# MATHEMATICAL LIMITS TO THE JUDGMENT FACTOR IN FIRE SCHEDULE RATING

BY

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## DISCUSSION BY LESTER B. DROPKIN

The present paper by Mr. McIntosh joins a growing list of recent contributions to the Society which are united in the belief that real value results from approaching a given subject of actuarial interest by theoretical means. With such a viewpoint, this reviewer is heartily in accord.

The actuary dealing with the charges of a fire rating schedule is faced with a mass of unknowns. To find order and useful relationships within this mass, the author sets up a mathematical model involving a system of linear equations, in which the number of unknowns exceeds the number of equations.

Consideration of the validity of the author's model, I leave to others who have a much greater knowledge of, and intimacy with the fire field—my remarks will concentrate on the more mathematical aspects of the paper.

With the author, we speak of m equations in n unknowns, $n > m$. There is essentially only one way to proceed and that is to consider n-m of the charges as being "independent variables" or parameters. Upon selecting the charges which are to serve as parameters, the original system of equations may be transformed and reduced to a system of m equations in which each of the "dependent variables," or non-parametric charges, is expressed as a linear combination of the n-m parameters.

It is this step, this transformation and reduction, this display in a concrete and specific way of the dependence relationship, that constitutes a key contribution by Mr. McIntosh. For, while we may know in a general and conceptual sense that it is possible to so transform and reduce the original equations, no progress is in fact possible until the deed is actually done and the specific relationships exhibited.

The author casually refers us to "conventional techniques" to accomplish the desired transformations. Such conventional techniques, however, often involve a good deal of laborious work. Set forth in an appendix to this review is an extremely convenient method for handling systems of linear equations. The method may have its advantages, since in checking out the numerical values of the paper there was at least one instance where the author's values checked out only by rounding while the values produced by the appended method checked out exactly.

It is, of course, clear that any n-m of the charges may be chosen as parameters. Different solutions emerge by choosing different sets of charges as the parameters, and by choosing different ways in which the parameters are to be fixed or limited.

A second contribution of the paper is the recognition by the author that if the several dependence relationships are made subject to certain restrictions, such as that each charge must be non-negative, and if they are considered conjointly, then delimiting the possible range of one charge results in a concommitant limitation of other charges.

It is somewhat unfortunate that because the author's presentation has been interlarded with verbal encrustations of unnecessary mathematical jargon, the paper has been made much more difficult to approach than, perhaps, need have been the case. The basic ideas of the paper are essentially simple and this reviewer would have preferred a more simple exposition of them.

Mr. McIntosh has, perhaps unwittingly, imposed a heavy burden upon himself for I am sure that this Society will be looking forward to future papers in which he will carry forward the ideas and conceptions of the present notable contribution.

## APPENDIX

The following procedure is based upon Crout's modification of the standard method of elimination in solving systems of linear equations.[1] It is an extremely simple and convenient procedure to use when calculations are performed on a desk-type calculator since such calculators enable one to find the sum of a series of products and, if desired, to make a final division in one continuous machine operation.

It will be recalled that we start with a system of m equations in n unknowns (with $n > m$) each of which may be expressed as:

$$R_i = \sum_{j=1}^{n} A_{ij} P_j \; ; \tag{1}$$

the m equations being developed as i runs from 1 to m. After selecting any n-m of the P's to serve as parameters, we wish to express each of the m remaining P's as a linear combination of the n-m parameters. If, conforming to the notation of the paper, we let $r = n\text{-}m$ and identify the parameters by the subscripts 1 to r, and the remaining m variables by the subscripts $r + 1$ to n, the resulting equations will be of the form:

$$P_j = \left( \sum_{l=1}^{r} w_{jl} P_l \right) + w_{j0}\,, \tag{2}$$

with $j = r + 1, \ldots, n$.

(While in the paper each of the n-m parameters is also expressed as a linear combination, this is a detail and unnecessary for the purposes of this appendix.)

We deal with the system of equations in a condensed shorthand form, writing down only coefficients and constants. The form is a rectangular array—otherwise known as a matrix. While it is convenient to describe an array by the term "matrix," knowledge of matrix algebra is neither necessary nor used here.

The method consists of writing down the given equations (1) in con-

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[1] For the mathematical basis of Crout's method see, for example, "Numerical Analysis" by K. S. Kunz, Chapter 10, McGraw-Hill Book Company, Inc., New York, 1957.

densed form—the given matrix; forming one matrix—the auxiliary matrix; and a set of final results—the final matrix.

To explain and illustrate the method, we use the hypothetical example given in the paper. In particular, we illustrate the case when charges $P_1$, $P_2$, and $P_4$ are taken as parameters. We proceed as follows:

*Step 1:* Mentally transpose the terms in the given equations so that the constants ($R_i$) and the parameters are on the right side; arrange the order of the variables on the left side so that a non-zero coefficient appears as the first coefficient of the first equation. Write down the coefficients and constants to get the given matrix.

In our example, we have:

| $P_5$ | $P_6$ | $P_3$ | R | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|---|---|
| .20 | 1.0 | 0 | .40 | −1.0 | − .25 | 0 |
| 0 | 1.0 | .50 | .55 | 0 | −1.0 | —.6 |
| 0 | 1.0 | 1.0 | .42 | − .40 | 0 | −.3 |

*Step 2:* Form the auxiliary matrix from the given matrix according to the following rules.

(a) The various numbers, or elements, are determined in the following order: elements of the first column, then elements of the first row to the right of the first column; elements of second column below first row, then elements of the second row to right of second column; elements of third column below second row, then elements of third row to right of third column; etc.

(b) The first column is identical with the first column of the given matrix. Each element of the first row, except the first, is obtained by dividing the corresponding element of the given matrix by that first element.

In our example, we have, to this point:

| $P_5$ | $P_6$ | $P_3$ | R | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|---|---|
| .20 | 5.0 | 0 | 2.0 | −5.0 | −1.25 | 0 |
| 0 | | | | | | |
| 0 | | | | | | |

(c) Each element on or below the principal diagonal is equal to the corresponding element of the given matrix minus the sum of those products of elements in its row and the corresponding elements in its column (in the auxiliary matrix) which involve only previously computed elements.

(d) Each element to the right of the principal diagonal is given by a calculation which differs from (c) only in that there is a final division by its diagonal element (in the auxiliary matrix).

Applying (c) and (d) step by step, we have:

| $P_5$ | $P_6$ | $P_3$ | $R$ | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|---|---|
| .20 | 5.0 | 0 | 2.0 | —5.0 | —1.25 | 0 |
| 0 | 1.0* | | | | | |
| 0 | 1.0† | | | | | |

| | | | | | | |
|---|---|---|---|---|---|---|
| * 1.0 = | 1.0 | − ( | 0 | × | 5.0 | ) |
| | from Row 2, Col. 2 of gv. matrix | | from Row 2, Col. 1 of aux. matrix | | from Row 1, Col. 2 of aux. matrix | |
| † 1.0 = | 1.0 | − ( | 0 | × | 5.0 | ) |
| | from Row 3, Col. 2 of gv. matrix | | from Row 3, Col. 1 of aux. matrix | | from Row 1, Col. 2 of aux. matrix | |

Then:

| $P_5$ | $P_6$ | $P_3$ | $R$ | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|---|---|
| .20 | 5.0 | 0 | 2.0 | —5.0 | —1.25 | 0 |
| 0 | 1.0 | .50* | .55† | 0‡ | —1.0‡ | —.60‡ |
| 0 | 1.0 | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| * .50 =( | .50 | − | 0 | × | 0 | ) ÷ | 1.0 |
| | Row 2, Col. 3 gv. matrix | | Row 2, Col. 1 aux. matrix | | Row 1, Col. 3 aux. matrix | | Row 2, Col. 2 aux. matrix |
| † .55 =( | .55 | − | 0 | × | 2.0 | ) ÷ | 1.0 |
| | Row 2, Col. 4 gv. matrix | | Row 2, Col. 1 aux. matrix | | Row 1, Col. 4 aux. matrix | | Row 2, Col. 2 aux. matrix |

‡ Determined in the same manner as the .50 and .55.

Then:

| $P_5$ | $P_6$ | $P_3$ | $R$ | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|---|---|
| .20 | 5.0 | 0 | 2.0 | —5.0 | − 1.25 | 0 |
| 0 | 1.0 | .50 | .55 | 0 | − 1.0 | − .60 |
| 0 | 1.0 | .50* | —.26† | − .80 | 2.0 | .60 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| * .50 = | 1.0 | − ( | 0 | × | 0 | + | 1.0 | × | .50 | ) | | |
| | R3 C3 gv. | | R3 C1 aux. | | R1 C3 aux. | | R3 C2 aux. | | R2 C3 aux. | | | |
| † —.26 = | [.42 | − ( | 0 | × | 2.0 | + | 1.0 | × | .55 | )] | ÷ | .50 |
| | R3 C4 gv. | | R3 C1 aux. | | R1 C4 aux. | | R3 C2 aux. | | R2 C4 aux. | | | R3 C3 aux. |

*Step 3:* Form the final matrix from the auxiliary matrix according to the following rules:

(a) The elements of each column to the right of the vertical line are determined in the following order: last, next to last, second from last, etc.

(b) The last element in each column is identical to the corresponding element in the corresponding column of the auxiliary matrix.

(c) Each element is equal to the corresponding element of the corresponding column of the auxiliary matrix minus the sum of those products of elements in its row in the auxiliary matrix to the right of the principal diagonal and corresponding elements in its column in the final matrix which involve only previously computed elements.

In our example, the final matrix is first:

| | W | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|
| $P_5$ | | | | |
| $P_6$ | .68* | .40† | −2.0 | −.90 |
| $P_3$ | −.26 | −.80 | 2.0 | .60 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| * .68 = | .55 (R2 C4 aux.) | − ( | .50 (R2 C3 aux.) | × | −.26 (R3 C4 final) | ) |
| † .40 = | 0 (R2 C5 aux.) | − ( | .50 (R2 C3 aux.) | × | −.80 (R3 C5 final) | ) |

Then:

| | W | $P_1$ | $P_2$ | $P_4$ |
|---|---|---|---|---|
| $P_5$ | −1.4* | −7.0† | 8.75 | 4.5 |
| $P_6$ | .68 | .40 | −2.0 | −.90 |
| $P_3$ | −.26 | −.80 | 2.0 | .60 |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| * −1.4 | = | 2.0 (R1 C4 aux.) | − ( | 5.0 (R1 C2 aux.) | × | .68 (R2 C4 final) | + | 0 (R1 C3 aux.) | × | −.26 (R3 C4 final) | ) |
| † −7.0 | = | −5.0 (R1 C5) | − ( | 5.0 (R1 C2) | × | .40 (R2 C5) | + | 0 (R1 C3) | × | −.80 (R3 C5) | ) |

For the purposes of this illustration, we have done quite a bit of writing; in actual use, it is simple and fast. The interested reader may wish to check the following derivation of eq. (3a) of the paper.

| $P_5$ | $P_6$ | $P_4$ | R | $P_1$ | $P_2$ | $P_3$ |
|---|---|---|---|---|---|---|
| *Given* | | | | | | |
| .20 | 1.0 | 0 | .40 | −1.0 | −.25 | 0 |
| 0 | 1.0 | .60 | .55 | 0 | −1.0 | −.50 |
| 0 | 1.0 | .30 | .42 | −.40 | 0 | −1.0 |
| *Aux.* | | | | | | |
| .20 | 5.0 | 0 | 2.0 | −5.0 | −1.25 | 0 |
| 0 | 1.0 | .60 | .55 | 0 | −1 | −.50 |
| 0 | 1.0 | −.30 | .433 | 1.333 | −3.333 | 1.667 |
| *Final* | | | | | | |
| | | $P_5$ | .550 | −1.0 | −6.25 | 7.5 |
| | | $P_6$ | .290 | −.80 | 1.0 | −1.5 |
| | | $P_4$ | .433 | 1.333 | −3.333 | 1.667 |