative number of cases over certain ranges.

Exhibit M is an extract of the computer output showing the charges and the savings (which are marked with an asterisk on the tabulation). Exhibit N is an extract of the tabulation of the first and second differences running down the column. For ease of examination these were multiplied by 1000 in order to get them to be whole numbers. Finally, Exhibit O shows the results of the third major calculation that was made on this single pass of the data. In order to test the requirement that the charges at a common entry ratio should decrease as the expected loss increases, we calculated the first differences (again multiplied by 1000) between adjacent columns. A quick examination of this run showed that there were no negative figures and, in fact, the pattern seemed to be relatively smooth from one set of differences to the next. Similar calculations were made in the areas in which the polynomial did not apply, but were made using the desk calculators.

Exhibit P is the statement in FORTRAN language of the heart of the computer calculation of the insurance charge in the areas where the polynomial is applicable. It is assumed that computations in a retrospective rating plan evaluation have reached the point where it is necessary to evaluate $\phi(r)$. The formula of Exhibit P is one of the numerous possible ways of writing the statement in IBM 7080 FORTRAN language so as to reproduce the Table M charge. Finally, Exhibit Q is a ready reference for the premium group numbers and the entry ratios over which each of the formulas applies. The possibility of computing Plan D ratings of retrospective risks through the use of computers now seems wholly feasible.

In conclusion, it might be of interest to note the effect of this revision of the Table. One way to do so is to compare $\phi(1.00)$ under the 1954 Table and the 1965 Table. Exhibit R shows this comparison. It can be seen that the change is largest in the small premium sizes and decreases as the size of the risk increases to the point where it is a reduction at the highest sizes. A similar comparison can be made at $\phi(1.60)$ by reference

to Exhibit D. A second means of comparison is shown in Exhibit S where we can visualize the underlying distributions. Notice the close agreement between the raw data and the 1965 Table M and the change in shape from the distribution underlying the 1954 Table M.

Work on the 1965 Table proceeded with a sense of urgency because the 1954 Table was known to be deficient. We made a number of quick decisions and resisted revising certain of them because of the time element involved. Despite all this, it took three years of elapsed time to get the revision into effect. The best time to start the next revision of Table M is *now*.

It was evident as we worked on this assignment, that an improved theory of risk variation would have been of great benefit. The approach used was highly empirical, and we were extremely fortunate to find as many haystacks containing needles as we did. To avoid the difficulties and the pitfalls of empiricism, we should try to borrow from the mathematical theory of risk, from Monte Carlo techniques and from operations research, especially in the area of anti-selection. Let's begin pushing out some frontiers today, so we'll be ready to solve tomorrow's problems.

ACKNOWLEDGEMENTS

I have only written about the phase of Table M in which I was principally interested and involved. There are many other facets to retrospective rating that were reviewed and revised at the same time. Others are much better equipped than I to comment on these areas.

There are many who have worked tirelessly on this overall project. I personally know of the substantial contributions by: Roy H. Kallop, John R. Bevan, Harry T. Byrne, Robert Pollack, Robert A. Bailey, George D. Morison, Harry R. Richards, Stephen S. Makgill, James P. Jensen, John P. Welch, Burton Covitz, Fred M. Chorpita, Daniel J. Flaherty, Joseph F. Martorana, Mrs. Joan M. Featherer and John Craig.

WORKMEN'S COMPENSATION INDIVIDUAL RISK EXPERIENCE
UNLIMITED LOSS DATA

EXHIBIT A

Old Premium Group Number	Premium Range	Number of Risks	Standard Premium	Loss Ratio	Raw Data Average Loss per Risk	E Expected Loss (Current Level)	¢(1.00) Charge at Actual/ Expected of 1.00	Proportion of Risks Falling Below the Mean	Median Value of Actual/ Expected
1	1678 - 1987	16,950	30,926,469	60.9%	1,112	1,500	.6240	79%	.19
2	1988 - 2429	16,983	37,272,090	60.3	1,324	1,790	.6005	78	.23
3	2430 - 2871	12,089	31,893,890	60.8	1,603	2,160	.5775	77	.26
4	2872 - 3311	8,907	27,443,432	62.7	1,933	2,610	.5592	76	.30
5	3312 - 3974	9,808	35,556,533	58.5	2,121	2,860	.5279	74	.35
6	3975 - 4856	9,153	40,115,384	59.8	2,621	3,540	.5154	73	.38
7	4857 - 5739	6,372	33,614,135	59.2	3,125	4,220	.4865	72	.42
8	5740 - 6623	4,694	28,954,076	62.0	3,823	5,160	.4758	71	.45
9	6624 - 7506	3,674	25,897,532	59.3	4,180	5,640	.4420	70	.53
10	7507 - 8388	2,856	22,636,320	57.9	4,591	6,200	.4320	70	.54
11	8389 - 9271	2,376	20,940,019	58.8	5,183	7,000	.4240	69	.56
12	9272 - 10155	1,874	18,182,410	59.0	5,727	7,730	.4019	65	.59
13	10156 - 11038	1,664	17,619,552	60.6	6,422	8,670	.4022	67	.61
14	11039 - 11920	1,336	15,314,562	60.2	6,898	9,310	.3781	65	.64
15	11921 - 12804	1,245	15,376,031	58.8	7,260	9,800	.3741	66	.66
16	12805 - 13687	1,040	13,771,549	60.3	7,987	10,800	.3743	66	.64
17	13688 - 14569	856	12,077,797	56.4	7,964	10,800	.3461	63	.71
18	14570 - 15454	798	11,980,889	57.7	8,658	11,700	.3531	64	.68
19	15455 - 16336	711	11,290,884	59.8	9,491	12,800	.3314	63	.72
20	16337 - 17219	658	11,042,869	60.8	10,207	13,800	.3594	67	.67
21	17220 - 19934	1,507	27,890,998	56.4	10,442	14,100	.3240	64	.75
22	19935 - 24736	1,842	40,981,259	57.2	12,735	17,200	.3113	65	.75
23	24737 - 29887	1,210	32,839,117	54.7	14,841	20,000	.2868	63	.80
24	29888 - 35429	899	29,240,852	56.7	18,458	24,900	.2788	60	.81
25	35430 - 41346	610	23,385,219	59.4	22,767	30,700	.2884	64	.78
26	41347 - 47340	492	21,821,261	61.3	27,182	36,700	.2755	64	.79
27	47341 - 53831	353	17,821,787	59.5	30,063	40,600	.2523	62	.83
28	53832 - 60821	334	19,096,583	55.3	31,619	42,700	.2482	59	.84
29	60822 - 72427	328	21,601,968	55.9	36,795	49,700	.2294	60	.85
30	72428 - 90603	319	25,640,792	59.4	47,716	64,400	.2340	60	.85
31	90604 - 117449	267	27,426,578	53.3	54,739	73,900	.1943	59	.90
32	117450 - 159395	178	24,179,267	60.7	82,440	111,000	.1842	62	.89
33	159396 - 226509	123	22,861,752	58.2	108,130	146,000	.1611	57	.96
34	226510 - 322986	75	20,381,237	57.4	155,931	211,000	.1655	64	.91
35	322987 - 608220	44	17,547,482	52.6	209,919	283,000	.1348	57	.92
36	608221 - 2307046	21	20,656,415	52.6	517,503	699,000	.1198	54	.99
		<u>112,646</u>	<u>855,278,990</u>	<u>58.5</u>					

TABLE M

COUNTRYWIDE WORKMEN'S COMPENSATION AVERAGE COST PER CASE
ALL CASES, INDEMNITY PLUS MEDICAL

<u>Policy Period</u>	<u>Average Cost</u>	<u>Ratio to Previous Year</u>	<u>Cumulative Change</u>
1956-57	181.03	Base	1.000
1957-58	197.77	1.092	1.092
1958-59	206.46	1.044	1.140
1959-60	223.00	1.080	1.231
1960-61	230.14	1.032	1.270
1961-62	242.40	1.053	1.337

EXHIBIT C

TABULATION OF RAW DATA

Old Premium Group 1

Standard Premium	Unlimited Losses	R	No. of Risks			r	$\phi(r)$ Charge
		Ratio A/ (.596P)		Sum 1	Sum 2	Adjusted Ratio	
1950	206614	177.78	1	1	.00	173.58	.0000
1903	100439	88.57	1	2	89.21	86.48	.0051
1684	63829	63.60	1	3	139.15	62.10	.0080
1958	59617	51.09	1	4	176.68	49.88	.0102
1797	44947	41.97	1	5	213.16	40.98	.0123
1820	43504	40.10	1	6	222.51	39.15	.0128
1937	43849	38.00	1	7	235.11	37.10	.0135
1868	41693	37.46	1	8	238.89	36.57	.0138
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
67236	42789	1.07	37	3474	10656.38	1.04	.6138
43350	27296	1.06	24	3498	10691.12	1.03	.6158
50055	31317	1.05	27	3525	10726.10	1.03	.6179
34125	21065	1.04	19	3544	10761.35	1.02	.6199
45683	27925	1.03	25	3569	10796.79	1.01	.6219
43608	26435	1.02	24	3593	10832.48	1.00	.6240
47706	28573	1.01	26	3619	10868.41	.99	.6260
32649	19412	1.00	18	3637	10904.60	.98	.6281
54544	32111	.99	30	3667	10940.97	.97	.6302
54854	31864	.98	30	3697	10977.64	.96	.6323
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
265980	38016	.24	146	7769	14801.98	.23	.8526
284133	38828	.23	155	7924	14879.67	.22	.8571
326246	42676	.22	179	8103	14958.91	.21	.8617
328726	41082	.21	179	8282	15039.94	.21	.8663
302686	36100	.20	166	8448	15122.76	.20	.8711
331543	37399	.19	183	8631	15207.24	.19	.8760
373757	40091	.18	205	8836	15293.55	.18	.8809
386279	39082	.17	212	9048	15381.91	.17	.8860
425014	40408	.16	233	9281	15472.39	.16	.8912
388311	34605	.15	213	9494	15565.20	.15	.8966
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
740516	30809	.07	405	12115	16408.07	.07	.9451
845263	30105	.06	463	12578	16529.22	.06	.9521
836277	24972	.05	460	13038	16655.00	.05	.9594
887957	21086	.04	487	13525	16785.38	.04	.9669
1061074	18927	.03	582	14107	16920.63	.03	.9747
1054247	12388	.02	576	14683	17061.70	.02	.9828
916101	5675	.01	500	15183	17208.53	.01	.9913
3215866	340	.00	1767	16950	17360.36	.00	1.0000
30926469	18844603		16950				

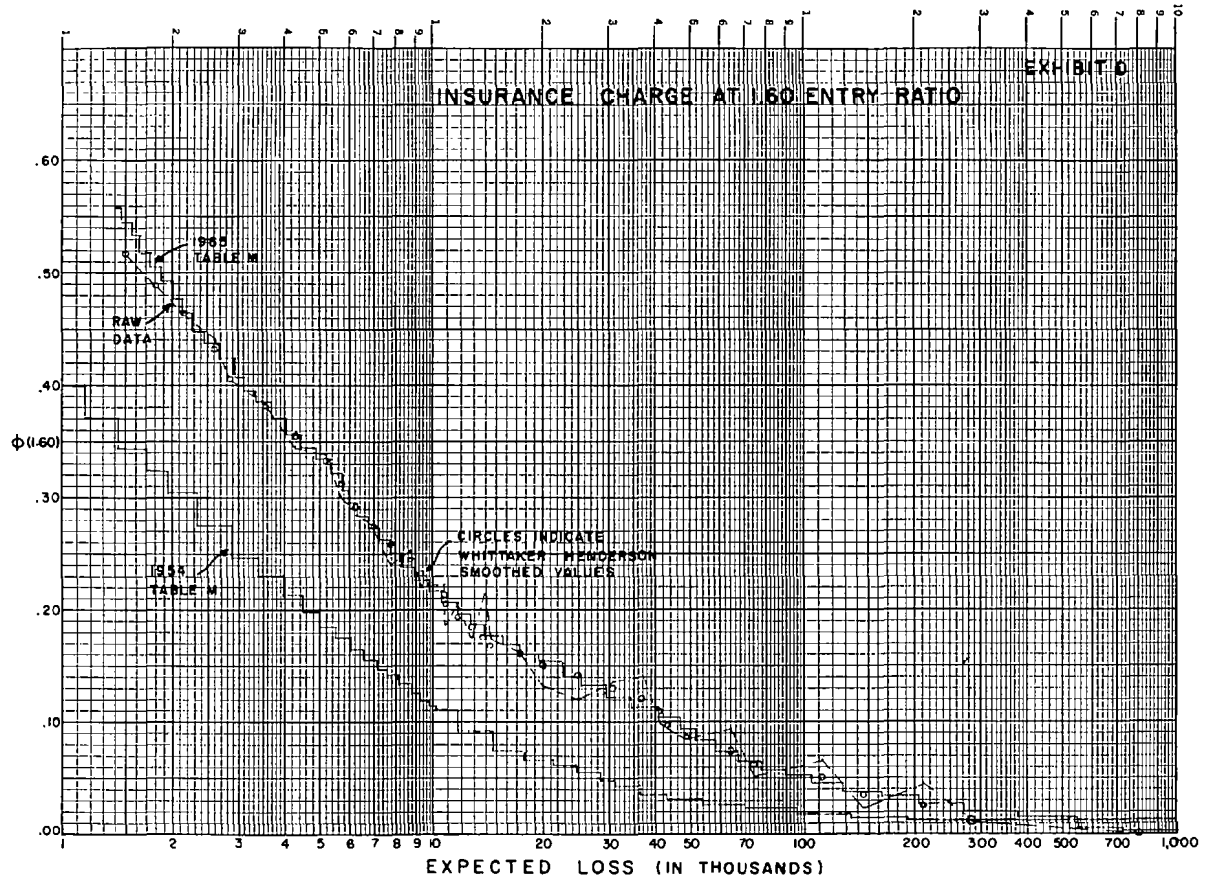


EXHIBIT E

EXAMPLE OF TEST RUN FOR SELECTING EQUATION

Old Premium Group 10

r	Raw	Smoothed	Fitted Equation (1)		Fitted Equation (2)	
	Data	Data	Value	% Error*	Value	% Error*
.20	.8270	.8249	.8248	-.2%	.8247	-.6%
.40	.6873	.6901	.6908	.8	.6911	1.1
.60	.5896	.5877	.5867	-.5	.5873	-.2
.80	.4976	.5051	.5042	-.3	.5045	-.2
1.00	.4320	.4377	.4371	-.1	.4368	-.2
1.20	.3752	.3799	.3812	.3	.3805	.1
1.40	.3302	.3325	.3339	.4	.3330	.1
1.60	.2898	.2924	.2934	.4	.2926	.1
1.80	.2579	.2576	.2586	.4	.2582	.2
2.00	.2297	.2293	.2285	-.3	.2288	-.2
2.20	.2067	.2039	.2028	-.5	.2035	-.2
2.40	.1848	.1823	.1808	-.8	.1819	-.2
2.60	.1667	.1631	.1621	-.6	.1633	.1
2.80	.1504	.1470	.1463	-.5	.1471	.1
3.00	.1373	.1329	.1328	-.1	.1331	.2
3.20	.1247	.1197	.1211	1.2	.1208	.9
3.40	.1141	.1094	.1109	1.3	.1100	.6
3.60	.1041	.1034	.1016	-1.7	.1005	-2.9
3.80	.0958	.0915	.0931	1.7	.0919	.5
4.00	.0882	.0837	.0850	1.6	.0843	.7
4.20	.0817	.0771	.0774	.4	.0774	.4
4.40	.0764	.0712	.0704	-1.1	.0713	.1
4.60	.0707	.0660	.0644	-2.4	.0658	-.3
4.80	.0659	.0610	.0600	-1.7	.0609	-.1
5.00	.0612	.0566	.0581	2.7	.0567	.1

* Percentage errors are calculated as the error in the insurance charge for values of $r \geq 1.00$, and as percentage errors in the saving for values of $r < 1.00$, where the saving equals the charge plus the entry ratio minus 1.00.

FINAL CURVE FITTING RUN

Example of Two Premium Groups

r	O.I.D. PREMIUM GROUP 10				O.L.D. PREMIUM GROUP 11			
	Raw Data	Smoothed Data	Fitted Equation (3) Value	% Error*	Raw Data	Smoothed Data	Fitted Equation (3) Value	% Error*
.20	.8270	.8249	.8248	-.5%	.8207	.8224	.8223	-.6%
.40	.6873	.6901	.6910	1.0	.6836	.6836	.6845	1.1
.60	.5836	.5877	.5872	-.3	.5810	.5779	.5772	-.4
.80	.4976	.5051	.5044	-.2	.4941	.4921	.4916	-.2
1.00	.4320	.4377	.4368	-.2	.4240	.4229	.4220	-.2
1.20	.3752	.3799	.3806	.2	.3655	.3637	.3643	.2
1.40	.3302	.3325	.3331	.2	.3179	.3154	.3160	.2
1.60	.2898	.2924	.2927	.1	.2763	.2747	.2752	.2
1.80	.2579	.2576	.2583	.3	.2419	.2401	.2406	.2
2.00	.2297	.2293	.2288	-.2	.2134	.2119	.2114	-.3
2.20	.2067	.2039	.2035	-.2	.1906	.1870	.1865	-.3
2.40	.1848	.1823	.1818	-.3	.1697	.1659	.1654	-.3
2.60	.1667	.1631	.1631	-.0	.1511	.1476	.1475	-.1
2.80	.1504	.1470	.1470	-.0	.1351	.1321	.1321	-.0
3.00	.1373	.1329	.1331	.1	.1222	.1188	.1189	.1
3.20	.1247	.1197	.1208	.9	.1110	.1058	.1074	1.6
3.40	.1141	.1094	.1101	.6	.1012	.0968	.0974	.7
3.60	.1041	.1034	.1005	-2.8	.0923	.0918	.0887	-3.4
3.80	.0958	.0915	.0920	.5	.0845	.0804	.0809	.6
4.00	.0882	.0837	.0843	.8	.0771	.0733	.0739	.8
4.20	.0817	.0771	.0774	.4	.0705	.0674	.0677	.5
4.40	.0764	.0712	.0712	.1	.0647	.0621	.0622	.1
4.60	.0707	.0660	.0657	-.4	.0597	.0575	.0573	-.4
4.80	.0659	.0610	.0609	-.2	.0556	.0532	.0530	-.3
5.00	.0612	.0566	.0567	.2	.0517	.0493	.0494	.2

COEFFICIENTS

b ₂	.3388717	.44044010
b ₃	-.16700810	-.22436769
b ₄	.15762138	.20788815
b ₅	-.042426274	-.058737127
b ₆	.0011206270	.0032547560
b ₇	.0011646563	.0011371418
b ₈	-.00012901262	-.00014028810

* Percentage errors are calculated as the error in the insurance charge for values of $r \geq 1.00$, and as percentage errors in the saving for values of $r < 1.00$, where the saving equals the charge plus the entry ratio minus 1.00.

EXHIBIT G

INTERPOLATION TO DETERMINE COEFFICIENTS
FOR NEW PREMIUM GROUPS

The coefficients for Premium Group .43 were found by interpolation between the values (shown on Exhibit F) for Old Premium Groups 10 and 11. Harmonic interpolation was used as follows:

	<u>Old Premium Group 10</u>	<u>Old Premium Group 11</u>	<u>New Premium Group .43</u>
$\phi(1.00)$.4368310	.4220344	.430
$1/\phi(1.00)*$	2.2892150	2.3694751	2.3255814
Interpolating Proportion	.5468932	.4531068	
b_2	.33887170	.44044010	.3848930
b_3	-.16700810	-.22436769	-.1929981
b_4	.15762138	.20788815	.1803976
b_5	-.042426274	-.058737127	-.0498168
b_6	.0011206270	.0032547560	.0020876
b_7	.0011646563	.0011371418	.0011522
b_8	-.00012901262	-.00014028810	-.0001341

* Determined as $2.0 (i.e., 1 + r)$ plus the sum of the coefficients b_2 through b_8 for the old premium group columns.

MATRIX OF COEFFICIENTS

Premium Group	<u>b₂</u>	<u>b₃</u>	<u>b₄</u>	<u>b₅</u>	<u>b₆</u>	<u>b₇</u>	<u>b₈</u>	Maximum Usable r
.64	-2.4031906	4.9946221	-5.0352823	2.6534276	-0.7476412	0.1066010	-0.0060366	1.74
.63	-2.2041460	4.5552874	-4.6030607	2.4345125	-0.6381637	0.0984249	-0.0055528	1.74
.62	-2.0171974	4.1473717	-4.1993122	2.2285352	-0.6319215	0.0905791	-0.0051517	1.74
.61	-1.8310552	3.7423805	-3.7973659	2.0229685	-0.5756088	0.0827421	-0.0047170	1.74
.60	-1.6570528	3.3682250	-3.4231837	1.8300687	-0.5223754	0.0752858	-0.0043009	1.74
.59	-1.4832753	2.9961886	-3.0500584	1.6371411	-0.4689893	0.0677905	-0.0038819	3.78
.58	-1.3210878	2.6538260	-2.7035217	1.4562566	-0.4185056	0.0606505	-0.0034801	3.78
.57	-1.1594855	2.3148689	-2.3592562	1.2758234	-0.3679712	0.0534825	-0.0030759	3.78
.56	-1.0269758	2.0494542	-2.0828563	1.1267103	-0.3251918	0.0472951	-0.0027214	3.78
.55	-0.8896476	1.7743882	-1.7964056	0.9721749	-0.2808567	0.0408827	-0.0023540	3.78
.54	-0.7549972	1.5141474	-1.5311006	0.8317601	-0.2411180	0.0351902	-0.0020300	3.78
.53	-0.6159643	1.2463364	-1.2586406	0.6878316	-0.2004408	0.0293691	-0.0016989	5.00
.52	-0.4911550	1.0191766	-1.0283315	0.5666651	-0.1663803	0.0245276	-0.0014256	5.00
.51	-0.3651269	0.7926783	-0.7988756	0.4460618	-0.1325207	0.0197223	-0.0011549	5.00
.50	-0.2502708	0.5970000	-0.5988208	0.3401507	-0.1025966	0.0154505	-0.0009130	5.00
.49	-0.1339706	0.4013138	-0.3983194	0.2338214	-0.0725095	0.0111496	-0.0006690	5.00
.48	-0.0330425	0.2511240	-0.2463034	0.1543364	-0.0502972	0.0080085	-0.0004926	5.00
.47	0.0688215	0.1034968	-0.0973794	0.0767784	-0.0286993	0.0049637	-0.0003220	5.00
.46	0.1540083	0.0062860	-0.0037170	0.0308102	-0.0165737	0.0033337	-0.0002345	5.00
.45	0.2380330	-0.0828200	0.0811232	-0.0097579	-0.0061649	0.0019723	-0.0001634	5.00
.44	0.3140541	-0.1462885	0.1387943	-0.0343862	-0.0006725	0.0013635	-0.0001375	5.00
.43	0.3848930	-0.1929981	0.1803976	-0.0498168	0.0020876	0.0011522	-0.0001341	5.00
.42	0.4520075	-0.2252639	0.2077841	-0.0572532	0.0025717	0.0012534	-0.0001473	5.00
.41	0.5105354	-0.2297982	0.2072574	-0.0497446	-0.0008845	0.0018417	-0.0001828	5.00
.40	0.5578036	-0.1952433	0.1636623	-0.0183729	-0.0108692	0.0032798	-0.0002603	5.00
.39	0.6000823	-0.1393633	0.0963798	0.0270763	-0.0250111	0.0053088	-0.0003703	5.00
.38	0.6373082	-0.0619243	0.0065091	0.0850057	-0.0426207	0.0078055	-0.0005046	5.00
.37	0.6699527	0.0362174	-0.1052290	0.1558492	-0.0643880	0.0109835	-0.0006831	5.00
.36	0.6988077	0.1534788	-0.2401878	0.2446196	-0.0935920	0.0156192	-0.0009676	5.00
.35	0.7225221	0.2925464	-0.3982271	0.3492669	-0.1290635	0.0214321	-0.0013340	5.00

MATRIX OF COEFFICIENTS

Premium Group	<u>b₂</u>	<u>b₃</u>	<u>b₄</u>	<u>b₅</u>	<u>b₆</u>	<u>b₇</u>	<u>b₈</u>	Maximum Usable r
.34	0.7505061	0.4063878	-0.5060154	0.4200164	-0.1536101	0.0254783	-0.0015867	5.00
.33	0.7820579	0.5135850	-0.5980510	0.4804099	-0.1748942	0.0290010	-0.0018055	5.00
.32	0.8143930	0.6003855	-0.6482440	0.5203436	-0.1920786	0.0322248	-0.0020243	5.00
.31	0.8512591	0.6585917	-0.6488186	0.5364166	-0.2044858	0.0350829	-0.0022395	5.00
.30	0.8943599	0.6784657	-0.5853285	0.5210722	-0.2100882	0.0372850	-0.0024331	5.00
.29	0.9448819	0.6556264	-0.4557299	0.4792926	-0.2129519	0.0398448	-0.0026880	5.00
.28	1.0031130	0.5638149	-0.1995380	0.3610718	-0.1933591	0.0390678	-0.0027418	5.00
.27	1.0682040	0.4044798	0.1691869	0.1904578	-0.1632730	0.0374271	-0.0027790	5.00
.26	1.1457775	0.1507550	0.6772723	-0.0367783	-0.1230709	0.0349965	-0.0027982	5.00
.25	1.2431208	-0.2287912	1.3574460	-0.3263899	-0.0751633	0.0326514	-0.0028739	5.00
.24	1.3523489	-0.6733113	2.0671535	-0.5402002	-0.0764166	0.0408023	-0.0037099	5.00
.23	1.4805248	-1.2073324	2.8012469	-0.6334556	-0.1521826	0.0647098	-0.0056850	5.00
.22	1.6498167	-1.9405544	3.6845402	-0.6430773	-0.3019227	0.1055812	-0.0089291	5.00
.21	1.8352317	-2.7436078	4.6519573	-0.6536154	-0.4659240	0.1503452	-0.0124822	5.00

TABLE M

FORMULAS FOR CHARGES BEYOND THE
RANGE WHERE EQUATION (3) APPLIES

General Formula: $\phi(r) = c - mr$

<u>PG.</u>	<u>Applicable to Values of r in the range</u>	<u>c</u>	<u>m</u>	<u>PG.</u>	<u>Applicable to Values of r in the range</u>	<u>c</u>	<u>m</u>
.64	1.75 - 3.78	.7190	.10	.44	5.01 - 6.15	.3077	.05
.64	3.79 - 10.59	.5297	.05	.43	5.01 - 6.05	.3027	.05
.63	1.75 - 3.78	.7040	.10	.42	5.01 - 5.96	.2982	.05
.63	3.79 - 10.29	.5147	.05	.41	5.01 - 5.88	.2942	.05
.62	1.75 - 3.78	.6900	.10	.40	5.01 - 5.79	.2897	.05
.62	3.79 - 10.01	.5007	.05	.39	5.01 - 5.70	.2862	.05
.61	1.75 - 3.78	.6750	.10	.38	5.01 - 5.65	.2827	.05
.61	3.79 - 9.71	.4857	.05	.37	5.01 - 5.59	.2797	.05
.60	1.75 - 3.78	.6600	.10	.36	5.01 - 5.54	.2772	.05
.60	3.79 - 9.41	.4707	.05	.35	5.01 - 5.50	.2752	.05
.59	3.79 - 9.12	.4562	.05	.34	5.01 - 5.47	.2737	.05
.58	3.79 - 8.77	.4387	.05	.33	5.01 - 5.44	.2722	.05
.57	3.79 - 8.46	.4232	.05	.32	5.01 - 5.41	.2707	.05
.56	3.79 - 8.19	.4097	.05	.31	5.01 - 5.38	.2692	.05
.55	3.79 - 7.95	.3977	.05	.30	5.01 - 5.35	.2677	.05
.54	3.79 - 7.68	.3842	.05	.29	5.01 - 5.33	.2667	.05
.53	5.01 - 7.45	.3727	.05	.28	5.01 - 5.30	.2652	.05
.52	5.01 - 7.25	.3627	.05	.27	5.01 - 5.27	.2637	.05
.51	5.01 - 7.09	.3547	.05	.26	5.01 - 5.25	.2627	.05
.50	5.01 - 6.91	.3457	.05	.25	5.01 - 5.21	.2607	.05
.49	5.01 - 6.76	.3382	.05	.24	5.01 - 5.19	.2597	.05
.48	5.01 - 6.62	.3312	.05	.23	5.01 - 5.15	.2577	.05
.47	5.01 - 6.50	.3252	.05	.22	5.01 - 5.13	.2567	.05
.46	5.01 - 6.38	.3192	.05	.21	5.01 - 5.11	.2557	.05
.45	5.01 - 6.26	.3132	.05				

TABLE OF CHARGES AND SAVINGS - TABLE M

Two Special Premium Groups

<u>r</u>	Premium Group .85 Expected Losses \$596 <u>Insurance Charges</u>	Premium Group .99 Expected Losses \$3 <u>Insurance Charges</u>
.01	.994	For $r \leq 100.00$, $\phi(r) = 1.000 - .01r$ (Retain all decimal places)
.02	.988	
.03	.982	
.04	.976	
.05	.970	
.06	.967	
.07	.963	
.08	.960	
.09	.956	
For $.10 \leq r \leq .35$	$\phi(r) = .9675 - .15r$	
For $.36 \leq r \leq 5.00$	$\phi(r) = .9500 - .10r$	
For $5.01 \leq r \leq 14.00$	$\phi(r) = .7000 - .05r$	

TABLE M

25

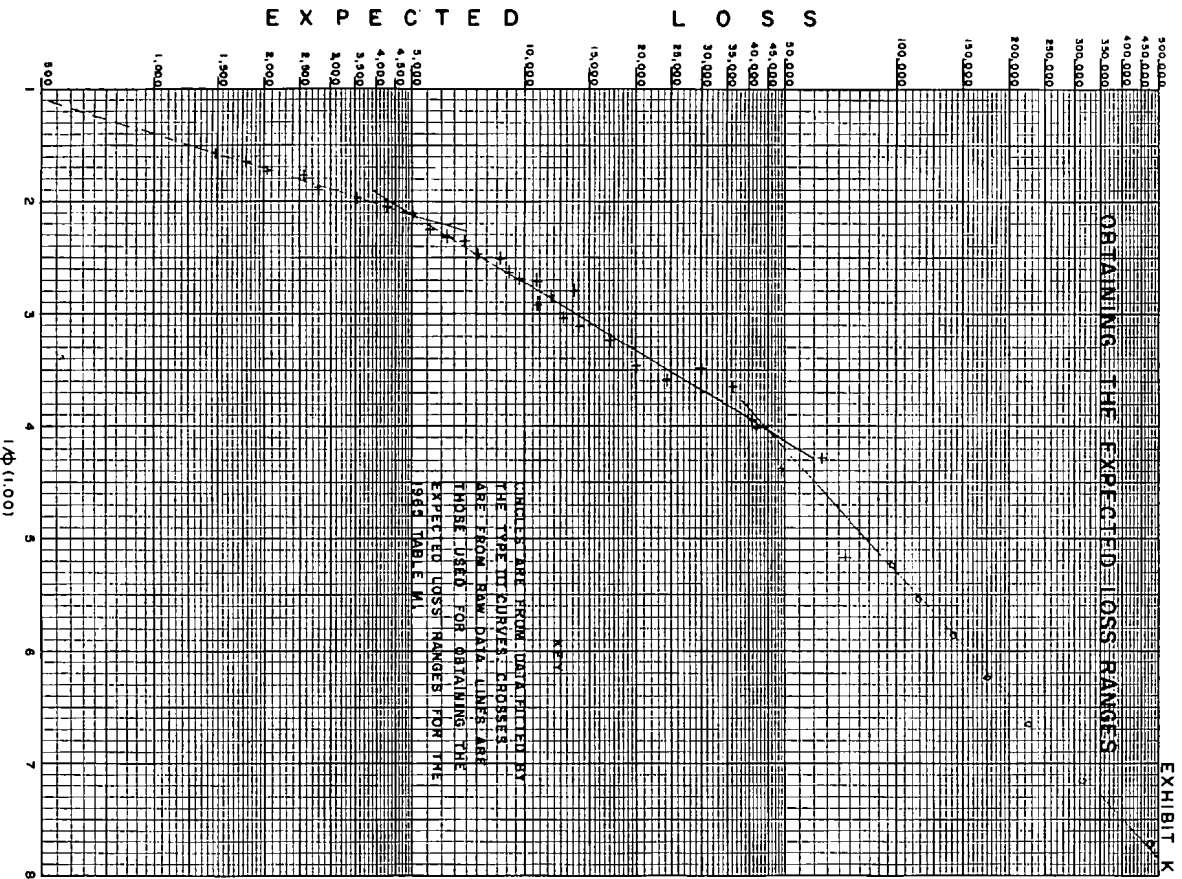


EXHIBIT L

EXPECTED LOSS RANGES

<u>Premium Group</u>	<u>Range</u>	<u>Premium Group</u>	<u>Range</u>
.64	\$1,400 - \$1,479	.32	\$15,500 - \$17,299
.63	1,480 - 1,569	.31	17,300 - 19,499
.62	1,570 - 1,659	.30	19,500 - 22,099
.61	1,660 - 1,769	.29	22,100 - 25,299
.60	1,770 - 1,879	.28	25,300 - 29,299
.59	1,880 - 2,009	.27	29,300 - 34,299
.58	2,010 - 2,149	.26	34,300 - 40,599
.57	2,150 - 2,299	.25	40,600 - 46,299
.56	2,300 - 2,469	.24	46,300 - 51,599
.55	2,470 - 2,659	.23	51,600 - 58,099
.54	2,660 - 2,869	.22	58,100 - 66,199
.53	2,870 - 3,109	.21	66,200 - 76,299
.52	3,110 - 3,379	.20	76,300 - 89,299
.51	3,380 - 3,689	.19	89,300 - 105,999
.50	3,690 - 4,029	.18	106,000 - 128,999
.49	4,030 - 4,429	.17	129,000 - 160,999
.48	4,430 - 4,879	.16	161,000 - 204,999
.47	4,880 - 5,379	.15	205,000 - 270,999
.46	5,380 - 5,679	.14	271,000 - 373,999
.45	5,680 - 6,009	.13	374,000 - 542,999
.44	6,010 - 6,369	.12	543,000 - 724,999
.43	6,370 - 6,779	.11	725,000 - 916,999
.42	6,780 - 7,229	.10	917,000 - 1,119,999
.41	7,230 - 7,739	.09	1,120,000 - 1,399,999
.40	7,740 - 8,309	.08	1,400,000 - 1,799,999
.39	8,310 - 8,959	.07	1,800,000 - 2,389,999
.38	8,960 - 9,689	.06	2,390,000 - 3,339,999
.37	9,690 - 10,499	.05	3,340,000 - 4,989,999
.36	10,500 - 11,499	.04	4,990,000 - 8,249,999
.35	11,500 - 12,599	.03	8,250,000 - 16,199,999
.34	12,600 - 13,899	.02	16,200,000 - 44,899,999
.33	13,900 - 15,499	.01	44,900,000 & Over

RETROSPECTIVE PLAN D - TABLE OF CHARGES AND SAVINGS - TABLE M
PREMIUM GROUP

ENTRY RATIO	64	63	62	61	60	59	58	57	56	55	54
0.01	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*	0.990 0.000*
0.02	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*	0.981 0.001*
0.03	0.973 0.003*	0.973 0.003*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.972 0.002*	0.971 0.001*
0.04	0.965 0.005*	0.965 0.005*	0.964 0.004*	0.964 0.004*	0.964 0.004*	0.964 0.004*	0.963 0.003*	0.963 0.003*	0.963 0.003*	0.963 0.003*	0.963 0.003*
0.05	0.957 0.007*	0.957 0.007*	0.957 0.007*	0.956 0.006*	0.956 0.006*	0.955 0.005*	0.955 0.005*	0.955 0.005*	0.954 0.004*	0.954 0.004*	0.954 0.004*
0.06	0.950 0.010*	0.950 0.010*	0.949 0.009*	0.949 0.009*	0.948 0.008*	0.948 0.008*	0.947 0.007*	0.947 0.007*	0.946 0.006*	0.946 0.006*	0.946 0.006*
0.07	0.944 0.014*	0.943 0.013*	0.942 0.012*	0.941 0.011*	0.941 0.011*	0.940 0.010*	0.940 0.010*	0.940 0.009*	0.939 0.008*	0.938 0.008*	0.937 0.007*
0.08	0.937 0.017*	0.936 0.016*	0.935 0.015*	0.935 0.015*	0.934 0.014*	0.933 0.013*	0.932 0.012*	0.931 0.011*	0.931 0.011*	0.930 0.010*	0.929 0.009*
0.09	0.931 0.021*	0.930 0.020*	0.929 0.019*	0.928 0.018*	0.927 0.017*	0.926 0.016*	0.925 0.015*	0.924 0.014*	0.923 0.013*	0.923 0.013*	0.922 0.012*
0.10	0.926 0.026*	0.924 0.024*	0.923 0.023*	0.922 0.022*	0.920 0.020*	0.919 0.019*	0.918 0.018*	0.917 0.017*	0.916 0.016*	0.915 0.015*	0.914 0.014*
0.11	0.920 0.030*	0.918 0.028*	0.917 0.027*	0.915 0.025*	0.914 0.024*	0.913 0.023*	0.911 0.021*	0.910 0.020*	0.909 0.019*	0.908 0.018*	0.907 0.017*
0.12	0.915 0.035*	0.913 0.033*	0.911 0.031*	0.910 0.030*	0.908 0.028*	0.906 0.026*	0.905 0.025*	0.903 0.023*	0.902 0.022*	0.901 0.021*	0.900 0.020*
0.13	0.910 0.040*	0.908 0.038*	0.906 0.036*	0.904 0.034*	0.902 0.032*	0.900 0.030*	0.899 0.029*	0.897 0.027*	0.896 0.026*	0.894 0.024*	0.893 0.023*
0.14	0.905 0.045*	0.903 0.043*	0.901 0.041*	0.898 0.038*	0.896 0.036*	0.894 0.034*	0.893 0.033*	0.891 0.031*	0.889 0.029*	0.887 0.027*	0.886 0.026*
0.15	0.901 0.051*	0.898 0.048*	0.896 0.046*	0.893 0.043*	0.891 0.041*	0.889 0.039*	0.887 0.037*	0.884 0.034*	0.883 0.033*	0.881 0.031*	0.879 0.029*
0.16	0.896	0.893	0.891	0.888	0.885	0.883	0.881	0.878	0.877	0.875	0.873

TABLE M

TABLE M - CUMULATIVE DISTRIBUTION AND UNDERLYING FREQUENCY DISTRIBUTION

ENTRY RATIO	64		63		62		61		60		59		58		57	
	D1	D2	D1	D2	D1	D2	D1	D2	D1	D2	D1	D2	D1	D2	D1	D2
0.01	9.		9.		9.		9.		9.		9.		9.		9.	
0.02	8.	1.	8.	1.	9.	0.	9.	0.	9.	0.	9.	0.	9.	0.	9.	0.
0.03	8.	0.	8.	0.	8.	1.	8.	1.	8.	1.	8.	1.	9.	0.	9.	0.
0.04	8.	0.	8.	0.	7.	1.	8.	0.	8.	0.	9.	-1.	8.	1.	8.	1.
0.05	7.	1.	7.	1.	8.	-1.	7.	1.	8.	0.	7.	2.	8.	0.	8.	0.
0.06	6.	1.	7.	0.	7.	1.	8.	-1.	7.	1.	8.	-1.	7.	1.	8.	0.
0.07	7.	-1.	7.	0.	7.	0.	6.	2.	7.	0.	7.	1.	8.	-1.	8.	0.
0.08	6.	1.	6.	1.	6.	1.	7.	-1.	7.	0.	7.	0.	7.	1.	7.	1.
0.09	5.	1.	6.	0.	6.	0.	6.	1.	7.	0.	7.	0.	7.	0.	7.	0.
0.10	6.	-1.	6.	0.	6.	0.	7.	-1.	6.	1.	6.	1.	7.	0.	7.	0.
0.11	5.	1.	5.	1.	6.	0.	5.	2.	6.	0.	7.	-1.	6.	1.	7.	0.
0.12	5.	0.	5.	0.	5.	1.	6.	-1.	6.	0.	6.	1.	6.	0.	6.	1.
0.13	5.	0.	5.	0.	5.	0.	6.	0.	6.	0.	6.	0.	6.	0.	6.	0.
0.14	4.	1.	5.	0.	5.	0.	5.	1.	5.	1.	5.	1.	6.	0.	7.	-1.
0.15	5.	-1.	5.	0.	5.	0.	5.	0.	6.	-1.	6.	-1.	6.	0.	6.	1.
0.16	4.	1.	4.	1.	5.	0.	5.	0.	5.	1.	5.	1.	6.	0.	5.	1.
0.17	4.	0.	5.	-1.	5.	0.	5.	0.	5.	0.	6.	-1.	6.	0.	6.	-1.
0.18	4.	0.	4.	1.	4.	1.	5.	0.	5.	0.	5.	1.	5.	1.	6.	0.
0.19	4.	0.	4.	0.	5.	-1.	4.	1.	5.	0.	5.	0.	5.	0.	5.	1.
0.20	4.	0.	4.	0.	4.	1.	5.	-1.	5.	0.	5.	0.	5.	0.	6.	-1.
0.21	4.	0.	4.	0.	4.	0.	4.	1.	4.	1.	5.	0.	6.	-1.	5.	1.
0.22	3.	1.	4.	0.	4.	0.	5.	-1.	5.	-1.	5.	0.	5.	1.	5.	0.
0.23	4.	-1.	4.	0.	4.	0.	4.	1.	4.	1.	5.	0.	5.	0.	6.	-1.
0.24	3.	1.	4.	0.	4.	0.	4.	0.	5.	-1.	4.	1.	4.	1.	5.	1.
0.25	4.	-1.	3.	1.	4.	0.	4.	0.	4.	1.	5.	-1.	5.	-1.	5.	0.
0.26	3.	1.	4.	-1.	4.	0.	5.	-1.	4.	0.	4.	1.	5.	0.	4.	1.
0.27	4.	-1.	4.	0.	4.	0.	4.	1.	5.	-1.	5.	-1.	5.	0.	5.	-1.
0.28	3.	1.	3.	1.	4.	0.	4.	0.	4.	1.	4.	1.	4.	1.	5.	0.
0.29	3.	0.	4.	-1.	4.	0.	4.	0.	4.	0.	5.	-1.	5.	-1.	5.	0.
0.30	3.	0.	3.	1.	3.	1.	3.	1.	4.	0.	4.	1.	4.	1.	4.	1.
0.31	4.	-1.	4.	-1.	4.	-1.	4.	-1.	4.	0.	4.	0.	5.	-1.	5.	-1.

TABLE M

TABLE M - CROSS COLUMN FIRST DIFFERENCES
PREMIUM GROUP

ENTRY RATIO	64	63	62	61	60	59	58	57	56	55	54	53	52	51	50
0.01	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.02	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	1.	0.
0.03	0.	1.	0.	0.	0.	0.	0.	0.	0.	1.	0.	0.	0.	0.	0.
0.04	0.	1.	0.	0.	0.	1.	0.	0.	0.	0.	1.	0.	0.	0.	0.
0.05	0.	0.	1.	0.	1.	0.	0.	1.	0.	0.	0.	1.	0.	0.	0.
0.06	0.	1.	0.	1.	0.	1.	0.	1.	0.	0.	1.	0.	1.	0.	0.
0.07	1.	1.	1.	0.	1.	0.	1.	1.	0.	1.	0.	1.	0.	1.	0.
0.08	1.	1.	0.	1.	1.	1.	1.	0.	1.	1.	0.	1.	0.	1.	1.
0.09	1.	1.	1.	1.	1.	1.	1.	1.	0.	1.	1.	1.	1.	0.	1.
0.10	2.	1.	1.	2.	1.	1.	1.	1.	1.	1.	1.	1.	0.	1.	1.
0.11	2.	1.	2.	1.	1.	2.	1.	1.	1.	1.	1.	1.	1.	1.	1.
0.12	2.	2.	1.	2.	2.	1.	2.	1.	1.	1.	2.	1.	1.	1.	1.
0.13	2.	2.	2.	2.	2.	1.	2.	1.	2.	1.	2.	1.	1.	2.	1.
0.14	2.	2.	3.	2.	2.	1.	2.	2.	2.	1.	2.	1.	2.	1.	2.
0.15	3.	2.	3.	2.	2.	2.	3.	1.	2.	2.	2.	1.	2.	1.	2.
0.16	3.	2.	3.	3.	2.	2.	3.	1.	2.	2.	2.	2.	2.	2.	1.
0.17	3.	3.	3.	3.	2.	3.	2.	3.	2.	2.	2.	2.	2.	2.	2.
0.18	4.	3.	3.	3.	3.	3.	2.	2.	3.	2.	2.	3.	2.	2.	2.
0.19	4.	3.	4.	3.	3.	3.	3.	2.	3.	2.	3.	2.	2.	3.	2.
0.20	4.	4.	3.	4.	3.	3.	3.	3.	3.	2.	3.	3.	2.	2.	3.
0.21	4.	4.	4.	4.	3.	3.	4.	3.	2.	3.	3.	3.	3.	2.	3.
0.22	4.	4.	4.	4.	4.	4.	3.	3.	3.	3.	3.	3.	3.	3.	2.
0.23	5.	4.	5.	4.	4.	4.	3.	4.	3.	3.	3.	3.	3.	3.	3.
0.24	5.	4.	5.	4.	5.	4.	4.	3.	3.	4.	3.	3.	3.	3.	3.
0.25	6.	4.	5.	5.	4.	4.	5.	3.	4.	3.	4.	3.	3.	4.	3.
0.26	5.	5.	5.	5.	5.	4.	5.	3.	4.	4.	3.	4.	3.	4.	3.
0.27	6.	5.	6.	4.	5.	5.	4.	4.	4.	4.	4.	4.	3.	4.	3.
0.28	6.	5.	6.	5.	5.	5.	4.	4.	4.	4.	5.	3.	4.	4.	3.
0.29	6.	6.	6.	5.	5.	5.	5.	4.	4.	5.	4.	4.	4.	4.	3.
0.30	7.	6.	6.	5.	6.	5.	5.	4.	4.	5.	4.	4.	5.	4.	3.
0.31	7.	6.	6.	6.	6.	5.	5.	5.	4.	5.	5.	4.	4.	4.	4.

FORTRAN CODING FORM

EXHIBIT P

Program DETERMINATION OF TABLE M CHARGE		Punching Instructions						Page	of
Graphic							Card Form #	Identification	
Programmer	JOAN FEATHERER	Date	1965	Punch				73 80	

C FOR COMMENT									
STATEMENT NUMBER	FORTRAN STATEMENT								
1	10 ASSUME THE 4,4 x 7 MATRIX OF COEFFICIENTS IS LOADED AS E(I, M) WHERE I								
2	1 IS THE PREMIUM GROUP AND M IDENTIFIES COEFFICIENTS B(2) THRU B(8).								
3	20 ASSUME AN OBTAINED VALUE OF R, FALLS WITHIN THE ACCEPTABLE RANGE								
4	1 FOR APPLICATION OF EQUATION (3) FOR THE OBTAINED VALUE OF I. (NOTE								
5	2 -- IF THE PAIR (R, I), IS NOT IN THE ACCEPTABLE RANGE, EITHER USE A								
6	3 DIFFERENT EQUATION THRU ADDITIONAL PROGRAMMING OR DETERMINE THE								
7	4 CHARGE MANUALLY.)								
8	30 COMPUTE THE CHARGE (CH) AND THE CHARGE, ROUNDED TO THREE DECIMAL								
9	1 PLACES (CHRND), AS FOLLOWS IN IO								
10	10 CH = 1. / ((1. + R + R * R * (E(I, 1) + R * (E(I, 2) + R * (E(I, 3) + R * (E(I, 4) + R * (E(I, 5) + R * (E(I, 6) + R * (E(I, 7))))))))								
11	CHRND = (AINT((CH + .0005) * 1000.)) * 1. E-3								
12	IF (R + CHRND - 1.) 20, 21, 21								
13	20 CHRND = 1. - R								
14	C 21 WRITE OUTPUT STATEMENT TO FIT INDIVIDUAL REQUIREMENTS								

TABLE M

* A standard card form, IBM electro 888157, is available for punching source statements from this form.

AREAS OF FORMULA APPLICATION

Equation (3) = $1/(1+r + b_2r^2 + b_3r^3 + b_4r^4 + b_5r^5 + b_6r^6 + b_7r^7 + b_8r^8)$

Equation (4) = $c - mr$

Premium Group Number:	.64-.60	.59-.54	.53-.21	.20-.12	.11-.01
Expected Loss Range(\$):	1,400-1,879	1,880-2,869	2,870-76,299	76,300-724,999	725,000 and Over

Critical Entry Ratios

.00	(3)	(3)	(3)	A linear interpolation between the charges at Premium Groups .21 and .11 using the premium group number as the argument.	No formula but the charges and savings are symmetrical about $r = 1.00$.
⋮	↓	↓	↓		
1.74	(4)	↓	↓		
1.75	↓	↓	↓		
⋮	↓	↓	↓		
3.78	(4)	(4)	↓		
3.79	↓	↓	↓		
⋮	↓	↓	↓		
5.00	↓	↓	↓		
5.01	↓	↓	(4)		
⋮	↓	↓	↓		
∞	↓	↓	↓		

∞ indicates that the formula works properly out to some limiting value and beyond this the charge is zero.

Equation (3) constants are found in Exhibit H.

Equation (4) constants are found in Exhibit I.

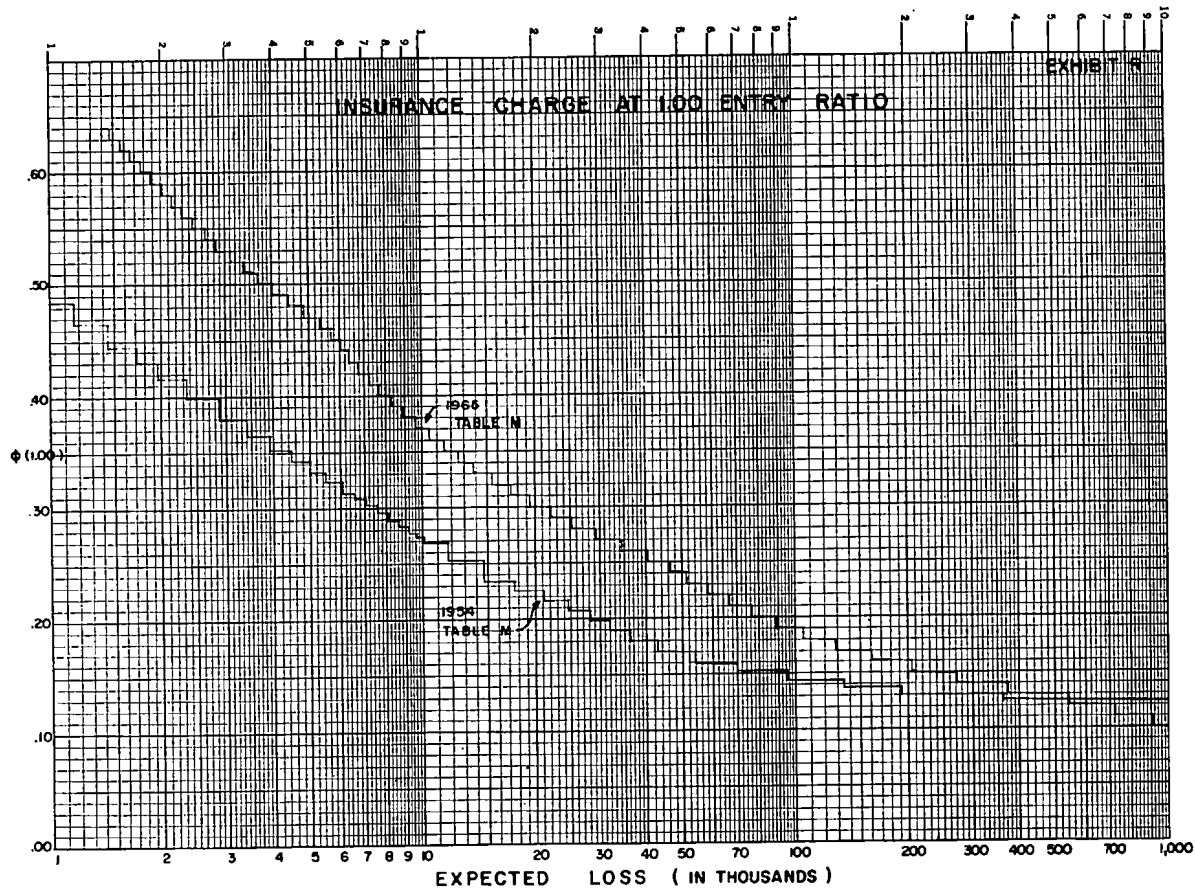
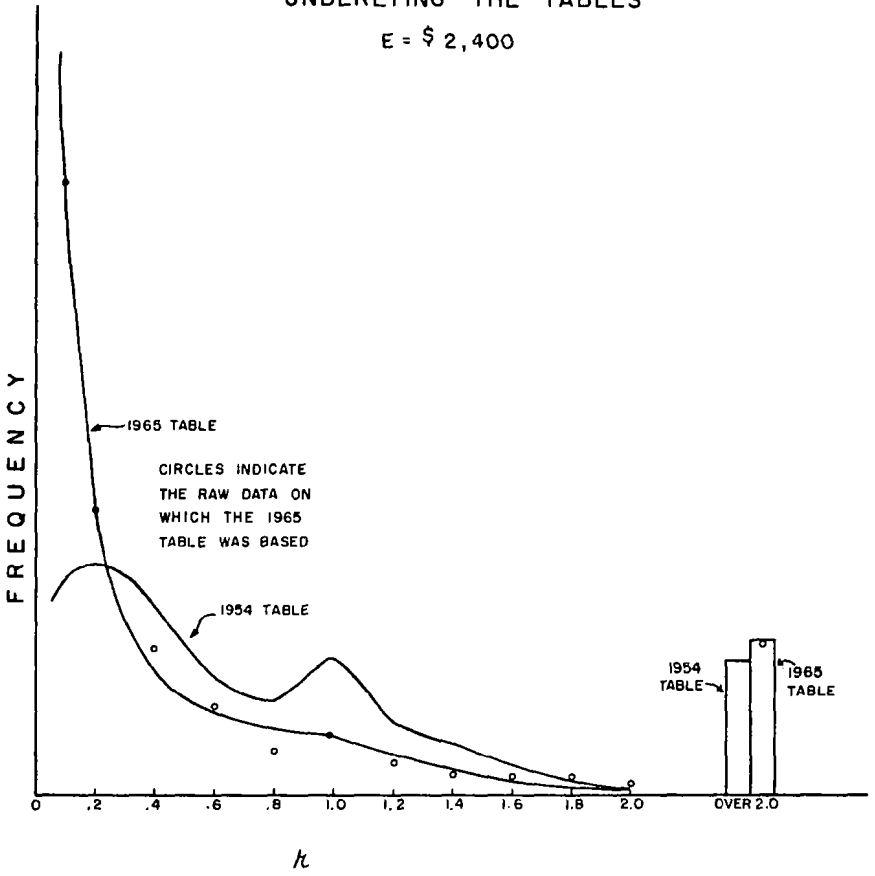


TABLE M

IMPLIED RISK DISTRIBUTIONS
UNDERLYING THE TABLES

$E = \$ 2,400$



APPENDIX A

The formula for the variance of the expected losses can be written¹⁰:

$$\sigma_{E_s}^2 = E^2(1 + V_c^2)/m_f \dots\dots\dots (A1)$$

where V_c is the coefficient of variation of the underlying claim distribution and m_f is the mean claim frequency. In terms of the coefficient of variation of the expected losses we'd write:

$$V_E^2 = (1 + V_c^2)/m_f \dots\dots\dots (A2)$$

Now, if between year 0 and year 5, we can assume the coefficient of variation of the claims remains constant then

$$\frac{V_{E_5}^2}{V_{E_0}^2} = \frac{m_{f_5}}{m_{f_0}} \dots\dots\dots (A3)$$

$$V_{E_5}^2 = \frac{m_{f_5}}{m_{f_0}} V_{E_0}^2 \dots\dots\dots (A4)$$

By general reasoning, we can equate $V_{E_5}^2$ to $V_{E_0}^2$ only if m_{f_5} equals m_{f_0} ; but if the average cost per claim increases by a factor of 1.35, we can equate m_{f_5} to m_{f_0} only if $E_5 = 1.35 E_0$. This straightforward logic completes the argument.

It is interesting to note that if the average cost per claim is brought about by an increase primarily in the higher cost claims (more long term medical, longer life expectancy for injured workers, etc.), a factor larger than 1.35 would be merited due to the increased coefficient of variation of the claims.

APPENDIX B

It is almost obvious that if we change the index from R to r in the manner described, the insurance charge, $\phi(r)$, is the same as the previously calculated $\phi(R)$. By definition we have for some specific R :

$$\begin{aligned} \phi(R) &= (\text{Sum } 2 \text{ at } R) / (\text{Sum } 2 \text{ at } 0) \\ &= \sum_{i=\omega}^n (R_i - R) / \sum_{i=\omega}^N R_i \end{aligned}$$

where R_i is the value of the ratio starting with the highest value (R_ω) going down to the case which has a value equal to R (R_n), and in the

¹⁰ Bailey, R. A. and Simon, L. J., "Two Studies in Automobile Insurance Ratemaking *PCAS XLVII*, p. 18, equation (14) with slight changes in notation to fit our present definitions.

denominator continuing to the smallest value (R_N). It is apparent from the above equation that if each of the values of R is multiplied by some constant, the value of ϕ remains unchanged except that it will now be shown as $\phi(kR)$, that is, $\phi(r)$.

Care must be exercised in interpolating when the value of r that is being sought is not found in the adjusted ratio column. In Premium Group 28 we were faced with the problem of obtaining the insurance charge for an entry ratio of .20 when the tab run showed the following information:

R	r	ϕ	<i>Number of Risks</i>	<i>Sum 1</i>	<i>Sum 2</i>
.20	.21	.7876	3	324	245.06
.18	.19	.8084	1	325	251.54

To obtain r , we had to multiply R by 1.07347 and Sum 2 at entry of θ was 311.14. We now observe that if we were seeking a value for $r = .20$ this would be equivalent to looking for a value of $R = .18631$ (that is, $.20/1.07347$). Between $R = .20$ and $R = .18631$ there would not be any risks and the value of Sum 1 would still be 324. The number of points of excess over $.18631$ would increase by $324 \times (.01369)$ since there would be 324 risks which would each contribute this amount in addition to what had already been accumulated as excess points beyond the value of $R = .20$. Hence, Sum 2 at $R = .18631$ would be 249.50. Thus, $\phi = 249.50/311.14 = .8019$. Another way to obtain this result a little more quickly is to do a straight linear interpolation on ϕ using R as the scale. In this case we would have:

$$(.00631/.02000) \times (.8084 - .7876) = .0066$$

Therefore, the value of ϕ would be $.8084 - .0066 = .8018$. Except for the fact that the values of ϕ are initially rounded off to four decimal places, these two methods produce the same result.

APPENDIX C

Mathematical Testing of Table M Functions

A set of working definitions was established first. Let r be defined as the ratio of actual losses (a random variable) to expected losses for a given risk. We will deal with samples of risks which have the same expected losses (or nearly so). In general $0 \leq r \leq \omega$, where ω is the maximum value that can be assumed and theoretically approaches infinity.

Define $f(r)$ as the density function as pictured in Sketch 1. Notice that $f(0)$ is not necessarily zero since in practice, many risks have no losses.

However, we are certain that $f(\omega) = 0$ (C1)

and that $f(r) \geq 0$ for $0 \leq r \leq \omega$(C2)

Further, since the area under the curve must equal 1, $\int_0^\omega f(r)dr = 1$.
.....(C3)

As discussed in the main text, distributions underlying Table M must have a mean of 1, hence, $\int_0^\omega rf(r)dr = 1$(C4)

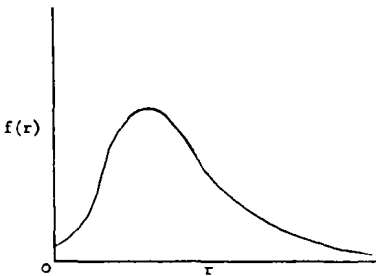
The distribution function, as shown in Sketch 2, will be given by

$$F(r) = \int_0^r f(r)dr \quad \text{.....(C5)}$$

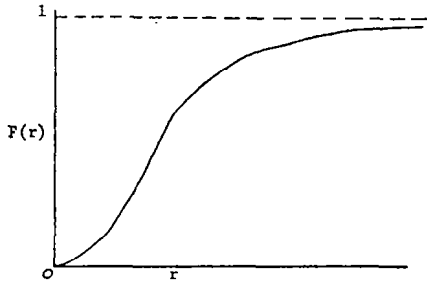
It follows that $F(0) = 0$ (C6)

and that $F(r) \geq 0$ for $0 \leq r \leq \omega$(C7)

Statement (C3) may also be restated as $F(\omega) = 1$(C8)



Sketch 1



Sketch 2

Define a special function, Sketch 3, which we will find very useful:

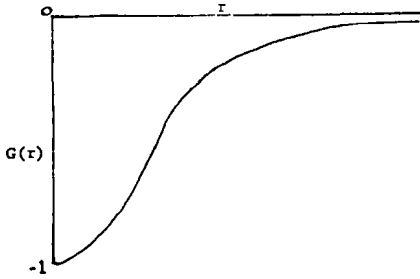
$G(r) = F(r) - 1$(C9)

Hence, $G(0) = -1$ (C10)

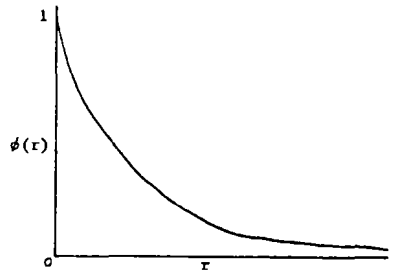
and $G(\omega) = 0$(C11)

Let the Table M function be called $\phi(r)$. See Sketch 4. The basic definition of the Table M function (variously called "the charge," "the column of charges" and "the excess pure premium ratio") is:

$$\phi(r_0) = \int_{r_0}^\omega (r - r_0)f(r)dr \Bigg| \int_0^\omega rf(r)dr \quad \text{.....(C12)}$$



Sketch 3



Sketch 4

which will ordinarily be denoted $\phi(r)$ for simplicity of notation. Note that by (C4), the denominator in (C12) equals unity.²⁰

It can be shown through straightforward proofs²¹ starting with (C12) that

$$\phi(r_0) = 1 - r_0 + \int_0^{r_0} \int_0^r f(r) dr dr \dots\dots\dots (C13)$$

$$= 1 - r_0 + \int_0^{r_0} F(r) dr \dots\dots\dots (C14)$$

$$= 1 + \int_0^{r_0} G(r) dr \dots\dots\dots (C15)$$

$$\phi(0) = 1 \dots\dots\dots (C16)$$

$$\phi(\omega) = 0 \dots\dots\dots (C17)$$

$$\phi'(r_0) = G(r_0) \dots\dots\dots (C18)$$

$$\phi''(r_0) = f(r_0) \dots\dots\dots (C19)$$

Further, it can be shown that the area under the $f(r)$ curve can be given by

$$[G(r)]_0^\omega \dots\dots\dots (C20)$$

and the mean of the $f(r)$ curve can be given by $[rG(r) - \phi(r)]_0^\omega$ (C21)

Another basic feature of Table M is the so-called "Saving" defined by:

$$\psi(r_0) = \left[r_0 \int_0^{r_0} f(r) dr - \int_0^{r_0} rf(r) dr \right] \Bigg| \int_0^\omega rf(r) dr \dots\dots\dots (C22)$$

Note that by (C4), the denominator in (C22) equals unity.

²⁰ The inclusion of the denominator in both (C12) and (C22) was suggested by Mr. Hewitt in his review. To assist the reader, we have agreed to incorporate this distinct improvement in the Appendix.

²¹ For a similar development see Carlson, Thomas O., "Observations on Casualty Insurance Rate-Making Theory in the United States," *PCAS; Vol. LI*; p. 294.

It can be shown, from (C22), that

$$\psi(r_o) = \phi(r_o) + r_o - I \quad \dots\dots\dots (C23)$$

Since $\psi(r_o)$ can never be negative, we have

$$\phi(r_o) \geq I - r_o \quad \dots\dots\dots (C24)$$

A slightly more difficult proof is the formula for the second moment of the underlying distribution when only the function $\phi(r)$ is known. The equation

$$\mu_2' = \int_0^\omega r^2 f(r) dr$$

can be shown to be

$$\mu_2' = \omega^2 \phi'(\omega) - 2\omega\phi(\omega) + 2 \int_0^\omega \phi(r) dr$$

by the use of Roberts' ingenious reduction formula.²² For later use we

observe that to keep μ_2' finite we must have $\lim_{\omega \rightarrow \infty} \omega\phi(\omega)$ finite(C25)

These equations and relationships were used in testing various mathematical equations for acceptability as expressions for $\phi(r)$. For example, a general polynomial was considered of the form

$$\phi(r) = (b_0 + b_1 r + b_2 r^2 + \dots + b_n r^n)^c \quad \dots\dots\dots (C26)$$

From (C16) we prove $b_0 = I$. From (C25) it is apparent that the highest ordered term of (C25), i.e., ω^{cn+1} must remain finite as ω approaches infinity which will only be true if $cn + 1 < 0$. Since n is positive, c must be negative. By (C10) we also show $b_1 = -I/c$. We decided to try an equation with $c = -I$ and this resulted in Equation (3) in the main text. A similar type of analysis led to Equation (1).

As another example, consider

$$\phi(r) = k^{-r^n}$$

where k and n are constants to be determined from the data. As long as k and n are positive real numbers, (C16) and (C17) are satisfied. It is also noted that $\phi(I) = k^{-I}$ and since $\phi(I)$ must lie between 0 and 1, k must be greater than 1. By (C18) we have

$$G(r) = \phi(r) [-n(1n k)r^{n-1}]$$

Now $1n k$ must be positive (since $k > 1$), and $\lim_{r \rightarrow \infty} G(r) = 0$ which is

²² Roberts, Lewis H., "Graduation of Excess Ratio Distributions between Method of Moments," *PCAS; Vol. XLIV*; p. 52.

in agreement with (C11). However, $G(0) = 0$ and not -1 as it should by (C10). Hence, this equation is inappropriate for expressing $\phi(r)$.

This general approach was used on each equation considered to test to see if it was at all usable and to determine the value of certain constants which were required by the *a priori* conditions of Table M. I am critical of the results found by the use of orthogonal polynomials for Premium Groups .64 through .21 because each value of b_s is negative. It can be seen that in order to meet condition (C17), we must have $b_s > 0$ unless we do not allow ω to go to infinity. This forced us to abandon Equation (3) above $r = 5.00$. The press of time did not permit further experimentation.

Many possible curves were discarded because the constants were difficult to determine, and the entire project necessitated a family of about 50 different curves. Two special cases of (C26) were considered carefully because of their simplicity.

$$\phi(r) = \left(1 + \frac{r}{d}\right)^{-d} \dots\dots\dots (C27)$$

One method of establishing d for a given premium group was to get an equation for the variance of (C27) and solve it for d to find $d = (\mu_2 + 1) / (\mu_2 - 1)$. Then by calculating the variance of the raw data, we could determine d for testing purposes. Although a few premium groups were tested this way, we turned to another method. This procedure was to obtain $\phi(1)$ from the raw data and solve (C27) for d . A very good fit was obtained for Old Premium Group 11, but as we tried groups toward the extremes, the system broke down. In fact,

$$\lim_{d \rightarrow \infty} \left(1 + \frac{r}{d}\right)^{-d} = \frac{1}{e} = .368$$

By (C25) we can show that $d > 1$ is necessary, so that (C27) could not be used except when,

$$.368 \leq \phi(1) < .500$$

Because this was such a limited range, we modified the formula to get more flexibility:

$$\phi(r) = \left(1 + \frac{r}{d} + b_2 r^2\right)^{-d} \dots\dots\dots (C28)$$

Due to (C2) at $r = 0$, it can be shown that the maximum value for b_2 is $(d+1)/2d^2$ and by (C25) the minimum value for b_2 is $1/2$. Work was progressing on the programming of this equation when we discovered that Equation (3) was producing satisfactory results.

APPENDIX D

After preliminary investigations, the first four moments were calculated for Old Premium Groups 28 through 36 having as our objective the use of Pearson curves to graduate the underlying risk distributions. These moments were calculated using desk calculators so the procedure was shortened as follows:

1. Premium Groups 28, 29, 30 and 31 used 100 values selected from the raw data at the percentile points .995 through .005.
2. Premium Group 32 used one-half of the available 178 cases.²³
3. Premium Groups 33, 34, 35 and 36 used all of the values available (two very exceptional risks were excluded) which gave us 123, 74, 43 and 21 cases respectively.

The results are as follows:

Old Premium Group	Expected Losses (E)	E^{-1}	Mean	V^2	β_1	β_2
28	42,700	23.42×10^{-6}	1.0050	.451167	3.613	7.914
29	49,700	20.12×10^{-6}	.9957	.375437	2.515	6.809
30	64,400	15.53×10^{-6}	.9907	.430462	5.259	11.771
31	73,900	13.53×10^{-6}	1.0000	.271702	2.101	7.020
32	111,000	9.01×10^{-6}	.9751	.277317	6.428	13.857
33	146,000	6.85×10^{-6}	1.0004	.173485	.920	4.719
34	211,000	4.74×10^{-6}	.9607	.145665	1.496	5.140
35	283,000	3.53×10^{-6}	1.0212	.107014	.696	3.002
36	699,000	1.43×10^{-6}	1.0000	.090419	.008	2.123

Roberts²⁴ formulas were used for smoothing the moments, except that equal weights were applied to each group. This was done because the total actual losses in each group was considerable and we were willing to assume that the true expected losses for the group was equal to the mean expected losses in the sample. Thus, we wrote $b_3 = \sum E^{-1} \beta_1 / \sum (E^{-1})^2$. Based on the data above this resulted in $b_3 = .2069 \times 10^6$; hence $\hat{\beta}_1 = .2069 \times 10^6 E^{-1}$. Similarly, we evaluated $b_4 = \sum E^{-1} (\beta_2 - 3) / \sum (E^{-1})^2 = .3247 \times 10^6$; hence $\hat{\beta}_2 = .3247 \times 10^6 E^{-1} + 3$. The test criterion for Pearson curves is $k = 2\hat{\beta}_2 - 3\hat{\beta}_1 - 6$. In this case we have $k = .287 \times 10^6 E^{-1}$.

²³ Every other case was taken in such a way that the smallest value case was retained in the sample and the largest value case was excluded. Subsequent investigation indicated that the largest case was quite exceptional and, therefore, the decision was a sound one.

²⁴ Roberts, Lewis H., "Graduation of Excess Ratio Distributions by the Method of Moments," *PCAS; Vol. XLIV*; p. 51-52.

Over the range of E^{-1} shown above, k goes from .67 to .04. It was concluded that, although a Pearson Type VI curve was indicated, k was close enough to zero to use the Type III curve.

Rather than using Roberts' equation for b_3 as indicated above (and assuming thus that b_4 would be forced to conform to the Type III curve) it was decided that the basic equations for b_3 and b_4 would be solved subject to the additional restriction that the resultant solution would produce a Type III curve.

Let

$$f(E^{-1}) = \sum(\beta_1 - b_3 E^{-1})^2$$

$$g(E^{-1}) = \sum(\beta_2 - b_4 E^{-1} - 3)^2$$

$$h(E^{-1}) = 2\hat{\beta}_2 - 3\hat{\beta}_1 - 6 = 2b_4 E^{-1} - 3b_3 E^{-1}$$

The conditions are that $f(E^{-1})$ is to be a minimum, $g(E^{-1})$ is to be a minimum, but this is subject to the restriction that $h(E^{-1}) = 0$. Introducing λ , the undetermined Lagrangian multiplier, these conditions will be met if:

$$\frac{\delta f}{\delta b_3} + \lambda \frac{\delta h}{\delta b_3} = 0$$

$$\frac{\delta g}{\delta b_4} + \lambda \frac{\delta h}{\delta b_4} = 0$$

and
$$2b_4 - 3b_3 = 0$$

Taking the partial derivatives and solving for b_3 , we get

$$b_3 = \frac{4\sum\beta_1 E^{-1} + 6\sum(\beta_2 - 3)E^{-1}}{13\sum(E^{-1})^2}$$

The raw data gives us

$$b_3 = .2135 \times 10^6; \text{ hence } \hat{\beta}_1 = .2135 \times 10^6 E^{-1} \dots\dots\dots (D-1)$$

Again, following Roberts' approach for the variance, we solve

$$Na + b_2 \sum E^{-1} = \sum V^2$$

$$a \sum E^{-1} + b_2 \sum (E^{-1})^2 = \sum E^{-1} V^2$$

Solving, we get $a = .072609$ and $b_2 = .017021 \times 10^6$.

Since the new curves will have their means at 1.00, $\hat{V}^2 = \hat{\sigma}^2$ and we have

$$\hat{\sigma}^2 = .072609 + .017021 \times 10^6 E^{-1} \dots\dots\dots (D-2)$$

A characteristic of the Incomplete Gamma Function (the Type III curve)

is that
$$p = \frac{4}{\beta_1} - 1 \dots\dots\dots (D-3)$$

Combining (D1), (D2) and (D3), we can write

$$\hat{\sigma}^2 = .072609 + .31889/(p + 1) \dots\dots\dots (D-4)$$

$$\text{and } E = 53400(p + 1) \dots\dots\dots (D-5)$$

We were now in a position to calculate the Table M charges using the Pearson Type III curve as the underlying distribution of risks. For a trial value of p as used in the Tables of the Incomplete Gamma Function²⁵ determine $\hat{\sigma}^2$ from Equation (D4) and thence determine u_r from:

$$u_r = \sqrt{p + 1} + (r + .005 - 1)/\hat{\sigma}$$

Enter the Tables of the Incomplete Gamma Function with u_r and p and read out $I(u_r, p)$. Calculate $1 - I(u_r, p)$. Get the accumulation upward times .01 and this will be $\phi(r)$. By trial and error, an appropriate value of p was determined so that $\phi(1.00)$ was obtained at the desired value. The values are as follows:

Premium Group:	.19	.18	.17	.16	.15	.14	.13	.12
p :	.8245	1.1687	1.6453	2.3	3.2	4.9	8.0	16.2

The first three groups represent averages of pairs of values found by two trial values of p as follows:

- Premium Group .19 is a 268 : 87 weighting of $p = .8$ and $p = .9$
- Premium Group .18 is a 77 : 169 weighting of $p = 1.1$ and $p = 1.2$
- Premium Group .17 is a 47 : 39 weighting of $p = 1.6$ and $p = 1.7$

A sample of the worksheet used for Premium Group .13 is as follows:

r	u_r	$I(u_r, 8.0)$	$1 - I$	$\phi(r)$
.
.
.
1.58	4.78	.947	.053	.01079
1.59	4.81	.950	.050	.01026
1.60	4.84	.952	.048	.00976
1.61	4.87	.954	.046	.00928
.
.
.

An interesting corollary to the work in this area of the Table is the relationship between Table M and the χ^2 distribution. From the values

²⁵ Pearson, Karl, *Tables of the Incomplete Gamma Function*, Cambridge University Press, 1957.

of p above, determine γ , the degrees of freedom used in tabling the χ^2 function by

$$\gamma = 2(p + 1)$$

Also from the above, the value of $\hat{\sigma}$ is available for a given premium group.

Then

$$r_{1-\alpha} = \frac{\hat{\sigma}}{\sqrt{2\gamma}} (\chi_{\gamma, \alpha}^2 - \gamma) + 1.00$$

where $r_{1-\alpha}$ is the value of r at the α percentile and $\chi_{\gamma, \alpha}^2$ is the tabled value for χ^2 with γ degrees of freedom and at a probability value of α . For example, on Premium Group .13,

$$\begin{aligned} \gamma &= 2(8.0 + 1) = 18 \\ \hat{\sigma} &= [.072609 + .31889/(8.0 + 1)]^{1/2} \\ &= .32870 \\ r_{1-\alpha} &= .05478 (\chi_{18, \alpha}^2 - 18) + 1.00 \end{aligned}$$

If we inquire as to the value of r for which 5% of the risks exceed it, we find $\chi_{18, .05}^2 = 28.8693$ and thus $r_{.95} = 1.595$. Notice how this agrees with the above example. Naturally, the converse question can be asked – for a given value of r , what is the probability that it will be attained or exceeded? In our example, a value of $r = 1.20$ will be exceeded by about 25% of the risks because

$$\chi_{18, \alpha}^2 = \frac{1.20 - 1.00}{.05478} + 18 = 21.6501$$

and a reference to the χ^2 table shows this value has a probability of approximately .250.

APPENDIX E

As the risk size increases, we expect the shape of the risk distribution to approach the normal curve and we expect the variance to approach zero ultimately. It was decided to use Old Premium Group 36, which had $E = 699,000$ and $V^2 = .090419$, as a starting point. Assuming (and this can only be true as an approximation) that the only variance remaining at these large risk sizes is the variance in the claim distribution, and the number of cases (N) is fixed, the ratio of the two standard deviations of

the expected losses will be the same as the ratio of the two standard errors of the mean, that is

$$\frac{\hat{\sigma}_E}{\hat{\sigma}_{699000}} = \frac{S/\sqrt{N_E}}{S/\sqrt{N_{699000}}}$$

where S is the standard deviation of the claim distribution. (Because N is so large, I have used N rather than $N-1$ in the above formula.)

This may be simplified and rearranged as follows:

$$\begin{aligned} \hat{\sigma}_E &= \hat{\sigma}_{699000} \sqrt{N_{699000} (Average\ claim\ cost)/N_E (Average\ claim\ cost)} \\ &= \sqrt{(.090419)(699000)/E} = \sqrt{63203/E} \end{aligned}$$

Thus, as the size of the risk increases, the standard deviation decreases and approaches zero as a limit. Solving for E we have

$$E = 63200/\hat{\sigma}^2 \quad \dots\dots\dots(E1)$$

which was used to set the expected loss ranges for the premium groups using the normal curve.

For a trial value of $\hat{\sigma}$, determine z_r by

$$z_r = (r + .005 - 1)/\hat{\sigma}$$

Enter a table of the normal curve and read out the probability integral value representing the area under the curve to the right of z_r . Get the accumulation (to 3 decimal places) upward times .01 which equals $\phi(r)$. Values of $\hat{\sigma}^2$ which produced the desired values at $\phi(1.00)$ were:

Premium Group:	.11	.10	.09	.08	.07	.06	.05	.04	.03	.02	.01
$\hat{\sigma}^2$:	.275	.250	.225	.200	.175	.150	.125	.100	.075	.050	.025

A sample of the worksheet used for Premium Group .11 is as follows:

r	z_r	Q_r	$\phi(r)$
.	.	.	.
.	.	.	.
.	.	.	.
1.32	1.1818	.119	.01660
1.33	1.2182	.112	.01541
1.34	1.2545	.105	.01429
1.35	1.2909	.098	.01324
.	.	.	.
.	.	.	.
.	.	.	.