REPORT

OBSERVATIONS ON CASUALTY INSURANCE RATE-MAKING THEORY IN THE UNITED STATES

by

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Writing in 1941, Mr. Arthur L. Bailey, probably the most profound contributor to casualty actuarial theory the United States has produced, observed as follows with regard to his entry into the actuarial profession:

> "The first year or so I spent proving to myself that all of the fancy actuarial procedures of the casualty business were mathematically unsound. They are unsound—if one is bound to accept the restrictions implied or specifically placed on the development of the classical statistical methods. Later on I realised that the hard-shelled underwriters were recognising certain facts of life neglected by the statistical theorists. Now I am convinced that casualty insurance statisticians are a step ahead of those in most fields. This is because there has been a truly epistemological review of the basic conditions of which their statistics are measurements."[1]

In elaboration of these remarks, Mr. Bailey refers to recognition of heterogeneity of populations as opposed to the homogeneity assumed in classical statistics, the imposition of restrictive conditions on groups of estimates considered in the aggregate rather than upon each individual estimate, with consequent reduction in the variances involved, and the development of "credibility" formulas to produce consistent weightings of statistical experience with prior knowledge in the form, for example, of existing rate schedules. While statistical theory has in more recent years been giving attention to the first two of these departures from the classical approach, the third area seems to have escaped investigation outside of actuarial circles.

It appears appropriate in this jubilee year of the Casualty Actuarial Society to review actuarial developments in this country, with particular attention to theory. Certain of these developments will be found to complement the growth of the science elsewhere for reasons which are peculiar to our insurance system. An attempt is made to avoid duplication of material presented in any previous Congress. Presentation will be as nearly as possible topical with illustrative material, where needed, drawn from the automobile lines which contribute more than $6,000,000,000 of the $15,000,000,000 of annual premiums on non-life insurance in our country.

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Editor's Note: This paper was reprinted from the *Transactions* of the XVIIth International Congress of Actuaries held in Great Britain. Mr. Carlson died shortly after returning from Europe where he presented this, his last, technical paper.

Errata are listed following the Bibliography.

Theory in the United States has usually followed and, happily almost invariably, supported practice, echoing the experience voiced by W. Perks of England at the 1951 Scheveningen Congress:

"I want to stress that the modern developments of statistical theory are less important for actuarial work than for providing a sounder theoretical basis for the traditional actuarial methods. More and more we are finding that our methods are justified by the more precise modern analysis."[2]

IMPACT OF RATE REGULATION

The development of actuarial science in the United States cannot be fully understood without appreciation of the impact of statutory regulation, necessity having nurtured invention.

Rate-making by companies in concert dates from the latter years of the nineteenth century on a basis of voluntary informal association. Co-operation in rate-making became formalised in a rating organisation only in 1910, with the advent of the first workmen's compensation legislation, later repealed, and it was the puzzle of establishing rates for a new compulsory coverage, rather than the attendant regulation, that provided the real incentive. Four years later, in 1914, the Casualty Actuarial Society was established as a direct outgrowth of the committee discussions on the theory of rate-making as applied to workmen's compensation insurance. One of the three papers presented at the historic inaugural meeting remains to this day the classic introduction to the problem of the credibility to be accorded to statistical experience in the determination of classification rates.

The rapid spread of workmen's compensation legislation and the existence of a rating organisation embracing all private and most public carriers of workmen's compensation insurance focused actuarial attention upon this line and made it the testing-ground of theory and practices that were later extended to other lines.

The 1920's and 1930's witnessed the limited *de facto* extension of rate regulation to other casualty lines, and the concentration of rate-making activities in large measure in the hands of rating organisations, with non-affiliated companies establishing rates for the most part by reference. This is in general still the pattern as respects rate determination.

In 1944 the epochal decision of the United States Supreme Court in the South-Eastern Underwriters Association case reversed 75 years of court rulings, and established jurisdiction over the business in the United States Congress. That body returned regulation to the individual states, however, with the threat of federal intervention to fill any gaps, and specifically exempted the insurance industry from the application of the various federal anti-trust laws, except for acts of boycott, intimidation, or coercion, thus permitting companies to continue to pool their statistics and to make rates in concert.

The aftermath was the passage of rate regulatory laws in all 52 jurisdictions within the United States, none of them alike in all particulars [3], creating a flood of problems, but still almost universally considered preferable to federal control.

In 1951 the scope of the Society's considerations and membership was extended to include fire insurance in recognition of the passage of laws permitting a company to write both fire and casualty coverages.

D. B. Martin of Canada remarked at the Brussels Congress that North American actuaries have of necessity stressed practical rather than theoretical aspects. The science has been hammered out upon the anvil of practical experience, with attention restricted largely to day-by-day exigencies. In virtually all the 52 jurisdictions, rates before they can be used must be approved by state officials, some of whom are elected by popular vote and many of whom have had no insurance background; in some states public hearings are held.

The actuary must in consequence be able to explain all formulas on the level of comprehension of the general public, and thus has need for articulateness and common sense in at least as great measure as for mathematical facility and comprehension.

RATE-MAKING—GENERAL

Ratemaking embraces (1) the determination of "manual" or class rates and (2) the development of rating plans for the modification of the class rates for those individual risks large enough so that the deviation of the risk's experience from the class experience, as summarised in the class rate, is significant.

Establishment of class rates is based fundamentally upon an annual review of averages, with any further analysis of distributions restricted to special studies on the fringes of such reviews.

Statistical reports include by class within territory for each coverage the premiums, the amount of losses, the number of claims and, for lines with a third-party interest, the exposures or number of units of the manual rate base, e.g. number of cars for automobile liability. From these are calculated the averages:

loss ratio	=	(amount of losses)/(premiums)
average claim cost	=	(amount of losses)/(number of claims)
claim frequency	=	(number of claims)/(exposures)
pure premium	=	(amount of losses)/(exposures)
	=	(average claim cost) × (claim frequency)

Premiums at present rates are calculated as the summed products of exposures times current basic manual rates (or, in lines like burglary and glass, for which exposures are not reported, as the collected premiums

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adjusted to reflect the current rate level) and the loss ratio on present level calculated, using the losses within the limits represented by the basic rates.

Supplementary data, such as average claim costs by state for current trend review, are collected, but since this paper is to deal essentially with theory, and such practical devices were fully presented, together with classification structure, statistical bases and other details in the illustrative automobile line, by Mr. Matthews and Mr. McGuinness at the 1960 Brussels Congress [4], [5], I shall refrain from going beyond these descriptive fundamentals.

RATE-MAKING—OBJECTIVES

The primary objective of ratemaking in the United States has always been the establishment of rates that will be proper for the period during which they will be effective. By "proper" we mean adequate to meet the losses and expenses which may normally be expected and to provide what the statutes prescribe as a "reasonable margin for underwriting profit and contingencies", which is almost universally established currently for casualty lines as 5% of the premium (too often not realised). L. Wilhelmsen of Norway said at the 1960 Brussels Congress: "... adjustment to the level of rates following changes in the level of claims ... are part of the rating system in Canada and in U.S.A. In other countries forecasting of the period during which rates should be valid seems not to take place." [6] American actuaries have never thought otherwise than in terms of rates proper for their effective term. An important secondary objective is the establishment in the rate-making procedures of a "best" compromise between the principles of (1) stability in rate-level and (2) responsiveness to current experience indications while maintaining consistency in the interpretation of the statistics. These two objectives of meeting anticipated costs and yet compromising reasonably and consistently between stability and responsiveness are interwoven throughout the entire development of actuarial science, and actuarial procedures can only be understood in the light of these objectives.

At the outset, before statistical information is available, rates have to be based upon underwriting judgement. When statistical information subsequently becomes available, two alternatives are open: either to consider that body of statistical data as the only true information available, i.e. classical theory, or to consider such data together with the information that is embodied in an already established rate structure. The latter, and theoretically unorthodox, approach has been followed in the United States for more than fifty years and has its own niche in actuarial theory known as "credibility theory", which will be examined in detail. As successive revisions develop, it is obvious that the rate structure partakes more and more of the statistical contributions and may become entirely founded upon them, but each revision takes the existing schedule of rates as a springboard with current statistical data providing the impetus for the leap into the unknown territory of loss-and-expense predictions.

CREDIBILITY THEORY

The simple formula for such a weighting with credibility factors is

$$M = ZA + (1 - Z)B \tag{1}$$

$$= B + Z(A - B) \tag{1a}$$

where A is a statistical observation while B is the corresponding value in the reference base with which A is being compared. B is commonly a broader population average, whether this concept be utilised directly as when B, for example, is a countrywide class pure premium with which a local class pure premium A is being compared, or whether the concept is used by implication as when B is a present rate representing a population average that is broader than A either in respect to space (as in individual risk rating where it is in effect a class rate with which the risk's indicated rate is compared) or in respect to time (as in manual rate-making where it embodies the results of previous years of experience with which the current indication is compared). Z is the credibility, the mathematical measure of the credence attached to the statistical observation A. In one branch of the theory Z is determined from specific probability assumptions as to the deviation of the sample average A from the true average and is called a "limited fluctuation" credibility. In another branch Z is determined from parameters of the variables being weighted and is called a "greatest accuracy" credibility.

The credibility principle thus appears in various guises and disguises. The rationale behind its use is several-fold:

(1) One has to use point-estimates rather than interval-estimates, regardless of the standard deviations of the estimates, and a credibility-weighting procedure connecting with some norm or frame of reference introduces in effect reflection of the comparative validity of the point-estimate and at the same time introduces a marked stabilising influence into the procedure—one of our objectives. This stabilising is accomplished not only by restricting the impact of fluctuations in the statistical data so weighted, but also by introducing on a weighted basis the frame of reference, whether this be the existing schedule of rates (e.g. in determining statewide rate level) or averages drawn from a more comprehensive population (e.g. in determining relationships between territories or between classes).

(2) In establishing relationships between classes within a territory, or between territories within a state, the credibility-weighting of the individual indications with the average of all indications produces a series of indices which can be applied to any determined average rate-level change to produce equitable rate-levels for the individual classes or territories, as the case may be, recognising the comparative validity of the point-estimates involved.

(3) Complete consistency in the interpretation of statistical data is obtained by the credibility procedures; they replace the vagaries of underwriting judgement, sound though such may be. This is of the utmost importance in the processing of rates for thousands of classification and territory divisions annually, which must have the stamp of approval from a regulatory official before they can be used; the avoidance of unfair discrimination in rate determination is a universal statutory requirement

(4) There is a five-pronged interrogatory to which every rate-maker must subject himself: what questions may be raised about the revision by (*a*) actuaries on technical aspects, (*b*) company officials who will use the rates, (*c*) regulatory authorities who must approve the rates, (*d*) agents who must sell the rates to the public, and (*e*) the public who, it is hoped, will buy, in a sharply competitive market. The last three of these groups are most of all interested in the relationship of the new rates to the old, and it would be a practical impossibility to obtain approval from the regulatory authorities if one were to discard the present rate structure as information of no value.

Although the question of the dependability of an observation, as deducible from Gauss's law of error, was explored by Woolhouse in England as far back as 1873 [7], the first application in ratemaking resulted from Mowbray's investigations in 1914 [8]. He assumed that probabilities of accident can be represented by the terms of a binomial expansion, which approximates the normal curve as the exponent (here the exposure) becomes very large, and thereby deduced from tables of the indefinite normal integral, values of the variable corresponding to a given probability P that the variation of the observed average from the most probable value will not exceed k per cent. For this problem, the variable limit in the normal integral

$$\mathrm{P}=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \tag{2}$$

is

$$x=\frac{knq}{\sqrt{2npq}} \tag{3}$$

where $n=$ exposure

$q=$ probability a claim $=1-p$.

The exposure required to satisfy the assumed probability level is then

$$n=2\cdot\frac{x^2}{k^2}\cdot\frac{1-q}{q}. \tag{4}$$

Replacement of $=$ by $\geqslant$ expresses the criterion for 100 % credibility. Perryman in 1932 [9] modified this approach to reflect (in effect although not by specific statement) the assumption generally accepted by then that accidents follow a Poisson rather than a binomial distribution, by taking a single trial as a unit of exposure for $\frac{1}{s}$ years rather than or 1 year, so that n becomes ns and q becomes $\frac{q}{s}$ in the binomial formula, then letting s (and consequently the binomial exponent ns) become very large.

Then
$$x = k\sqrt{nq} \tag{5}$$
and
$$nq \geqslant 2 \cdot \frac{x^2}{k^2} \tag{6}$$
in the foregoing notation. Since nq is the annual number of claims this assumption simplifies the credibility criterion. Since $\frac{1}{\sqrt{nq}} = \frac{\sigma}{\mu} =$ coefficient of variation, the above criterion is equivalent to $\frac{\sigma}{\mu} \leqslant \frac{1}{\sqrt{2}} \cdot \frac{k}{x}$.

It should be noted that the selection of assumptions, i.e. of P and k, is arbitrary. The important point is that, once P and k are selected, consistency in the interpretation of statistical data between classes and between territories is ensured.

The criterion so determined is assumed to provide 100% credibility. For partial credibilities, the approach has varied. The procedure commonly adopted is that the relative weights of two experiences, one with exposure entitled to 100% credibility and the other with exposure $\frac{n}{r}$, would be in the ratio of the reciprocals of their standard deviations or as $\frac{\sqrt{n}}{\sigma}$ to $\frac{\sqrt{n/r}}{\sigma}$, that is, as 1 to $\frac{1}{\sqrt{r}}$. Thus credibility $Z(\leqslant 1)$ would be assigned to $Z^2 n$ claims; i.e. $Z = \frac{1}{\sqrt{r}}$.

Further considerations on partial credibilities are set forth in Longley-Cook's excellent recent booklet for students [10] where a complete bibliography will be found together with an extended discussion of other aspects of the theory.

Bailey, in 1943, in discussing this "limited fluctuation" credibility, cast light upon the problem of partial credibility without pursuing it to a conclusion [11]. He constructed a table of the normal sampling range of the ratio of actual to expected number of claims (or of frequencies) corresponding to given values of the probability that a smaller value will occur, assuming that the number of claims C follows a Poisson distribution approximated for large numbers of claim by a Pearson Type III distribution with the same skewness. If V_1 and V_2 are values in such a table, entered with E(C), corresponding to $\left(1 - \frac{1-P}{2}\right)$ and $\left(\frac{1-P}{2}\right)$, P being interpreted as in the foregoing,

$$2k = Z(V_1 - V_2)$$

and
$$Z = \frac{2k}{V_1 - V_2}.$$

Values of Z (<1) produced by this evaluation approximate (to the second decimal place for very small values of C and more accurately for large) those produced by the square root rule for the same values of P and k. The explanation lies in the fact that the credibility determined from the sampling range tables varies, for any given P and k, approximately with the reciprocal of the standard deviation of C, i.e. with $\sqrt{C}$; for skewness $=$ 0, the correspondence is of course exact. The same factor $\sqrt{C}$ is the numerator of the expression for the determination of Z by the square root rule, since the actual number of claims is assumed to be equal to the expected in the establishment of distribution tables; the denominator is $\sqrt{\text{no. of claims for 100\% credibility}}$, which equals $\frac{x}{k}$ from (5).

Thus the square-root rule is justified as the closest simple approximation, and a very close approximation to the theoretically correct values.

It should be noted that the foregoing formulas, strictly interpreted, refer not to the actual number but to the expected number of claims, although in practice the actual number is customarily used. The reason is not only that the actual number is an unbiassed estimate of the expected number, but also a matter of expediency, since the actual number is immediately available whereas the expected number involves not only an assumption as to the expected frequency but also a subsequent calculation. There is a very fundamental difference in the results of application: the actual number increases credibility for a class with relatively high loss frequency (adverse experience) while giving a low credibility for a class with relatively low loss frequency (favourable experience), while the expected number will give more weight compared with the class with low frequency. Further refinement from theoretical considerations produces formulas impracticably complicated, since these credibilities are used in thousands of calculations annually.

Expected losses are sometimes used, but theoretically this will relax the assumptions for a given credibility level because of the greater variances of distributions of loss amounts as compared with those of numbers of claims. When it is considered, however, that the criteria are determined empirically, and that the important results are consistency in the interpretation and use of statistics and the restriction of the influence of chance fluctuations, the broadening or narrowing of the underlying assumptions are minor considerations. R. A. Bailey in Longley-Cook's booklet [10] sets forth in Appendix C a comparison of the number of claims to maintain, for claim cost, pure premiums and trends, a given level of credibility determined from claim frequencies.

A. L. Bailey has developed the theory of the "greatest accuracy" credibility from regression theory [12] by obtaining the best unbiassed linear estimate of the population mean of a certain characteristic in terms of the observations of this characteristic and of all characteristics. In reviewing the following, consider, for example, the ith characteristic as an individual risk's loss hazard and the combination of all characteristics as the loss hazard of the class comprising all such risks.

Let X_i = population mean (i.e. true value of ith characteristic)

$x_i = X_i/\overline{X}$ where the bar signifies mean value, as usual

m = units of exposure in each observation

$a = \overline{X}/m$, so that $amx_i = X_i$

w_{ij} = jth observation of ith characteristic

y_{ij} = deviation of jth observation of ith characteristic from X_i

so that $w_{ij} = amx_i + y_{ij}$ $(j = 1, 2, \ldots k)$.

Then, since $E(y_i) = 0$, $E(x_i y_i) = 0$, and $\bar{w} = am$,

$$\sigma_w^2 = a^2 m^2 \sigma_x^2 + \frac{\sigma_y^2}{k}$$

and $$E(x_i - \bar{x})(w_i - \bar{w}) = am\sigma_x^2.$$

The linear regression equation of X_i on w_i gives as the best unbiassed linear estimate of X_i,

$$E(X_i|w_i) = X_i' = \frac{a^2 m^2 \sigma_x^2}{\sigma_w^2} \cdot w_i + \left(1 - \frac{a^2 m^2 \sigma_x^2}{\sigma_w^2}\right) . \bar{w}. \tag{7}$$

In other words,

$$E(X_i|w_i) = Zw_i + (1 - Z)\bar{w}$$

which is a weighted average of w_i, the average of all observations of the characteristic, with $\bar{w}$, the average of the observations of all characteristics, where the weight-factor, or credibility, attached to w_i is

$$Z = \frac{a^2 m^2 \sigma_x^2}{\sigma_w^2} = \frac{k}{k + \dfrac{\sigma_y^2}{a^2 m^2 \sigma_x^2}} . \tag{8}$$

It is easily shown that the variance of such an estimate is Z times that of w_i. So that we have illustrated here two of the areas of innovation in theory mentioned at the outset of this paper, credibility-weighting and the reduction of variance through the use of unbiassed estimates of averages for characteristics in the aggregate as compared with each of the characteristics individually.

The expression (8) for Z is easily translated to the familiar $\dfrac{P}{P+K}$ formula (where P refers to premium or expected losses) for the credibility used in the rating of individual risks obtained by A.W. Whitney in 1918

[13] and still in use. In such ratings, M in formula (1) is the modification of the class rate resulting from applying the formula to the risk, B represents the class rate, and A the rate indicated by the risk's own experience; if A is the risk's loss ratio and B the expected loss ratio, M becomes a percentage modification,

$$M = 1 - Z + Z \cdot \frac{A}{B} \tag{9}$$

from which it is seen that the credibility is at the same time equal to the credit if there are no losses. This characteristic is utilised in an elegant derivation of $Z = \frac{P}{P+K}$ by R. A. Bailey (A. L. Bailey's son) by developing the expected claim frequency for risks accident-free for n or more years as compared with all risks, assuming that the inherent hazard remains constant for each risk, following a Poisson distribution, with the risk parameter following a Pearson Type III distribution, producing the familiar negative binomial distribution of total claims [14]. The relative frequency so obtained is in the form $\frac{a}{a+n}$ whence the indicated discount from manual rates is $1 - \frac{a}{a+n}$ or $\frac{n}{a+n}$, and multiplication of numerator and denominator by the annual premium produces the $\frac{P}{P+K}$ form. This result was independently obtained by F. Bichsel in 1959 [15].

The relationship to the Gauss theory of error and weighting of observations should be noted. Weights proportional to the reciprocals of standard deviations of w_i and $\bar{w}$, i.e. in the ratio $\frac{1}{\sigma_{w_i}^2} : \frac{1}{\sigma_{\bar{w}}^2}$ would be of the form $Z = \frac{P}{P+K}$ and $1 - Z = \frac{K}{P+K}$.

In another paper [16] Bailey derives the same form (1) from Laplace's generalisation of Bayes' Rule in determining the expected value of a statistic which corresponds to the origin or cause of an observed event H, and shows that, if $P(H,x)$ represents the *a priori* probability connecting H and x, and $K(x)$ the *a priori* probability of the existence of x, then the regression of x on H, or $E(x|H)$, is linear

(a) when $P(H,x)$ follows the Binomial distribution, only when $K(x)$ follows the Beta distribution;

(b) when $P(H,x)$ follows the Poisson distribution, only when $K(x)$ follows the Pearson Type III distribution (producing a negative binomial form).

In summary, the credibility-weighting process, with theoretical justification even in the streamlined way in which it is used, is admirably adapted to provide the necessary balance between stability and responsiveness in the rate structure, and at the same time to provide the necessary link between statistical experience and prior information (whether in the form of the existing rate structure or of broader statistical averages) while ensuring utter consistency in the treatment of the various bodies of statistics involved in the determination of rates.

CREDIBILITY VARIABLES—EXPERIENCE RATING

It will be noted that formulas (4) and (6) show that the "limited fluctuation" credibility depends upon both volume of statistics and frequency of loss, but that the "greatest accuracy" formula in its customary appearance reflects only the volume directly, any reflection of loss frequency being restricted to variation of the K. Ideally we can postulate that any credibility factor should be a function $F(v, q)$ of the volume and loss frequency (volume being understood here in a general sense) such that

I. $0 \leqslant z \leqslant 1$

II. For $z < 1$,

$$\text{(a) } \frac{dz}{dv} > 0, \ \frac{dz}{dq} > 0; \qquad \text{(b) } \frac{d^2z}{dv^2} < 0, \ \frac{d^2z}{dq^2} < 0.$$

The square root formula for partial credibilities related to the 100% criterion determined by (6), or

$$Z = \sqrt{\frac{nq}{(nq)_{100\%}}} = \sqrt{\frac{1}{r}} \tag{10}$$

satisfies both postulates, as does the formula

$$Z = \frac{P}{P+K} = \frac{f(n).q}{f(n).q+K}. \tag{11}$$

The failure of (11) to reflect loss frequency variations by size of loss (i.e. for different loss severities) has resulted in the development of experience rating plans (wherein the credibility (11) is most often encountered) with a split of the losses into a primary portion which includes the first T_P dollars of each loss and an excess portion T_E which includes the balance of the losses, with credibility on T_P higher than on T_E for $T_P = T_E$. Because of the greater frequencies of loss on property damage liability as compared with bodily injury, credibilities for experience rating of property damage are correspondingly increased by variation of the K value.

Perryman in 1937 refined the theory of credibilities for experience rating and developed the multi-split principle to introduce a diminishing credibility for successive increments of a single loss by including as primary losses the first t dollars of each loss plus $r\%$ of the next t dollars plus $r^2\%$ of the next t dollars and so on in geometric progression, the balance being the excess losses [17]. The maximum primary loss is thus $t(1-r)$ dollars. The proportions of losses thus designated as primary and excess vary by class so that expected losses must be split by class correspondingly. The plan in this form has in actual application been restricted to workmen's compensation risks.

R. A. Bailey has recently re-examined experience rating theory [18], viewing the credibility as a multiple correlation coefficient between the frequencies of losses of different sizes and the total expected losses as modified by the rating plan, which might be called a multi-multi-split approach; with an assumption of comparative ignorance as to the correct tariff rate (which is realistic for a newly established coverage as in the present multiple-peril developments in the United States) he has the paradoxical result of a one-split plan with primary losses self-rated and with zero credibility on the excess losses, conclusions which he notes support the findings of Professor Karl Borch of Norway on excess of loss reinsurance presented in 1960 at Brussels [19].

A significant contribution to credibility theory reflecting the greater variances of distribution of loss amounts by size as the limit of loss increases is a study made by L. H. Roberts of the effect on credibility of using in manual ratemaking the now common automobile liability limits of 10/20 ($10,000 per claim subject to a maximum of $20,000 per accident) as compared with the old 5/10 basic limits [20]. He calculates "that 10/20 experience would require at least 40% more claims for full credibility to retain the same statistical reliability as 5/10 experience". Space does not permit a summary here of the details of the calculation.

RETROSPECTIVE RATING

The rating formula (9) applied to individual risks illustrates "prospective" experience rating, that is, a rating modification developed from past experience on the risk to apply for the coming year. In the 1930's a type of plan termed "retrospective" rating was developed under which the risk's experience is reviewed after expiration and a modification developed for retro-active application. Under such plans the risk's losses are as a rule self-rated within minimum and maximum limits in accordance with the formula

$$\mathrm{M}=\begin{cases}\mathrm{H} & (t\leqslant h)\\ \mathrm{B}+\mathrm{C}t & (h<t<g)\\ \mathrm{G} & (t\geqslant g)\end{cases} \tag{12}$$

where, if M, H, B, and G are ratios to class rates,

M = modification

H = minimum premium

G = maximum premium

t = risk loss ratio

h = minimum loss ratio reflected in rating

g = maximum loss ratio reflected in rating

C = loss conversion factor to include rate variables dependent on the losses

B = basic premium $= e+t'(g)-t''(h)$

where e = provision for other expenses (selling, administration, servicing)

$t'(g)$ = average class ratio of losses in excess of g to total losses

$t''(h)$ = average class ratio of losses less than h to total losses.

Now t' and t'' depend upon the distribution of loss ratios by risk about the average loss ratio (after adjustments to reflect equality between actual and expected losses).

Let E( ) = expected value, as usual

u = risk premium

R_s = excess pure premium ratio over a loss ratio of s

$$= \frac{\Sigma(\text{losses} > su)}{\Sigma E(tu)}$$

t = total loss ratio on each risk.

$$\text{Then } R_s = \frac{\Sigma_{tu=su}^{\infty}(tu - su)}{\Sigma E(tu)} \tag{13}$$

or, for a given size of risk u, writing $t_1 = \frac{t}{E(t)}$ and $s_1 = \frac{s}{E(t)}$, and further considering t as a continuous variable with distribution $F(t)$,

$$R_s = \frac{\int_{s_1}^{\infty} tF(t)dt - s_1\int_{s_1}^{\infty} F(t)dt}{\int_0^{\infty} F(t)dt = 1}$$

Since $\int_{s_1}^{\infty} tF(t)dt = \int_0^{\infty} tF(t)dt - \int_0^{s_1} tF(t)dt = \mu_t - \int_0^{s_1} tF(t)dt$

where μ has the usual signification of the mean value,

and further $\int_0^{s_1} tF(t)dt = s_1\int_0^{s_1} F(t)dt - \int_0^{s_1}\int_0^{s_1} F(t)dt\,dt$

$$\text{it follows that} \qquad R_s = \mu_t - s_1 + \int_0^{s_1}\int_0^{s_1} F(t)dt\,dt \tag{14}$$

as derived by A. L. Bailey [11].

Formula (12) may be modified so as to sectionalise the range of self-rated losses, or even so as to modify them by a credibility-weighting process, but these variations have not been utilised in common practice.

Excess pure premium ratios vary by size of risk, and the loss ratio variances are so great on small risks that the size of $t'(g)$ makes application impracticable. As u increases, however, $t'(g)$ approaches 0, and the plan is widely used on the larger risks. Carlson [21] has developed the theory in so far as it is concerned with the interrelationships of the variables involved, and has explored various types of plan, and Dorweiler [22] has developed procedures for producing graduated tables of excess pure premium ratios.

DISTRIBUTION THEORY

Little will be said here about distribution theory, because the subject has been so fully developed in the International Congress Transactions and the ASTIN Bulletins. Rate-making in the United States has had a much larger statistical volume at its command than elsewhere, and this has been a factor in what appears to be a more pragmatic approach to rate-making with less dependence upon mathematical models—notwithstanding the fact that the major factor has been, as already indicated, the impact of statutory regulation.

Distribution theory in casualty insurance statistics commonly stems from the mathematical model assumed for the distribution of the number of claims. For years this was assumed to follow the Poisson form and it is only recently that the negative binomial has come generally to supersede the Poisson for this purpose. Bailey's paper on sampling theory [11] uses the Poisson distribution to reflect chance fluctuations in claim frequency distributions, but makes allowance for other distributions in the development of expressions for the moments of other statistics, with analytic ramifications that covered almost the entire field of casualty actuarial science twenty years ago.

It is interesting to note that the negative binomial distribution was presented, including its generalised form, in the Proceedings of the Casualty Actuarial Society as early as 1942 as a model reflecting variation of the Poisson parameter to recognise differences from individual to individual in the inherent risk hazard [23]. Bailey derived it again in 1950 in his study of credibility theory as developed from Bayesian considerations, as already noted in this paper. But it did not come into general use in the United States until its application in 1959 by Harwayne [24] and Dropkin [25] in automobile driver record studies, since when there have been a number of papers relating to theory and applications [26]-[29]. Varied interpretations of this model, which is of surprisingly wide versatility, have been reviewed in PCAS [30] and [31], but the recent ASTIN article by Campagne [32] seems to be the most complete in this respect.

SCHEDULE RATING—LINEAR PROGRAMMING APPLICATION

Schedule rating is used as a classification refinement reflecting physical characteristics of individual risks. It was once applied universally in workmen's compensation insurance, but is now retained in only one state since experience rating has almost entirely superseded it. In fire insurance it still constitutes the core of rate differentiation.

The extreme refinement of classification effected by schedule rating and the interaction of the multitude of factors involved has to date placed such factors virtually beyond analysis. The theory of an approach to this very difficult problem has been developed in a recent paper by McIntosh [33] utilising linear programming techniques which with the new electronic speeds of data processing for the first time open the door to the possibility of undertaking the solution of such complex multivariate problems of factor-interdependence.

RISK THEORY

Collective risk theory, which has played such a large role in the literature of the science in Europe and the origination of which dates back close to the inception of the International Congress, has received little attention in America but is currently being examined by a rapidly increasing number of actuaries, and the Casualty Actuarial Society has organised a committee on the mathematical theory of risk. Again, the explanation for previous scanty consideration lies principally in the history of insurance developments in our country, with statutory measures forcing primary attention to the propriety of rates for individual risks. The two approaches need to be blended, for both involve important concepts without which the science is incomplete.

CONCLUSION

In reviewing the theoretical contributions of the Casualty Actuarial Society only those aspects which it is believed may not be familiar ground to actuaries in other countries have been emphasised. It has been possible to mention only a few of the papers that still are significant, and it should be emphasised that the greater number have dealt with practical problems and their solution rather than with theory. We welcome these international exchanges, and assure you of our increasing participation both in the International Congresses and in ASTIN.

I should like to acknowledge the suggestions of a number of colleagues in the Society, but shall name only four: the present immediate-past presidents, L. H. Longley-Cook and William Leslie, Jr., who proposed the general subject; and R. A. Bailey and L. H. Roberts, discussions with both of whom have been more helpful than they can realise.

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ERRATA

Page	*Location*	*Correction*
284	Bottom of page	Delete a*
287	Third paragraph	For "*k* per cent" read "100*k* per cent"
288	Line 3	For "or 1 year" read "for 1 year"
289	Line 9	For $\sqrt{C}$ read $\frac{1}{\sqrt{C}}$
291	Lines 20 and 21	For "standard deviations" read "variances"
292		For $\frac{dz}{dv}$, $\frac{d^2z}{dz^2}$ and $\frac{d^2z}{dq^2}$ read $\frac{\partial z}{\partial v}$, $\frac{\partial^2 z}{\partial z^2}$ and $\frac{\partial^2 z}{\partial q^2}$
294	Lines 2 and 3	For "average class ratio" read "average ratio"
294	Equation (13)	The lower limit of summation should be zero and the condition $tu \geqq su$ should be stated