THE OPTIMAL MANAGEMENT POLICY OF AN INSURANCE COMPANY

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1. INTRODUCTION

1.1 In this paper we shall discuss some of the decision problems which occur in insurance companies. We shall try to indicate how these problems may be solved by combining the familiar ideas of actuarial mathematics with those of modern theories of scientific management.

In these theories it is generally accepted that the essential function of management is to make decisions. In an insurance company management has to decide what kind of risks the company shall underwrite and if (or how) these risks shall be reinsured. When the results of an underwriting period become known, management will have to decide whether the profits – if any – shall be distributed as dividend or added to the "special reserves" or "catastrophe funds" of the company.

In general, management will have some rules as to how these decisions shall be made. We shall refer to the body of such rules as the *management policy* of the company.

1.2 If a policy shall be general, it must specify which decision should be taken in every possible situation. Mathematically this means that a policy is a function or a mapping from the set of all situations to the set of all possible decisions. A decision may lead to an action which will bring the company into a new situation.

In this paper we shall not consider all aspects of a complete management policy. We shall study only decisions concerning reserve funds and reinsurance. These decisions have particular actuarial interest, and they can be formulated mathematically in a fairly simple way.

In general there will obviously be an infinity of possible policies. This naturally leads us to consider the problem of determining the best among these policies. However the term "best" has no meaning without a scale of values, or a preference ordering. We must therefore assume that management has a preference ordering over the set of situations in which the company can be. The *objective* of management will then be to select the decision which will bring the company to the most preferred among the attainable situations.

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2. DISCUSSION OF A SIMPLE MODEL

2.1 In this Section we shall discuss an extremely simple example, in order to illustrate and clarify the ideas we have presented in general and rather vague terms in the Introduction.

We shall consider an insurance company which in each operating period underwrites a portfolio of insurance contracts. We shall assume the total claim payment in this portfolio can be:

Either 0 with probability p
or 2 with probability $q = 1 - p$

We shall further assume that the company receives a premium of 1 by underwriting the portfolio.

Our assumptions mean that the company in each period engages in a game, where the gain is either $+1$ or -1, with probabilities p and q respectively. We shall assume that the game is favorable to the company, i.e. that $p > q$.

2.2 Let us now assume that the company's initial capital is S_0. There is obviously a possibility that the capital may be lost after some periods of operations, i.e. that the company becomes insolvent or "ruined." However as the game is favorable to the company, the capital can be expected to grow as time goes by.

It is clear that an increase in the capital will reduce the probability of ruin, so that the company will seek to keep a certain amount of capital as a "special reserve." There must however in practice be some limit to the funds which an insurance company is willing to accumulate to meet such contingencies. In the following we shall assume that this limit is given by a constant Z, i.e. if the company's capital should exceed Z, the excess will be paid out as dividends. Z can then be interpreted as the reserve which management considers necessary to conduct insurance in a manner which will meet all possible demands of prudence and security.

2.3 When management decides on a value Z, is really decides on a *dividend policy,* or a rule stating when dividends should be paid. If the reserves of the company amount to S at the end of an underwriting period, the company will according to the rule pay a dividend

$$s = S - Z \text{ if } S > Z$$

and

$$s = 0 \qquad \text{if } S \leqq Z$$

This is obviously a very special rule. More generally we could consider dividend policies given by a rule

$$s = s(S)$$

where $s(S)$ is a function giving the amount s which will be paid as dividend if reserves at the end of an underwriting period are S.

2.4 Decisions concerning reserve funds are naturally linked to decisions with regard to reinsurance arrangements.

In the following we shall for the sake of simplicity assume that the only kind of reinsurance arrangements open to our company is quota share cession on "original terms." This means that management will have to decide on a quota k which shall be retained of the portfolio underwritten by the company.

If the company retains a quota k, and cedes a quota $1-k$, it will engage in a game where the stakes are $+k$ and $-k$, and not $+1$ and -1 as in the original game discussed in paragraph 2.1.

In our simple example the company's *risk* policy will consist of a set of rules stating how the numbers Z and k should be chosen when S is given. When the results of an underwriting period become known, these rules will determine the amount, if any, which shall be paid as dividend, and how the portfolio underwritten in the next period shall be reinsured.

2.5 Any pair (Z,k) will determine a complete risk policy in our simple model. It is however desirable, if possible, to single out one pair which is "best" according to some basic principle or objective which the company wants to reach. In the following we shall assume that the company's objective – at least in part – is to maximize the *expected discounted value of the dividend payments* which the company will make during its lifetime.

In itself this assumption does not appear unreasonable. Insurance is a business, and the ultimate purpose of putting money into business is usually to make it grow and return in the form of dividends. However the real test of an assumption lies in deriving its consequences, or implications and checking if these agree with the observations we can make. Introspection may tell us that the objective of an insurance company may well be to maximize expected dividend payments over a finite or infinite period. However we should not accept this unless we can observe that the company actually behaves as it would if it pursued this objective in a rational manner.

2.6 It is easy to see that the problem we have outlined is the classical problem of "the gambler's ruin." This is solved in many textbooks of probability, so we shall just restate the main results in our own terms, following on most points the presentation given by Feller [3].

Let S be the special reserve fund of the company, and asume $0 < S < Z$.

The game described in paragraph 2.1 can then "terminate" in two ways:

(i) S becomes negative, i.e. $S = -1$ in our simple model. In this case the company is ruined and the game is terminated for good.

(ii) S exceeds Z, i.e. in our simple model $S = Z + 1$. In this case the company pays a dividend of 1, and the game continues with reserve funds equal to Z.

2.7 Let now $w(S,n)$ be the probability that the game terminates with a dividend payment after n periods, i.e. that the reserve fund does not become negative, and reaches $Z + 1$ for the first time after n periods.

It is easy to see that this probability must satisfy the condition

$$w(S,n+1) = pw(S+1,n) + qw(S-1,n)$$

This is a difference equation in two variables with the boundary conditions:

$$\begin{array}{lll} \text{(i)} & w(S,0) = 0 & \text{for } 0 \leq S \leq Z \\ \text{(ii)} & w(-1,n) = 0 & \\ \text{(iii)} & w(Z+1,0) = 1 & \\ \text{(iv)} & w(Z+1,n) = 0 & \text{for } 0 < n \end{array}$$

The equation can be solved directly by classical means. We shall however find it more convenient to introduce the generating function

$$W_S(t) = \sum_{n=0}^{\infty} w(S,n)t^n$$

If we multiply our difference equation by t^{n+1} and sum over all $n \geq 0$, we obtain

$$W_S(t) = pt\, W_{S+1}(t) + qt\, W_{S-1}(t)$$

This is a difference equation in only one variable, with the obvious boundary conditions

$$W_{-1}(t) = 0 \text{ and } W_{Z+1}(t) = 1$$

2.8 The last difference equation has the general solution

$$W_S(t) = A_1 r_1^S + A_2 r_2^S$$

where r_1 and r_2 are the roots of the characteristic equation

$$r = ptr^2 + qt$$

i.e.

$$r_1(t) = (1 + \sqrt{1 - 4pqt^2})/2pt$$

$$r_2(t) = (1 - \sqrt{1 - 4pqt^2})/2pt$$

A_1 and A_2 are functions of t which must be determined so that the boundary conditions are satisfied, i.e.

$$A_1 r_1^{-1} + A_2 r_2^{-1} = 0$$

$$A_1 r_1^{Z+1} + A_2 r_2^{Z+1} = 1$$

From these we obtain

$$A_1 = \frac{r_1}{r_1^{Z+2} - r_2^{Z+2}} \text{ and } A_2 = \frac{-r_2}{r_1^{Z+2} - r_2^{Z+2}}$$

which gives the following expression for the generating function

$$W_S(t) = \frac{r_1^{S+1} - r_2^{S+1}}{r_1^{Z+2} - r_2^{Z+2}}$$

2.9 Let us now assume that the company has established the policy of paying dividend only when its capital exceeds a fixed amount Z, and let $V(S,Z)$ be the expected discounted value of the dividends which will be paid under this policy The probability that the first dividend shall be paid after n periods is $w(S,n)$. If this event occurs, the company will then enter the next period with a capital equal to Z. Hence the expected value of the first payment will be

$$w(S,n) \{1 + V(Z,Z)\}$$

which we will discount by the factor v^n. The first payment can take place after 1, 2, . . . , n, . . . periods, so that we have

$$V(S,Z) = \sum_{n=1}^{\infty} V^n w(S,n) \quad \{1 + V(Z,Z)\}$$

or if we introduce the generating function for $w(S,n)$

$$V(S,Z) = \{1 + V(Z,Z)\} \; W_S(v)$$

Putting $S = Z$, we obtain

$$V(Z,Z) = \frac{W_Z(v)}{1 - W_Z(v)}$$

and for $0 \leq S \leq Z$

$$V(S,Z) = \frac{W_S(v)}{1 - W_Z(v)}$$

or inserting the expressions for the generating function found in paragraph 2.8

$$V(S,Z) = \frac{r_1^{S+1} - r_2^{S+1}}{(r_1^{Z+2} - r_2^{Z+2}) - (r_1^{Z+1} - r_2^{Z+1})}$$

This result has been derived in different contexts by a number of authors, i.e. by Shubik and Thompson [6] who applied it to a problem very similar to the one considered in this paper.

2.10 Let us now consider reinsurance. We noted in paragraph 2.4 that reinsurance of a quota $1-k$ on original terms was the same as reducing the stakes of the game from -1 and $+1$ to $-k$ and $+k$. This is nothing but a change of unit in the original game, i.e. we have to replace S and Z by $\frac{1}{k}S$ and $\frac{1}{k}Z$. For typographical convenience we shall write $\frac{1}{k} = X$. Hence the expected discounted value of the dividend payments when the company selects a policy (Z,k) is given by

$$V(S,Z,X) = \frac{1}{X} \; \frac{r_1^{XS+1} - r_2^{XS+1}}{r_1^{XZ+1}(r_1 - 1) - r_2^{XZ+1}(r_2 - 1)} \text{ where } X \geq 1$$

In the following section we shall discuss this result in some detail, and determine the optimal policy.

3. THE OPTIMAL POLICY IN THE SIMPLE MODEL

3.1 Our problem can now be formulated as follows:

For a given $S \geq 0$, determine the values of X and Z which will maximize:

$$V(S,Z,X) = \frac{\frac{1}{X}\left\{ r_1^{XS+1} - r_2^{XS+1} \right\}}{r_1^{XZ+1}(r_1 - 1) - r_2^{XZ+1}(r_2 - 1)} = \frac{N(X)}{M(XZ)}$$

subject to

$$X \geq 1 \text{ and } Z \geq 0$$

Differentiating the denominator with respect to XZ we find

$$M'(XZ) = r_1^{XZ+1}(r_1 - 1) \log r_1 - r_2^{XZ+1}(r_2 - 1) \log r_2$$

From the expression found in paragraph 2.8 we note that for $v < 1$ we have $r_1 > 1$ and $r_2 < 1$. Hence $M'(XZ)$ is either always positive, or it has a single zero. This means that $M(XZ)$ takes its minimum value, either for $XZ = 0$, or for the single real root of the equation:

$$\left(\frac{r_1}{r_2}\right)^{XZ+1} = \frac{r_2 - 1}{r_1 - 1} \; \frac{\log r_2}{\log r_1}$$

In the following we shall write Y for this root, and we shall assume that it gives the minimum. The case where the minimum is $M(0)$ is actually trivial. It will occur in situations where the best policy is to pay out the initial capital as dividend immediately, without risking it in the insurance business.

3.2 We now consider the numerator. Differentiating with respect to X we find

$$N'(X) = -\frac{1}{X^2}\left\{ r_1^{XS+1} - r_2^{XS+1} \right\} + \frac{S}{X}\left\{ r_1^{XS+1} \log r_1 - r_2^{XS+1} \log r_2 \right\}$$
$$= \frac{1}{X^2}\left\{ (XS \log r_1 - 1)\, r_1^{XS+1} - (XS \log r_2 - 1)\, r_2^{XS+1} \right\}$$

It is easy to see that $N'(X)$ is negative for small values of X, and that it is steadily increasing with X toward $+\infty$. Hence $N(X)$ takes its maximum value either for $X = 1$, or for the largest attainable value of X.

To get an upper limit for X, we note that the equation in paragraph 3.1 gives us $XZ = Y$, where Y depends only on the given parameters. It then follows that Z will decrease with increasing X, but Z cannot become smaller than S, so that we have

$$Z = \frac{Y}{X} \geq S, \text{ or } X \leq \frac{Y}{S}$$

For $S > Z$ our formula is not valid, since we have by definition

$$V(S,Z) = S - Z + V(Z,Z)$$

Hence the largest value of $N(X)$ is either

$$N(1) = r_1^{S+1} - r_2^{S+1} \text{ or } N\left(\frac{Y}{S}\right) = \frac{S}{Y}\left(r_1^{Y+1} - r_2^{Y+1}\right)$$

It is easy to show that

$$N(1) > N\left(\frac{Y}{S}\right) \text{ for all } S < Y$$

Hence $N(X)$ takes its maximum value for $X = 1$, i.e. when the company retains the whole portfolio. This means that in our simple model, reinsurance will not pay, i.e. it is not possible to increase the expected value of the dividend payments by reinsuring on original terms.

3.3 It may be useful to illustrate the preceding results by a simple numerical example.

We shall take $r_1 = 1.1$ and $r_2 = 0.7$ This corresponds to:

$$p = 0.565,\ q = 0.435 \text{ and } v = 0.983$$

Ignoring reinsurance for the time being, we find that the necessary reserves Z_0 are given by

$$\left\{\frac{11}{7}\right\}^{Z_0+1} = \frac{r_2 - 1}{r_1 - 1}\ \frac{\log r_2}{\log r_1} = 11.23$$

which gives $Z_0 = 4.368$

Table 1 gives the value of $V(S,Z)$ for some selected values of S and Z.

TABLE 1

EXPECTED DISCOUNTED VALUE OF DIVIDEND PAYMENTS

S = Initial Funds	Z = Reserves considered necessary							
	0	1	2	3	4	Z_0	5	6
0	1.25	1.49	1.70	1.83	1.89	1.90	1.89	1.82
1	2.25	2.69	3.05	3.30	3.40	3.41	3.40	3.27
2	3.25	3.69	4.19	4.52	4.67	4.68	4.67	4.49
3	4.25	4.69	5.19	5.56	5.79	5.80	5.79	5.56
4	5.25	5.69	6.19	6.56	6.81	6.83	6.82	6.55
Z_0	5.62	6.05	6.55	6.93	7.18	7.21	7.19	6.98
5	6.25	6.69	7.19	7.56	7.81	7.84	7.69	7.50

To illustrate the meaning of this table, let us assume that our insurance company finds itself with funds $S = 3$ at the end of an underwriting period, and that the management considers paying a dividend.

If management decides that $Z = 2$ is sufficient as a special contingency reserve for the future operations of a company, a dividend $s = 1$ will be paid immediately. This decision means that the expected discounted value of the dividends which the company will pay is equal to $V(3,2) = 1 + V(2,2) = 5.19$. If management is prepared to exercise some patience, and postpone dividend payments until the reserves reach $Z = 4$, this expected value will increase to $V(3,4) = 5.79$. However unlimited patience does not pay. If management should decide that reserves in excess of $Z_0 = 4.37$ are necessary, the expected value of dividend payments will decrease from its maximum value of 5.80. For instance if management should set its target as high as $Z = 6$, the expected value of the dividend payments will be reduced to $V(3,6) = 5.56$.

3.4 To illustrate the effect of reinsurance let us assume that the company reinsures 50% of its portfolio on original terms. According to paragraph 3.1, this will reduce reserve requirements by 50%, so that expected dividend payments will be maximized if the company decides to hold an amount 2.18 in reserve.

Using the notation of paragraph 2.11 we find for some values of S

$$V(0, 2.18, 2) = 0.95$$
$$V(1, 2.18, 2) = 2.34$$
$$V(2, 2.18, 2) = 3.42$$

These are considerably smaller than the corresponding values in Table 1, i.e. $V(0,Z_0) = 1.90$, $V(1,Z_0) = 3.41$, $V(2,Z_0) = 4.68$. This illustrates the point made in paragraph 3.2, that reinsurance does not pay.

3.5 Reinsurance plays an important part in real life, so we ought to explain why it does not appear to have any place in our simple model.

Our paradoxical result may be due to the very simplicity of the model. If we consider claim distributions of a more general form, it is possible that reinsurance arrangements may help to increase the expected value of the dividend payments. We shall not take up this problem here, although it certainly merits further study.

To find a solution to our paradox, we shall try to modify our assumptions about the company's objectives. In actuarial literature much – probably too much – attention has been given to the "probability of ruin." This probability has not proved particularly useful in practical work. In the following we shall consider a related concept, the company's "expectation of life," or in less actuarial terms, the "expected duration of the game." We shall assume that this concept enters into the company's objective function.

3.6 Let $D(S,Z)$ be the expected number of periods our company will stay in business, if the initial capital is S, and the company follows the dividend policy determined by Z.

It is easy to see that $D(S,Z)$ must satisfy the difference equation

$$D(S,Z) = pD(S+1,Z) + qD(S-1,Z) + 1 \quad \text{for } 0 \leq S \leq Z$$

with the boundary conditions

$$D(-1,Z) = 0$$
$$D(Z,Z) = D(Z+1,Z)$$

This equation can be solved by methods similar to those used in paragraph 2.8 (see [3] p. 317), and we find:

$$D(S,Z) = \frac{p}{(p-q)^2}\left\{\left(\frac{p}{q}\right)^{Z+1} - \left(\frac{p}{q}\right)^{Z-S}\right\} - \frac{S+1}{p-q}$$

Putting $p = 0.565$ and $q = 0.435$ as in our numerical example, we obtain $D(S,Z) = 33.4\ \{(1.3)^{Z+1} - (1.3)^{Z-S}\} - 7.7(S+1)$

Table 2 gives the values of the function $D(S,Z)$ for some selected values of S and Z.

TABLE 2

EXPECTED DURATION OF THE GAME

S = Initial Capital	Z = Reserves considered necessary					
	0	1	2	3	4	5
0	2.3	5.3	9.2	14.3	20.9	29.5
1	2.3	7.6	14.6	23.5	35.2	50.2
2	2.3	7.6	16.9	29.0	44.4	64.7
3	2.3	7.6	16.9	31.2	50.0	74.0
4	2.3	7.6	16.9	31.2	52.3	79.3

This table shows that some patience in paying dividend may increase the company's expectation of life in a dramatic manner.

3.7 To compare the Tables 1 and 2, let us consider an insurance company within initial capital 1.

If this company wants to maximize the expected discounted value of the dividends it will pay during its lifetime, it may decide on the policy of not paying any dividend before its capital exceeds 4 (considering this a sufficient approximation to the optimal value 4.368).

The expected value of the dividend payments will then be:

$$V(1, 4) = 3.40$$

This policy will give the company an expected life $D(1, 4) = 35.1$

If the company decides to reinsure 50% of its portfolio, the expected value of the dividend payments will be maximized if the required reserves is set at 2. This maximum value is

$$V(1,2,2) = \frac{1}{2} V(2,4,1) = \frac{1}{2} V(2,4) = 2.34$$

and the expected life of the company is $D(2,4) = 44.7$

3.8 The example just considered illustrates the point we want to make.

If the company reinsures a part of its portfolio, the expected value of the dividend payments will be reduced, but the company will obtain a longer expected life. It is not unreasonable to assume that the policy of an insurance takes both these elements into consideration.

In the terms of paragraph 2.5 this means that the company will select the policy (Z,k) which maximizes some function of two variables

$$kV\left(\frac{S}{k}, \frac{Z}{k}\right) \text{ and } D\left(\frac{S}{k}, \frac{Z}{k}\right)$$

In this paper we shall not embark on a general discussion of the possible shape of this function. We shall however note that one possible rule would be to maximize V subject to the restraint $D \geq M$ where M is some number, which for instance may be imposed by the government as a solvency requirement.

3.9 Returning to our numerical example, let us assume that for some reason we have fixed $M = 50$. From Table 2 we see that this will lead the company to set its reserve requirements at 4, i.e. whenever the company's reserves exceed 4, the excess will be paid out as dividend. From Table 2 we also see that if reserves should fall to 3, the company will not need reinsurance in order to satisfy the restraint $D \geq 50$, since $D(3,4) = 50$.

If however, reserves should fall to 2, something has to be done, because $D(2.4) = 44.4$, so that the restraint is no longer satisfied. If the company reinsures a quota $1-k$, i.e. retains a quota k, its expected life will become $D\left(\frac{2}{k}, \frac{4}{k}\right)$. By rough interpolation in Table 2, we see that the company can satisfy the restraint by reinsuring approximately 10% of its portfolio, i.e. by choosing $k = 0.9$. Should reserves fall to 1, the company's expected life without reinsurance will be $D(1,4) = 35.2$. In this case the company must reinsure a larger quota in order to satisfy the restraint.

It is worth noting that a restraint of the type $D \geq M$ can always be satisfied by reinsurance, since the company can obtain an infinite expectation of life by adopting the policy of always reinsuring its whole portfolio on original terms. However with this policy the company will never be able to pay any dividend.

If the objective of an insurance company is to maximize the expected discounted value of its dividend payments, subject to a restraint of the form $D \geq M$, the company will reinsure heavily when reserves are low, and reinsure less as reserves accumulate after a number of successful underwriting periods. This is very much the way in which insurance companies seem to behave, so our simple model may contain some of the essential elements of the problem which we set out to study.

4. RELATIONS TO THE COLLECTIVE THEORY OF RISK

4.1 The problems we have discussed in the two preceding sections were first studied in a systematic manner by Filip Lundberg at the beginning of this century. Lundberg's ideas are usually referred to as the "collective theory of risk." This name seems rather unfortunate today, but it appeared quite natural 50 years ago, when a term was needed to distinguish Lundberg's radically new approach from the now almost forgotten theory of risk developed by actuaries in the 19th century.

Lundberg attacked the problem in its fullest generality, and this naturally led to a theory of extreme mathematical complexity. Some recent papers by Cramer [2] and Kahn [4] give short surveys of the main results of the theory and fairly complete bibliographies.

It appears from these surveys that most work on collective risk theory has been concerned with mathematical details rather than the basic ideas behind the theory. In this Section we shall apply these ideas to our simple model, and try to show that the ideas also are fairly simple when stripped of their mathematical superstructure.

4.2 Let $u(S,n)$ be the probability that a company with initial capital S shall be ruined after n periods of operations.

Using the same methods as in paragraph 2.8 we find that the generating function

$$U_S(t) = \sum_{n=0}^{\infty} u(S,n)t^n$$

satisfies the difference equation

$$U_S(t) = ptU_{S+1}(t) + qtU_{S-1}(t)$$

with the obvious boundary condition

$$U_{-1}(t) = 1$$

If the company's policy is to pay out as dividend any capital in excess of Z, we get a second boundary condition

$$U_{Z+1}(t) = U_Z(t)$$

The solution of the difference equation is then

$$U_S(t) = \frac{r_1^{Z+1}\, r_2^{S+1}\,(r_1 - 1) - r_1^{S+1}\, r_2^{Z+1}\,(r_2 - 1)}{r_1^{Z+1}\,(r_1 - 1) - r_2^{Z+1}\,(r_2 - 1)}$$

4.3 For $t = 1$ the generating function becomes the probability that the company eventually shall be ruined, $R(S,Z)$.

From paragraph 2.8 we see that for $t = 1$ we have

$$r_1 = 1 \text{ and } r_2 = \frac{q}{p}$$

Inserting these values in the expression for $U_S(t)$, we find

$$U_S(1) = R(S,Z) = 1$$

This means that the company is certain to be ruined – sooner or later. The result holds for all finite values of S and Z, i.e. regardless of how large the initial capital is, and of how high the reserve requirements are set, as long as they are finite.

Our expression of $U_S(t)$ can be written

$$U_S(t) = \frac{(r_1 - 1)\, r_2^{S+1} - (r_2 - 1)\, r_1^{S+1} \left(\dfrac{r_2}{r_1}\right)^{Z+1}}{r_1 - 1 - (r_2 - 1)\left(\dfrac{r_2}{r_1}\right)^{Z+1}}$$

From paragraph 2.8 it follows that $r_1 > r_2$, so that as $Z \to \infty$ we have

$$\lim_{Z \to \infty} U_S(t) = r_2^{S+1}$$

For $t = 1$ we obtain the probability of ruin

$$R(S) = \lim_{Z \to \infty} R(S,Z) = \left(\frac{q}{p}\right)^{S+1}$$

which has played such an important part in the collective risk theory. The basic idea is that the company must maintain reserves S, which are so large that the ruin probability $R(S)$ is smaller than a certain acceptable maximum. Should this be impracticable, the ratio q/p must be reduced, either by reinsurance arrangements, or by "loading" the premium.

4.4 The collective risk theory has never found any significant applications in practice. The reasons are fairly obvious. Most insurance companies pay dividends or declare that they would do so if they had sufficient reserves. They will therefore have little use for a theory which presupposes that the company has a firm policy of never paying any dividend – neither to shareholders nor to policyholders.

In practice insurance companies follow policies which ultimately must lead to bankruptcy. Most actuaries realize this, and accept it. Often they add a remark to the effect that it does not really matter if their company is virtually certain to go out of business within the next 10,000 years. This remark, which really dismisses the whole collective risk theory as useless, also points to a more fruitful formulation of the problem. When ruin is certain, like death and taxes, it is natural to ask when it is likely to occur.

This question was first asked by Segerdahl [5], but he has apparently not followed up the idea. In this paper we have tried to show that it may be possible to create a theory of risk which can be used in practice, if we switch our attention from the traditional ruin probability to the time of ruin.

5. CONCLUDING REMARKS

5.1 The main purpose of this paper has been to study the objectives which insurance companies seek to achieve. If the objectives can be spelled out clearly, it will be possible to determine the operating policy which is "best" or "most efficient" in the company's pursuit of these objectives. A set of objectives may however appear quite reasonable on inspection, but imply an operating policy obviously different from the policy followed by any insurance company.

The enthusiastic expert on operations research may then conclude that management has got it all wrong, and insist that the policy should be changed. On this point the expert is right – if the stated objectives completely represent what the managers at the bottom of their hearts want to achieve.

A more mature social scientist may take a different attitude when confronted with management decisions which are obviously irrational under a stated set of objectives. He may admit the possibility that these decisions are quite rational, but under a set of more subtle objectives than the managers have been able to, or bothered to state explicitly. He may be right on this point, although he will probably not rule out the possibility that managers, like other people, may consistently make foolish decisions.

5.2 In the paper we have tried to illustrate these points by discussing a model which represents a drastic simplification of the real insurance world. By this simplification we may have lost, or "assumed away" some aspects which are essential to the real problem.

The methods of difference equations which we used in Section 2 can obviously be applied also when the discrete stochastic variable can take more than two values, but the mathematics will become very cumbersome as the number of possible values increases. In such cases the characteristic equation will be an algebraic equation of high degree, and may have both complex and multiple roots. The function $M(Y)$ introduced in paragraph 3.1 will then contain terms of the form Y^n and sin Y in addition to the terms r^Y, and may have several local minima. This may clearly mean that there is no unique value of Z which maximizes expected dividend payments. In such models there may well be room for reinsurance.

If we consider continuous stochastic variables, the method of difference equations will obviously break down. However the problem can then be formulated in terms of integral equations, an approach which has been explored in another paper [1].

5.3 The assumption that a firm seeks to maximize the expected discounted value of its dividend payments seems a very natural one. The purpose of business is, almost by definition, to make profits, and the earlier the better.

It should be noted that the discount factor v used in our model does not necessarily have anything to do with the market rate of interest. The discount factor $v < 1$ expresses the assumption that an early dividend payment is preferred to a later one. Put another way we can say that $v < 1$ means that the company looks to first things first, i.e. that it attaches greater weight to secure the dividend payment of 1965 than that of 1970.

The assumption implies that the firm assigns some value to "staying in business." This value is however equal to the expected value of the dividends which the firm will be able to pay during its remaining life. It is not unreasonable to assume that some firms, such as insurance companies, may attach a higher value to "staying alive," and this naturally leads us to assume that the expectation of life, i.e. the function $D(S,Z)$

introduced in paragraph 3.6, enters into the objective function of an insurance company.

5.4 In our model we assumed that the probability p was completely known – or in the terminology of American actuaries – that p had 100% credibility. This is probably more unrealistic than any of our simplifying assumptions.

In practice p will not be completely known, and the company's estimate of p may change as experience accumulates. In this case it is not very reasonable to assume, as we did in paragraph 2.3, that the dividend payment at the end of period n depends only on the reserves at that time, i.e. that the dividend policy is given by a function of one variable

$$s_n = s(S_n)$$

The reasonable assumption would be that the whole accumulated experience of the company is taken into account when a dividend payment is considered. This will give us a dividend rule determined by a function of the form:

$$s_n = s(S_n, S_{n-1}, \ldots, S_1, S_0)$$

To some extent credibility theory has been developed apart from the main body of actuarial mathematics. It appears however that if we want a complete and realistic theory for the management of insurance companies, credibility theory must be brought in as an essential element.

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