ON SOME ESSENTIAL PROPERTIES OF A TARIFF CLASS

EDWARD FRANCKX

The celebration of the Casualty Actuarial Society's Golden Jubilee at first seemed an excellent occasion for discussing certain aspects of credibility theory. There is a pertinent reason for that. Credibility theory is a branch of actuarial science created and developed by our American colleagues with the object of justifying the techniques of rating non-life risks. We have, however, abandoned our original project, because the problems of experience rating and credibility theory will be discussed at the next ASTIN Colloquium which will be held at Lucerne in 1965. But the principal reason is more fundamental. In effect the primary problem in the insurance industry is "to be able to meet the expenses resulting from the arrival of claims"; the secondary problem, (because it is in fact complementary) is "the practical realization of the assembly of resources to meet the obligations of the assurer." This second problem, which includes both the problem of rating and the problem of reassurance is from a logical point of view dependent on the first. The first defines the basic fact "what must one expect?" – the second poses the question "how can one prepare for it?"

The object of this note is to reply in a very general form to the first question.

In his excellent introduction to credibility theory Longley-Cook insists on the necessity of basing the theory on an adequate mathematical model. This model must be on the one hand sufficiently precise to represent with the necessary approximation the reality of the risks run and on the other hand sufficiently practical to enable both the particular actuarial cases and the problems of forecasting or security to be treated.

The model of life assurance – Dodson's model based on attained age – is not appropriate for non-life insurance. Nevertheless, the life model belongs to a particular class of stochastic processes and it is within the framework of this general theory that the actuary finds quite naturally the basis of his mathematical calculations. But this theory is complicated and very large; in particular Lundberg's theory of risk and its different developments are also contained in the general theory.

On the other hand, as we have often pointed out, it is remarkable that

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for years practitioners and businessmen have found the reply to the two questions posed in our introduction above without having recourse to any very sophisticated theories. This fact gives food for thought and suggests that a mathematical model can be found which justifies at least to the first approximation current experience and the usual rules of action.

As in every constructive theory – and a model has no other end than to define the structure – we must start from certain basic elements. In non-life insurance we choose our point of departure in the theory of tariff classes.

By definition a tariff class is obtained: (1) By the juxtaposition of a finite number of risks N which belong to k homogeneous classes of risk designated by $C_1, C_2, \ldots, C_k$. (2) However, in spite of the fundamental heterogeneity of the class, we admit that for any specified class the mean claim remains constant. Let S be this amount.

The second hypothesis is equivalent to saying that if one considers the "claims department" as an independent financial organization and if on each arrival of a claim in a class the department is "endowed" with the amount of the mean claim S, then, by the application of the law of large numbers, the statistical equilibrium of the budget of this financial organization will be assured in mean.

The loading to be added to S to meet fluctuations and to provide a security loading is a complementary problem belonging to the theory of risk which we do not touch in this note. In this branch the notion of "feed back control" could probably produce algorithms capable of progressive adaptation which have not been studied up to the present.

On the other hand the notion of a homogeneous class is worthy of further attention. A homogeneous class is composed of elements which are interchangeable from the point of view of risk, that is to say indistinguishable on the basis of statistical observation and a fortiori by the assurer.

Consider such a risk. What distinguishes non-life insurance from life assurance is the fact that in a specific interval of time, located on the time axis by two arbitrary times t_1 and t_2, the number of claims produced by any one contract can vary from nought up to any number n.

This means, for example, if t_1 and t_2 correspond to the beginnings of financial years, that in the course of one year of insurance, the number of claims is a random variable N, such that

$$\text{prob } (N = n) = p_n > 0$$

$$\text{with } \sum_0^\infty p_n = 1 \qquad n = 0, 1, 2, \ldots$$

It is the choice of this (p_n) which is the decisive act of the actuary. In fact stochastic events are also met with in other domains. The arrival of claims in a company is, to a large extent, comparable to the arrival of telephone calls at an exchange or the arrival of clients at a service hatch. These are the types of events studied by operational research. In this domain, the majority of real situations have been approached with success by means of Poisson variates. This means putting

$$p_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

t being the time interval of observation, λ the parameter of the Poisson process, which is also the mean value of the number of events which occur in an interval of unit time.

Let us recall that the hypotheses which lead to the Poisson law are:

1. Markov's hypothesis: or the hypothesis that the number of events in two separate time intervals are independent.
2. The hypothesis of rare events, which is that in a very small time interval the probability of the occurrence of two or more events is arbitrarily small in comparison with the occurrence of one event only. Note that this limitation excludes the application of the model to risks with conflagration where the claims occur in groups.

It is remarkable that these hypotheses alone allow a precise answer to the question "what can one expect?"

Let us examine the question mathematically. The global *endowment* which must be made to the claims department is a random figure at the beginning of a financial year depending on the stochastic number of claims which are notified to the Company in the course of a year.

We can, therefore, define this endowment by a random variable D. The question posed, namely, "what can one expect?" is the same as the fixing of this stochastic variable D. It determines the stochastic endowment for a tariff class in the course of the current accounting period and, in a general sense, the stochastic endowment for this same class in the course of an interval of any time t.

Now, because of the hypotheses made for the tariff class, whatever the value of t,

$$\text{prob } (D = nS) = \text{prob (total number of claims} = n)$$

The random variable D is thus completely known as soon as the random variable giving, T, the total number of claims of the tariff class has been ascertained. This is expressable in a very simple manner. If two Poisson variates are added the sum is also a Poisson variate of which the parameter is the sum of the parameters of the variates composing it.

Thus, suppose a contract belongs to a homogeneous class C characterized by a Poisson variate N_1 of parameter λ_i

$$p_{n_1} = \frac{(\lambda_i t)^{n_1}}{n_1!} \exp\{-\lambda_i t\} \tag{1a}$$

and that a second contract belongs to a homogeneous class C_j characterized by the Poisson variate N_2 of parameter λ_j

$$p_{n_2} = \frac{(\lambda_j t)^{n_2}}{n_2!} \exp\{-\lambda_j t\} \tag{1b}$$

then the total number of claims expected under the two contracts will be determined by a Poisson variate N

$$p_n = \frac{[(\lambda_i + \lambda_j)t]^n}{n!} \exp\{-(\lambda_i + \lambda_j)t\} \tag{2}$$

or parameter $\lambda_i + \lambda_j$

This property can be generalized by induction. It is thus sufficient to add the parameters of the ensemble of risks belonging to the tariff class.

Suppose that:

the classes	$C_1, C_2, \ldots, C_i, C_j, \ldots, C_k$
of parameters	$\lambda_1, \lambda_2, \ldots, \lambda_i, \lambda_j, \ldots, \lambda_k$
comprise	$n_1, n_2, \ldots, n_i, n_j, \ldots, n_k$ different

risks with $\sum_1^k n_i = N$

N being the total number in the tariff class. Under these conditions we arrive at the essential property of the endowment variate D for the tariff class.

The variable global endowment B for a tariff class is a Poisson variate.

$$p\ (B = nS) = \frac{(t \sum_1^k n_i \lambda_i)^n}{n!} \exp\{-t \sum_1^k n_i \lambda_i\} \tag{3}$$

of parameter $S = \sum_1^k n_i \lambda_i$ for $t = 1$

This result, valid under the hypothesis defined above, but certainly true as a first approximation represents a solution whose value should be carefully tested. Apart from the fact that it emphasizes once more the fundamental part played in practice by the Poisson law it throws into relief a very important result: *one single parameter is sufficient to characterize the endowment variate for a tariff class.* This reduction to a single parameter, now that we have departed from a multiplicity of homogeneous classes is, perhaps, the most important consequence of the above-mentioned theoretical result.

In practice, the heterogeneity of a tariff class is not always known. Generally, in fact, very little is known about it and what is more it varies from year to year according to the general underwriting policy of the assurer.

But heterogeneity in effect only plays a very relative part in the problem "what can one expect?" Consider two assurers possessing different portfolios of the same tariff class. We can express this fact by the sequence (n_i'), giving the distribution of the insured of the first company into homogeneous classes which is different from the sequence (n_i) of the second company.

However, if $\sum_1^k n_i' \lambda_i = \sum_1^k n_i'' \lambda_i$

the "global endowment" variates of the two classes of risks are strictly identical, which is to say that *from the global point of view, in spite of the differences in heterogeneity, the sums of the endowments are, from the point of view of the two assurers, indistinguishable.*

They are financially equivalent even if the global complements of the two classes are different i.e. $\sum_1^k n_i' \neq \sum_1^k n_i''$.

The possible equivalence of global variates is a remarkable property.

We shall make use of the property of the equivalence of the global variables of the allocations. To this, we give the name *principle of substitution of equivalent partitions.*

Let us consider the population N of the tariff class. Effecting a partition of this population consists of sub-dividing into l sub-classes $E_1, E_2, \ldots, E_j, \ldots, E_l$ comprising the populations $n_1'', n_2'', \ldots, n_j'', \ldots, n_l''$ such that $\sum_{j=1}^{l} n_j'' = 1$.

Let us suppose, *as a working hypothesis,* that each sub-class E_j is conventionally regarded as a homogeneous class. Then, from the risk point of view, we attach to it a Poisson variable with parameter λ_j.

Under these conditions, there corresponds to this partition a global allocation variate of the tariff class, defined by the parameter λ^0

$$\lambda^0 = \sum_{i=1}^{l} n_j'' \tilde{\lambda}_j \tag{4a}$$

From the point of view of the insurer, the global variate defined by (4a) will be identical to that which he must experience in reality, provided that

$$\lambda^0 = \sum_{i=1}^{l} n_i' \lambda_i \tag{4b}$$

n_i' *and* λ_i characterizing the *real partition* of the insurer.

In conclusion, if the partitions satisfy the system (4a) (4b), they must be considered equivalent "from the point of view of allocations to the claims department." This implies that one can substitute for the real partition any equivalent hypothetical partition; and it is this rule which we call the principle of the substitution of equivalent partitions. Let us straightaway give a specific and important application of this in practice. Among the multiplicity of substitutions which one can imagine there is one which possesses a special characteristic. It results from the hypothesis that the total population N of the tariff class constitutes one whole homogeneous class E of parameter $\tilde{\lambda}$.

By virture of the system (4a) (4b), this homogeneous class is equivalent from the allocation point of view if and only if:

$$N \tilde{\lambda} = \sum_{i=1}^{k} n_i' \lambda_i$$

or if $\tilde{\lambda} = \dfrac{n_1' \lambda_1 + \ldots + n_i' \lambda_i + \ldots + n_k' \lambda_k}{n_1' + \ldots + n_i' + \ldots + n_k'}$ (5)

Thus, to each class of the tariff there corresponds from the allocation point of view an equivalent homogeneous class having the same population N, if and only if the parameter λ of the homogeneous class is the mean weighted by the population n_i' of the parameters of each homogeneous subclass of the tariff class.

Let us revert to the notion of equivalent partition which we have noted:

$$\begin{cases} \{ n_j'' \lambda_j \} & j = 1, 2, \ldots l \\ \text{with } \sum n_j'' = N \end{cases}$$

From (5) it is clear that the equivalence is only achieved if:

$$\breve{\lambda} = \frac{n''_1 \breve{\lambda}_1 + \ldots + n''_i \breve{\lambda}_i + \ldots + n''_l \breve{\lambda}_l}{n''_1 + \ldots + n''_j + \ldots + n''_l} \tag{6}$$

The totality of all the equivalent partitions is thus characterized by the fact that, whatever the partition, the weighted mean (6) remains invariant.

This invariant, which is the parameter of the equivalent homogeneous class, determines the numerical value of the compatibility relationship (6). It plays, as we shall see further on in this note, a preponderant role in the problem of rating.

In fact, the above considerations result solely from the mathematical model which we have considered. It is time that we established the link between theory and practice. This will be given to us by the classical statistical theory of estimation.

We know from the law of large numbers that during a series of independent tests in relation to one and the same variable the mean value of the values observed converges in probability towards the mean value of this variable. This is particularly so, if this variable is the Poisson variable

$$N_i : p_n^i = \frac{(\lambda_i)^n}{n!} e^{-\lambda_i}$$

with its mean value $En_i = \lambda_i$

On the other hand, if R_i is the total number of claims observed in the course of the year for a homogeneous sub-class the mean of the values observed will be $\frac{R_i}{n'_i}$

Therefore, within the meaning of the law of large numbers, $\frac{R_i}{n'_i}$ is the "statistical estimation" of En_i, that is of the parameter λ_i.

If we substitute these estimations in the relation (5) we shall obtain the estimated value of $\breve{\lambda}$ of the numerical invariant of all the equivalent partitions, which we will denote by λ^*

$$\lambda^* = \frac{\sum_{i=1}^{k} n'_i \frac{R_i}{n'_i}}{\sum_{1}^{k} n'_i} = \frac{\sum_{i=1}^{k} R_i}{\sum_{1}^{k} n'_i} = \frac{R}{N} \tag{7}$$

Thus, from the statistical point of view, *the estimated value of the invariant of the equivalent partitions is precisely the frequency observed in the tariff class.*

This result is remarkable, because the influence of the real heterogeneity of the tariff class has been totally eliminated in (7). On the other hand, we have returned to the "working" figures which practitioners have always used.

Let us demonstrate that they were perfectly right in using such figures from the point of view of rating which, as we have mentioned in our introduction, is the quest for admissible solutions as regards the collection of the necessary means for the financing of the allocation.

When the insurer effects a rating, he in effect achieves a partition of the tariff class, in respect of which he covers the claims. Each sub-class of this class is defined by the property that the *premium asked for is identical for all the insured of this sub-class.*

Rightly or wrongly, the insurer considers that each risk of such a sub-class is for him equivalent. This amounts to saying that this sub-class is *"considered as being homogeneous."*

Let us therefore use the theory of equivalent partitions.

If λ_i is the parameter of the class, the pure premium required for each risk of the sub-class will be:

$$\Pi_i = \sum_{n=1}^{\infty} p_n^i \, n \, S = S \sum_{n=1}^{\infty} n p_n^i = S\lambda_i \tag{8}$$

If we take into account the relationships (6) and (7), we obtain the statistical condition of compatibility:

$$S\frac{R}{N} = \frac{n_1'' \Pi_1 + n_2'' \Pi_2 + \ldots + n_l'' \Pi_l}{n_1'' + n_2'' + \ldots + n_l''}$$

Let us suppose:

$$\theta_i = \frac{n_i''}{n_1'' + n_2'' + \ldots + n_l''} = \frac{n_i''}{N}$$

and let us call this number the *coefficient of credibility belonging to the risk at the level of premium* Π_i.

We thus obtain the *general principal of rating: any method of rating* (Π_1, Π_2, . . . , Π_l) *is admissible in practice, if the mean value of the premiums charged within the terms of credibility, is equal to the mean premium of the tariff class.*

$$S\frac{R}{N} = \theta_1 \Pi_1 + \theta_2 \Pi_2 + \ldots + \theta_l \Pi_l \tag{9}$$

We have not invented anything; we consider that rating at the average premium is admissible in practice, and in Europe this method has many advocates. But we have justified, without much difficulty, the attitude of our American colleagues who wish to introduce nuances into rating, by introducing different levels of premium. In actual fact, the relationship (9) implies a constraint, the coefficients of credibility cannot be chosen arbitrarily and they must satisfy the condition of statistical compatibility.

In fact, we are pushing open a door which is ajar. For by multiplying (9) by N, we find again the condition of the accounting equilibrium: *the gathering in of premiums en masse must balance the expenses resulting from claims.* And it is this obvious fact with which we conclude our note.