to permit its discovery. Also, much of the world's work is done with tables prepared from simple functions like that of the normal curve. Thus, it's difficult to say that practical applications prefer formulas and accept tables only when formulas can't be found. What then is the fascination of the search for simple formulas to fit empirical data?

One motive might be to find or test an explanation of why the empirical data are as they are. The distinction between "to explain" and "to describe" may have become blurred at some levels of epistemology, but for immediate purposes I want to use the word "explanation" to cover something that helps me visualize a model within which I can see what produces the result.

Does the Type III Pearson curve purport to be the frequency distribution that can be expected when some definable factors are working on the individual items? In other words, is there a model that underlies it? I do not know whether there is or is not such a model. Has an analysis of the sources of hazard differences among exposure items suggested that they should be subject to analogous factors? In other words, does the Type III model, if it exists, look promising? With affirmative answers to both questions, a good fit would tend to support the inferences drawn from the analysis. Absent affirmative answers to either or both questions, the fit would seem to be coincidental. Moreover, searches for such fits, prior to dealing with such questions, would seem to be searches for such coincidences.

Such searches may be well worthwhile and yield many useful results, including those turned up through serendipity. However, some questions suggest themselves to which answers would be interesting: Do the conventional tests of Goodness of Fit apply to an undirected or trial and error search for a formula to fit some empirical data? Does testing a single hypothesis against some data call for different testing mathematics than starting with the data and then drawing at random from an infinite (or very large) available supply of formulas until one is found that seems suitable? Was the chi-square test built on the latter model? There is the intuitive notion that the random search should be shorter if the data are too thin to carry much information about the higher moments. Probably the notion is unfounded.

I hope these comments have some bearing on Mr. Carlson's concern with the rationale and the utility of models. Certainly his paper will stimulate others on claim count distributions.

DISCUSSION BY KENNETH L. McINTOSH

In this paper, deceptively simple in concept though perhaps not simple in mathematical detail, Mr. Carlson has accomplished three things, one of which possibly exceeds the limits of his own original objectives. First, the paper constitutes an excellent historical summary of various approaches to the negative binomial distribution in general, including presentation of one such approach in some detail. Secondly, the use of the factorial moment generating function is demonstrated. This extremely powerful mathematical tool is ignored by many authors,[1] yet, as this paper shows, with remarkably little effort the function yields results obtainable by other means only at the cost of considerable difficulty.

Thirdly and finally, in pursuing the rationale of the negative binomial, Mr. Carlson has gone far beyond that distribution to open for actuarial exploration the entire area of the general compound Poisson, of which the negative binomial is but a specific example. He then notes casually that the area is "fertile." It might be noted casually that The Bomb is "noisy."

This paper complements rather than supplements the negative binomial derivations presented earlier by Mr. Dropkin[2] and (independently) by Dr. Bichsel.[3] As it is only when Mr. Carlson's derivation is brought together with these earlier presentations that we approach critical mass, it seems necessary to bring Mr. Dropkin's derivation again under discussion despite the scrutiny to which it already has been subjected. This will serve to include Dr. Bichsel also, since his derivation parallels Mr. Dropkin's so closely that, for all present purposes, the latter may be considered representative of both.

To compare and contrast the two developments, Mr. Carlson's and Mr. Dropkin's, it first must be noted that the general compound Poisson distribution assumes either of two equivalent forms represented respectively by the left and right members of the identity:

$$\left[Q_1(r;\lambda) = e^{-\lambda} \sum_{r} \frac{\Pi\, \lambda_j^{\,k_j}}{\Pi\, k_j\,!} \right] \equiv \left[e^{-\lambda} \sum_{k} \frac{\lambda^k}{k!} \left\{ \frac{\lambda_j}{\lambda} \right\}^{k*} = Q_2(r;\lambda) \right] \qquad (1)^{4}$$

$$(\Pi = \prod_j)$$

In present specific context:

$j =$ the number of claims arising from a single accident; briefly: "claims-per-accident."

$k_j =$ the number of accidents each producing exactly j claims; briefly: "j-claim accidents."

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[1] *E.g.*, Cramér, in his *Mathematical Methods of Statistics* (Princeton, 1946) recognizes this function only by means of a single problem buried in fine print on p. 257. But see Feller, W. (*An Introduction to Probability Theory and Its Applications.* Vol. I (2nd Ed.) Wiley (1957) Chs. XI & XII), who concentrates on it to the exclusion of the more–commonly–encountered characteristic function and moment generating function. To be honest, before beginning this review I knew almost nothing of the function beyond the fact of its existence.

[2] Dropkin, Lester B., Some Considerations on Automobile Rating Systems Utilizing Individual Driving Records. *PCAS* XLVI (1959), p. 165.

[3] Bichsel, Dr. F., Une Méthode pour Calculer une Ristourne Adéquate pour Années sans Sinistres. *The ASTIN Bulletin*. I (1960), p. 107.

[4] Mr. Carlson's notation is not compatible with that of Mr. Dropkin, and neither system is entirely adequate for what follows here. Hence, it has been expedient to introduce notation as shown. However, notational equivalents will be obvious in cross-reference to original equations of either author except possibly in certain specific cases explained as they occur.

$k = \sum k_j$ = the total number of accidents; briefly: the "total-of-accidents."

$r = \sum jk_j$ = the total number of claims; briefly: the "total-of-claims."

λ_j = the parameter of a Poisson distribution of k_j.

$\dfrac{\lambda_j}{\lambda}$ = the probability that exactly j claims arise from any single accident.

$\left\{\dfrac{\lambda_j}{\lambda}\right\}$ = the distribution of claims-per-accident. (This is *not* the cumulative distribution function, but is the distribution itself, *i.e.* the sequence of the several probabilities $\dfrac{\lambda_j}{\lambda}$).

$\left\{\dfrac{\lambda_j}{\lambda}\right\}^{k*}$ = the k-fold convolution of $\left\{\dfrac{\lambda_j}{\lambda}\right\}$ with itself.

It will be convenient to have:

$$g_j = \frac{\lambda_j}{\lambda} \tag{2.a}$$

$$\left.\begin{aligned} p(k;\lambda) &= \frac{e^{-\lambda}\lambda^k}{k!} \\ P(k;\lambda) &= e^{-\lambda}\sum_k \frac{\lambda^k}{k!} \end{aligned}\right\} \tag{2.b}$$

The validity of Identity (1) which shows the so-called "multiple Poisson" to be the equivalent of a "compound Poisson," is demonstrated in Appendix C.

In present context, $Q_1(r;\lambda)$ and $Q_2(r;\lambda)$ are alternative expressions of the cumulative distribution function of the total-of-claims distribution. But for change of notation, the left member of Identity (1) is exactly Mr. Carlson's Eq. (4).[5] On the assumption that the relationship:

$$\lambda_j = \frac{\lambda_1}{j}\,\beta^{j-1} \qquad (\beta = \text{constant}) \tag{3}$$ [6]

holds among the parameters λ_j, the development presented by Mr. Carlson leads to a negative binomial total-of-claims distribution:

$$b(r; n_r, \pi_r) = \binom{-n_r}{r}\pi_r^{n_r}\,(-p_r)^r \tag{4}$$ [7]

---

[5] Let: $Q_1(r;\lambda) = P(r)$; $\lambda = a_1 + a_2 + \ldots$; $\Pi\lambda_j^{k_j} = a_1^{x_1}\cdot a_2^{x_2}\ldots$; $\Pi k_j! = x_1!\,x_2!\ldots$ Mr. Carlson's Eq. (4) then follows.

[6] Let: $j = k$; $\lambda_j = a_k$; $\beta = b$. Eq. (3) then becomes Mr. Carlson's Eq. (2).

[7] Let: $\pi_r = (1 - b)$; $\rho_r = b$; $n_r = \dfrac{a}{b}$; $r = r$. Mr. Carlson's Eq. (8a) then follows.

On the other hand, Mr. Dropkin has concerned himself entirely with accident frequency, and has not become involved with the claim distributions with which Mr. Carlson deals. On the assumption that inhomogeneity of the automobile driver population may be reflected by variation of the Poisson parameter, Mr. Dropkin's basic equation is (with notational changes):

$$f(k) = \int_{0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot \phi_\lambda d\lambda \tag{5}$$

where ϕ_λ is the probability density function of the distribution of λ among individuals of the population. Assuming the p.d.f. ϕ_λ to be specifically the Pearson Type III, Eq. (5) leads to the negative binomial total-of-accidents distribution:

$$b(k; n_k, \pi_k) = \binom{-n_k}{k} \pi_k^{n_k}(-\rho_k)^k \qquad (6)^{8}$$

as Mr. Dropkin demonstrates.

Though exhibiting identical mathematical properties, it can be shown that Mr. Carlson's negative binomial *claim* distribution, $b(r; n_r, \pi_r)$, and Mr. Dropkin's negative binomial *accident* distribution, $b(k; n_k, \pi_k)$, are actuarially incompatible. They cannot ever both be applicable simultaneously to data arising from the same population.

The negative binomial is a form of the compound Poisson[9], therefore, Identity (1) holds for that distribution.[10] Assuming the relations:

$$\lambda = \log \frac{1}{\pi_r}$$

$$\frac{\lambda_j}{\lambda} = \frac{\rho_r^j}{\lambda j}$$

Identity (1) becomes:

$$\left[B_1(r; n_r, \pi_r) = \sum_r \binom{-n_r}{r} \pi_r^{n_r} \; (-\rho_r)^r \right] \equiv$$

$$\left[e^{-\lambda} \sum_k \frac{\lambda^k}{k!} \left\{ \frac{\rho_r^j}{\lambda j} \right\}^{k*} = B_2(r; n_r, \pi_r) \right] \qquad (7)^{11}$$

---

[8] Let: $\pi_k = \dfrac{a}{1+a}$; $\rho_k = \dfrac{1}{1+a}$; $n_k = r$; $k = x$. Mr. Dropkin's form then follows.

[9] This is demonstrated by Mr. Carlson's derivation of the total-of-claims distribution $b(r; n_r, \pi_r)$. In general, see, for example, Feller, op. cit. (1) p. 271, Example (c).

[10] *Cf.* Feller, op. cit. (1) Ch. XII, Sects. 1 & 2. Specifically see Eqs. (1.2), (2.1) and (2.4).

[11] Identity (7) is by no means obvious, but see Feller, op. cit. (1), Ch. XII, Sect. 2. The distribution $\{ \rho_r^j/\lambda j \}$ is the logarithmic distribution here assumed applicable to the claims-per-accident. Letting: $\lambda_j = a_k$; $\rho_r = a_1 = b$; $j = k$; Mr. Carlson's power series Eq. (2) follows immediately.

The Poisson components in the right members of Identity (1) and of Identity (7) represent the total-of-accident distributions underlying respectively both the general claim distribution $Q_1(r;\lambda) \equiv Q_2(r;\lambda)$ and the specific claim distribution $B_1(r;n_r,\pi_r) \equiv B_2(r;n_r,\pi_r)$. It can be shown that the substitution of Mr. Dropkin's negative binomial accident distribution, or in general of any other distribution whatever for the Poisson accident frequency distribution, destroys Identities (1) and (7)[12]. And since the validity of Identity (1) is a necessary (and sufficient) condition that the total-of-claims distribution be a compound Poisson, it follows that specifically Mr. Carlson's and Mr. Dropkin's respective negative binomials are mutually incompatible, as stated above. More generally, *no compound Poisson (or "multiple Poisson") total-of-claims distribution is compatible with any but a simple Poisson total-of-accidents distribution.*

In other words, if the total-of-claims distribution follows any form whatever of the compound Poisson (saving the trivial case of always exactly one claim per accident), the population is homogeneous with respect to the accident-expectancy which Mr. Dropkin's entire development assumes to be variable within the population. This is true regardless of any assumptions whatever concerning inter-parameter relationships among the several λ_j of the left member of Identity (1).

If the logic of Mr. Dropkin's assumption of an inhomogeneous driver population is self-evident, the logic of assuming Mr. Carlson's population of potential victims of railway accidents to be homogeneous as regards accident-expectancy can be demonstrated. The idiosyncracies of individual passengers can have no influence upon accident frequency. Moreover, variation among railroad operating personnel will have been reduced to a minimum by selection, training, and experience, and whatever variation remains will be masked into virtual insignificance by safety rules and safety equipment (*e.g.* automatic block signals). Homogeneity with respect to accident-expectancy (demanded by Mr. Carlson's fatality distribution) logically follows.

No purely actuarial analysis of actual loss data ever can *rationalize* either Mr. Dropkin's Pearson Type III or Mr. Carlson's equally arbitrary inter-parameter power series, though either or both of these assumptions can be *validated* (or, alternatively, *invalidated*) by actuarial analysis in a given case. Mr. Carlson's power series can be rationalized only if it can be shown that the distribution not of total-of-claims but of claims-per-accident logically should be the logarithmic distribution.[13] Obviously, this leads away from

[12] See Appendix A.

[13] *Cf.* Feller, op. cit. (1), p. 271, Eq. (2.4) and see *Note 11*, above. Mr. Carlson notes as "interesting" that a compound Poisson with three "unrelated" parameters fits certain railway fatality data better than does the negative binomial. These parameters *cannot* be "unrelated," since λ_j is directly proportional to the probability of exactly j deaths in a single accident, hence a relationship among the parameters must follow from the fatalities-per-accident distribution. I have not had opportunity to refer to the original studies of Lüders which Mr. Carlson cites. It may be that Lüders' data was too thin to reveal the claims-per-accident distribution, thus giving the appearance of "unrelated" parameters.

purely actuarial considerations into safety engineering analysis of railway accidents and the circumstances attendant upon them. It is possible that the Pearson Type III assumption someday may be rationalized by the psychologist, whose attempts to correlate driving record with the psychological pattern of the individual already have been partially successful. The most that any purely actuarial analysis can accomplish, however, is to validate this assumption empirically, as Mr. Dropkin and Dr. Bichsel have done.

Mr. Carlson notes that his "observations on rationale by no means exhaust the subject." If the negative binomial specifically did not offer a broad enough field of inquiry, the field of the general compound Poisson in actuarial application appears inexhaustible. And it is into exactly that unbounded area that Mr. Carlson has led.

APPENDIX A

From the right member of Identity (1):

$$Q_2(r;\lambda) = P(k;\lambda)\left\{g_j\right\}^{k*} \tag{1.R}$$

where:

$$g_j = \frac{\lambda_j}{\lambda} \tag{2.a}$$

$$\left.\begin{aligned} p(k;\lambda) &= \frac{e^{-\lambda}\lambda^k}{k!} \\ P(k;\lambda) &= e^{-\lambda}\sum_k \frac{\lambda^k}{k!} \end{aligned}\right\} \tag{2.b}$$

Let λ vary in accordance with a distribution function $\Phi(\lambda)$ with corresponding probability density function ϕ_λ.[1] Let:

$$f(k) = \int_0^\infty p(k;\lambda)\cdot\phi_\lambda d\lambda \tag{5}$$

$$F(k) = \sum_k\left(\int_0^\infty p(k;\lambda)\cdot\phi_\lambda d\lambda\right) \tag{8}$$

Transform the distribution $Q_2(r;\lambda)$ into a distribution $\mathbf{Q}_2(r;\lambda)$ by substitution of $F(k)$ for $P(k,\lambda)$ in Eq. (1.R):

$$\mathbf{Q}_2(r;\lambda) = F(k)\left\{g_j\right\}^{k*} \tag{9}$$

If $\mathbf{Q}_2(r;\lambda)$ is any compound Poisson whatever, we must have by Identity (1) a distribution $Q_1(r;\mu)$ such that:

$$\mathbf{Q}_2(r;\lambda) \equiv Q_1(r;\mu) \equiv Q_2(r;\mu) \tag{1.A}$$

---

[1] Obviously, if $\Phi(\lambda)$ is discrete, ϕ_λ is the frequency function rather than the p.d.f., and the integral of Eqs. (5) and (8) becomes a summation.

where:

$$\left[\, Q_1(r;\mu) = e^{-\mu} \sum_r \frac{\prod \mu_j^{k_j}}{\prod k_j!} \,\right] \equiv \left[\, e^{-\mu} \sum_k \frac{\mu^k}{k!} \left\{\frac{\mu_j}{\mu}\right\}^{k*} = Q_2(r;\mu) \,\right] \tag{1.B}$$

$$(\Pi = \prod_j) \qquad (\mu = \Sigma\mu_j) \qquad (k = \Sigma k_j)$$

and, since stochastic independence between j and k is assumed:

$$\left\{\frac{\mu_j}{\mu}\right\} = \left\{\frac{\lambda_j}{\lambda}\right\} = \left\{ g_j \right\} \tag{2.c}$$

From the right member of Identity (1.B), the total-of-accidents distribution underlying the distribution $Q_2(r;\mu)$ is:

$$\left.\begin{aligned} p(k;\mu) &= \frac{e^{-\mu}\mu^k}{k!} \\ P(k;\mu) &= e^{-\mu} \sum_k \frac{\mu^k}{k!} \end{aligned}\right\} \tag{2.d}$$

whence by Identity (1.A) and Eq. (9):

$$F(k) \equiv P(k;\mu) \tag{10}$$

Let:

$\rho(z;\zeta) =$ the generating function of $P(k;\zeta)$ $(\zeta = \lambda \; or \; \zeta = \mu)$

$f(z) =$ " " " " $F(k)$

$\phi(z) =$ " " " " $\Phi(\lambda)$

then:

$$p(z;\mu) = e^{-\mu+\mu z} \tag{11.a}$$

$$f(z) = \phi\,[p(z;\lambda)] = \phi\,[e^{-\lambda+\lambda z}] \qquad \text{(11.b)}^{2}$$

where the brackets of the right member of Eq. (11.b) indicate the compound function obtained by substitution of $p(z;\lambda) = e^{-\lambda+\lambda z}$ for z in $\phi(z)$. It then follows from Identity (10) that:

$$\phi[e^{-\lambda+\lambda z}] \equiv e^{-\mu+\mu z} \tag{12}$$

whence, immediately:

$$f(z) = \phi\,[e^{-\lambda+\lambda z}] = (e^{-\lambda+\lambda z})^c = e^{-c\lambda+c\lambda z} \tag{13}$$

$$\left(c = \frac{\mu}{\lambda}\right)$$

whence:

$$F(k) = e^{-c\lambda} \sum_k \frac{(c\lambda)^k}{k!} \tag{14}$$

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[2] See Appendix B, following, and *cf* Feller, *op. cit.* p. 269, *Theorem*.

whence ϕ_λ must be:

$$\phi_\lambda = \varphi_\lambda = \begin{cases} 1;\ for\ \lambda = \dfrac{\mu}{c} \\ 0;\ for\ \lambda \neq \dfrac{\mu}{c} \end{cases} \qquad (15.a)$$

and it further follows from Eq. (2.c) that:

$$c\lambda_j = \mu_j \quad \text{for all } j. \qquad (15.b)$$

The rabbit is now nicely out of the hat. It follows from Eqs. (14), (15.a), and (15.b) that although the level of hazard exhibited by a given population *in toto* may vary with time, any form of compound Poisson total-of-claims distribution (*e.g.* Mr. Carlson's negative binomial) implies homogeneity of the population as regards accident-expectancy and, therefore, is incompatible with any total-of-accidents distribution derived on assumption of inhomogeneity (*e.g.* Mr. Dropkin's negative binomial), save in the trivial case where each accident produces exactly one claim.[3]

APPENDIX B

There is an alternative derivation of the negative binomial accident frequency. In the particular instance, the following offers no advantage whatever over Mr. Dropkin's original derivation, however not only has it some theoretical interest, but the method in general may save calculation where all necessary generating functions are known in advance and need not themselves be calculated individually in the course of deriving a given distribution.

The Pearson Type III assumption is retained. Then:

$$\phi_\lambda = \frac{a^{n_k} \lambda^{n_k - 1} e^{-a\lambda}}{\Gamma(n_k)}$$

and by Eq. (5):

$$f(k) = \int_0^\infty \frac{e^{-\lambda}\lambda^k}{k!} \cdot \frac{a^{n_k}\lambda^{n_k - 1} e^{-a\lambda}}{\Gamma(n_k)}\, d\lambda \qquad (16)$$

(Eq. (16) is, but for notation, identical to Eq. (5) of Mr. Dropkin's Appendix A.)

Now the factorial moment generating function of the Pearson Type III is:

$$h(z) = \int_0^\infty \frac{z^\lambda a^{n_k}\lambda^{n_k - 1} e^{-a\lambda}}{\Gamma(n_k)}\, d\lambda = \frac{a^{n_k}}{\Gamma(n_k)} \int_0^\infty \lambda^{n_k - 1} e^{-(a - \log z)\lambda} d\lambda \qquad (17)$$

[3] It should be emphasized that homogeneity as regards accident-expectancy does *not* necessarily imply homogeneity of the population with regard to expected severity, *i.e.* individual claims-per-accident expectancy.

Repeated integration by parts[4] gives:

$$\int_0^\infty \lambda^{n_k - 1} e^{-(a - \log z)\lambda} d\lambda = \frac{\Gamma(n_k)}{(a - \log z)^{n_k}}$$

whence:

$$h(z) = \left[\frac{a}{a - \log z}\right]^{n_k} \tag{18}$$

Substitute $p(z;\lambda)$ or z in Eq. (18):

$$f(z) = h\,[p(z;\lambda)] = \left[\frac{a}{a - \log\,(e^{-\lambda + \lambda z})}\right]^{n_k} = \left[\frac{a}{a + \lambda - \lambda z}\right]^{n_k} \tag{19}$$

Let:

$$\lambda = \rho_k$$
$$a = \pi_k = 1 - \rho_k = 1 - \lambda \tag{20}$$

Substitute in Eq. (19):

$$f(z) = \left[\frac{\pi_k}{1 - \rho_k z}\right]^{n_k} \tag{21}$$

But the right member of Eq. (21) is the generating function of the negative binomial:[5]

$$b(k;n_k,\pi_k) = \binom{-n_k}{k} \pi_k^{n_k} \; (-\rho_k)^k \tag{6}$$

Hence it follows immediately that:

$$F(k) = \mathrm{B}(k;n_k,\pi_k) = \pi^{n_k} \sum_k \binom{-n_k}{k} (-\rho_k)^k \tag{22}$$

APPENDIX C

Let:

$$a_1 + a_2 + \ldots = \lambda$$
$$a_1^{x_1} \cdot a_2^{x_2} \ldots = \Pi\lambda_j^{k_j}$$
$$x_1!\, x_2! \ldots = \Pi k_j!$$
$$P(r) = Q_1(r;\lambda)$$

[4] Or see any standard table of definite integrals, *e.g.* Korn & Korn, *Mathematical Handbook for Scientists and Engineers.* McGraw-Hill (1961), p. 820, Integral #380.

[5] See, *e.g.*, Feller, *op. cit.* p. 271, Eq. (2.3)

Mr. Carlson's Eq. (4) then becomes the left number of Identity (1):

$$Q_1(r;\lambda) = e^{-\lambda} \sum_r \frac{\Pi\lambda_j^{k_j}}{\Pi k_j!} \qquad (\Pi = \prod_j) \tag{1.L}$$

Mr. Carlson has developed the generating function associated with his Eq. (4) to be (in his notation):

$$f(z) = e^{-a_1 - a_2 - \ldots + a_1 z + a_2 z^2 + \ldots} \tag{23}$$

(See his Eq. (5)

Let:

$f(z) = q_1(z;\lambda) =$ the generating function of $Q_1(r;\lambda)$

$a_j = \lambda_j$

Then Eq. (23) becomes:

$$q_1(z;\lambda) = e^{-\lambda + \Sigma\lambda_j z^j} \tag{24}$$

Turning to the right member of Identity (1):

$$Q_2(r;\lambda) = e^{-\lambda} \sum_k \frac{\lambda^k}{k!} \left\{\frac{\lambda_j}{\lambda}\right\}^{k*} \tag{1.R}$$

the generating function of the Poisson component is:

$p(z;\lambda) = e^{-\lambda + \lambda z}$

and the generating function of $\left\{\frac{\lambda_j}{\lambda}\right\}$ is (by definition of that function):

$$g(z) = \sum \frac{\lambda_j z^j}{\lambda} = \frac{1}{\lambda} \sum \lambda_j z^j$$

By a fundamental theorem[6], if $q_2(z;\lambda)$ is the generating function of $Q_2(r;\lambda)$ then:

$q_2(z,\lambda) = p[g(z);\lambda] = e^{-\lambda + \lambda g(z)}$

whence:

$$q_2(z;\lambda) = e^{-\lambda + \lambda\left(\frac{1}{\lambda}\Sigma\lambda_j z^j\right)} = e^{-\lambda + \Sigma\lambda_j z^j} \tag{25}$$

But by Eqs. (24) and (25):

$q_1(z;\lambda) \equiv q_2(z;\lambda)$

Therefore:

$Q_1(r;\lambda) \equiv Q_2(r;\lambda)$

Q. E. D.

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[6] Feller, *op. cit.*, p. 269, *Theorem.* Also see Knopp, Konrad, *Elements of the Theory of Functions.* Dover #S154 (1952), p. 88.