MATHEMATICAL LIMITS TO THE JUDGMENT FACTOR IN FIRE SCHEDULE RATING

BY

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INTRODUCTION

E. G. Richards has stated: "* * * if *experience* is to measure fire insurance costs, it will show that the rate upon a specific risk should be the same as the average rate of its class * * *"[1]. Mr. Richards continues to the effect: "By the [schedule rating] method the charge or credit for each separate part or use of a risk is of necessity theoretical, its cost being purely an estimate unsubstantiated by actual experience, because no way has yet been discovered for subdividing the underwriter's outgo into separate parts corresponding to the separate structural parts or uses of the risk as provided for in existing rating schedules."[2] More recently, Longley-Cook has summarized the schedule rating process thusly: "* * * a schedule rating plan with numerous credits and debits for favorable and unfavorable features may be established * * *. Rate level adjustments, based on loss ratio developments will be made to insure the overall adequacy of the rates, but the individual debits and credits continue to be based on judgment *alone*."[3] (Emphasis added.) These authors, writing independently some 45 years apart, have expressed a concept echoed by others and long accepted as an axiom of the fire schedule rating process, that no mathematical basis whatever exists for the individual charge or credit of the rating schedule, hence that the specific rate of the individual schedule-rated risk must rest solely upon judgment.

Any suggestion that the charges and credits of any fire rating schedule are or can be rigorously computed from actuarial data would be absurd under present circumstances. However, anyone who has ever been faced with the problem of actually assigning values to the specific charges and credits of a specific schedule for actual application in the field has sooner or later (usually sooner) faced the choice either of modifying his "judgment" or of disregarding completely the "overall adequacy of the rates", if the term "rates" is understood to mean the average rates respectively indicated for the several classes to which a schedule may apply.

The earliest recorded instance of mathematical bounds to the schedule-maker's judgment is, to the author's best knowledge, to be found in the writings of A. F. Dean, perhaps the most vociferous advocate of judgment ever known to the fire insurance industry. Following detailed explanation and defense of the analytical basis for the occupancy charges of the Dean Schedule, we find this confession:

[1] E. G. Richards, *The Experience Grading and Rating Schedule.* The National Board of Fire Underwriters. New York. (1915). P. 14.
[2] *Ibid.* P. 16.
[3] Laurence H. Longley-Cook, *Notes on Some Actuarial Problems of Property Insurance.* Reprinted in *Fire Insurance Rate Making and Kindred Problems.* C.A.S. (1960). P. 89 ff.

"The same basic occupancy charges for D buildings as for B buildings *would materially disturb the relations* established by usage in the rates [of certain categories of risks].

*"In lieu of the classification of combustibility the exigencies of the case have been met by selecting a basic occupancy charge and apportioning same * * * in such a manner as to produce a charge in cents * * * approximately equal to that which has been sanctioned by years of usage."*[4]

(Emphasis added.)

The point to be emphasized here is that certain rate *relationships* had to be met, and it was possible to meet them only by selecting and apportioning certain charges in a certain manner. The fact that the target rates to which Dean referred were themselves "sanctioned by years of usage", hence were based upon judgment, is in present context both irrelevant and immaterial. The simple fact is that the target rates forced a modification of judgment in selection and apportionment of individual charges. Had there been actuarial justification for the target rates, there would have been actuarial support for the schedule charges thereby indicated.

More modern examples of the interlock between target class rate levels and the charges of the applicable schedule may be taken from the operation (still in progress) of revising the *Uniform Grading Schedule,* or *"U.G.S."*, of the Middle Department Association of Fire Underwriters into the *Louisiana Uniform Grading Schedule,* or *"La. U.G.S."*

Detailed description of this schedule is not necessary, but certain of its characteristics should be explained. There is no "basis rate" as the term is commonly used; separate charges are provided for individual hazards. All charges are in "points" (to avoid decimals in the body of the rate calculation) and the final point total is multiplied by a so-called "rate conversion factor" to produce the rate.[5]

The exact values of charges are not important here, but certain ratios between charges are significant. In the original *U.G.S.*, the ratio of the frame wall charge divided by the joist floor-roof charge is 1.25. Also, the *U.G.S.* occupancy charges in frame construction are, on the average, about 1.4 times the corresponding occupancy charges in brick. In the *La. U.G.S.* the wall/floor-roof ratio is 0.8 and the frame/brick occupancy ratio is 1.2. The inversion of the one ratio from 1.25 to 0.8 and reduction of the other from 1.4 to 1.2 appear to reflect conflict of judgment as between Philadelphia and New Orleans. What these revisions actually reflect is *not* conflict of judgment but significant differences in the rate levels required in Pennsylvania and Louisiana respectively.

When the *La. U.G.S.* was originally filed (1953), classified experience of

[4] A. F. Dean, *The Philosophy of Fire Insurance,* edited by W. R. Townley. 3 Vol. Edward B. Hatch. Chicago. (1925). Vol. I. P. 281. (Original reference unknown to this author.)

[5] Arithmetically, the rate is the same as would be obtained if the rate conversion factor were taken as a "basis rate", and the point charges converted to appropriate percentages thereof.

more than minimal credibility indicated that pre-existing brick rate levels should be continued, but that pre-existing frame rate levels should be reduced by about 20%. Although the *overall* rate level could easily be adjusted by the rate conversion factor, the original *U.G.S.* could not be made to produce in Louisiana the *comparative* class rate levels required, until the indicated changes were accomplished.[6] Further revisions necessary to meet the class rate levels required in Louisiana included, among others, increasing the ratio between ordinary mercantile occupancy and office occupancy from 3.75/2.25 in the original to 3.75/1.60 in the *La. U.G.S.*, major adjustment of certain exposure charges, extension of a credit table by which structural charges are modified for internal exposure from occupancy and extremely drastic reduction in charges determining differentials between the rate on contents and the rate of the containing building.

It is obvious that once the point charges had been adjusted to meet comparative target rate levels, adjustment of the rate conversion factor to meet the required overall rate level then in effect automatically adjusted individual charges to absolute values which were definitely related to the class rate levels. The overall operation was by no means judgment-free, but the final result cannot be said to rest upon judgment *alone*.

It is not the purpose here to discuss methods of establishing the target class rate levels in the first place.[7] We here assume that a definite pattern of target levels has been pre-determined by appropriate methods, and consider only the problem of designing a schedule to produce this pre-determined rate pattern.

The existence of mathematically rigorous limits to the value of an individual charge can be demonstrated very easily, but these are not the final bounds to the fire ratemaker's judgment. It will be shown that certain complete combinations of charges are forbidden *as combinations* even though the individual charges may all be estimated within their respective individual limits. It is the existence of such forbidden combinations which constitutes the final and sometimes narrow restriction upon the exercise of judgment in preparing the schedule.

By analogy, a fire rating schedule may be likened to a house of cards. Incautious movement of one card can result in collapse of the entire interlocked pile. The rigorous consequences of incautious tampering with a single schedule charge may snowball into completely unacceptable distortions of the entire rate structure. This fact is not obvious, though the practicing ratemaker soon learns it by experience. We here propose to demonstrate that such is the case, and in so doing, will have displayed the limits within which judgment must be exercised if the overall pattern of rates is to exhibit both adequacy and consistency.

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[6] To have placed a lower rate conversion factor on frame than on brick would have led to serious complications with mixed construction.

[7] The interested reader is referred to the several excellent articles on this subject which appear in *Fire Insurance Rate Making and Kindred Problems*. C.A.S. New York. (1960).

FIRE SCHEDULE ALGEBRA

Statement of Theory

The author has noted previously that the class rate levels produced by a fire rating schedule may be expressed as a set of simultaneous equations in which the several charges of the schedule appear as variables, the co-efficients reflect actually existing field conditions and the class rate levels appear as the constant terms.[8] Assuming these equations to be consistent and not redundant, it is immediately apparent that if the number of equations were at least equal to the number of schedule charges, a unique solution would follow by elementary (though tedious) algebra, thereby eliminating all further exercise of judgment once the class rate levels had been pre-determined. In practice, however, the number of charges invariably exceeds the number of rating classes. Not only is unique solution impossible, the number of solutions will be infinite. A little reflection shows that the number of free choices permitted the ratemaker will be equal only to the difference between the number of charges and the number of equations, not to the full number of the charges themselves, but the equations impose no limits whatever upon the exercise of any or all of the choices permitted. So far, judgment is still unbounded for all practical purposes.

Any attempt to refine the classification plan to increase the number of classes to a figure equal to the number of schedule charges can only result in the loss of all statistical credibility in the classified loss experience. The number of equations, therefore, cannot be increased without impairing and perhaps destroying all ratemaking significance of the loss experience itself. We can, however, supplement the equations with inequalities. The system of m equations in n unknowns where $(n > m)$ is readily converted to a system of m' *significant* inequalities where $(m' \geqq n)$. Solution of the simultaneous inequalities does not yield a unique set of charges to produce the required rate levels (except possibly in special cases). It does, however, establish: (a) mathematically rigorous limits to the exercise of judgment; (b) practical limits somewhat elastic but considerably narrower than the rigorous limits. The solution also will display in mathematical expression the "house of cards" structure of the schedule as an entity.

Inequalities are derived from two sources, one mathematical and the other engineering. Mathematically, all probabilities must be non-negative. A rigorous implication is that in theory all charges of the schedule must be non-negative.[9] This fact serves to establish inequalities equal in number to the number of charges in the schedule. An additional series of inequalities is

[8] Kenneth L. McIntosh. *The Rationale of the Fire Schedule—Part I, Theory. The Annals of the Society of C.P.C.U.*, Vol. 13, P. 8 ff. (Summer, 1960).

[9] The negative "charges" (*i.e.* credits) in many schedules are empirical. A multiplicative credit can be and for certain manipulations must be converted to the positive equivalent by subtracting the credit from 100% (or from 1.00). An additive credit reflects the absence of a hazard elsewhere blanketed with other hazard(s) under a single compound charge. *E.g.*, where the basis rate of a brick building contemplates joist floors and roof, the schedule may contain additive credits for concrete floor and for incombustible roof. If the compound charge in such cases is broken down into its several specific components, these are all non-negative. The additive credit, or negative "charge", will be no longer necessary.

based upon axioms such as that "wood burns more readily than concrete", the denial of which seems less a matter of "judgment" than an excursion into absurdity. Inequalities from this source, which might be termed "engineering" inequalities, may in certain cases be superfluous, but in general they will not all be superfluous. Between the mathematical requirement that charges be non-negative and the engineering axioms as exemplified, we will normally wind up with the number of significant inequalities greater than the number of schedule charges.

Unfortunately, simultaneous inequalities are not so easily manipulated as simultaneous equations. To solve the problem, it is convenient to turn to matrix algebra. If we express the several charges of the schedule as components of a column vector, we will find bounds to the set of all such vectors whose components satisfy the rate level equations, the mathematical requirement that all charges be non-negative and the engineering axioms. The properties of and bounds to the set of vectors will be found to constitute the final limitations upon the exercise of the ratemaker's judgment in the evaluation of the charges of the schedule once the target class rate levels have been pre-set.

By geometric analogy, we may think of an empty box and may pick one corner of it as the origin of a coordinate system. We take a marble and place it anywhere we please with respect to the origin. Now let any three of the components of the vector be the coordinates of the position of the marble. The remaining components will be functions of the three coordinate-components, and thus there will be one combination of components, *i.e.* one specific vector, associated with any given position of the marble in all space. Any certain one of these vectors represents a combination of schedule charges which will produce the required rate levels, but we find that *if the marble is placed outside of the box at least one component of the associated vector will be negative. Therefore, to avoid violating the axiom that all schedule charges must be non-negative, we must keep the marble inside of the box at all times.*

This restriction obviously limits the values assumed by the three coordinate-components, which in turn limits the values which any of the remaining components may assume as functions of the coordinates. Thus there will be limits to the values assumed by each charge of the schedule.

Furthermore, it must be remembered that one specific vector will be uniquely associated with any given position of the marble, and the required rate levels will be produced *only* by a *combination* of charges displayed as the components of one of these vectors. The combinations displayed depend upon the functions which relate the balance of the components to the three coordinate-components; thus certain entire combinations are forbidden *as combinations* regardless of all other considerations.

The shape and size of the box within which the marble must be kept, *i.e.* the mathematical bounds of the vector set, are determined by the pattern of target class rate levels, by the actually existing distribution of fire hazards among the risks to be rated and by the amounts of insurance carried on individual risks. The ratemaker can control only the first of these, and by the introductory assumption of pre-determined class rate levels we have denied him even that measure of freedom. We have placed him in a box and nailed the lid down. He may jump around inside, but he cannot get out.

The analogy may be extended to incorporate the engineering axioms ("wood burns more readily than concrete", etc.) if we now imagine the inside of the box to be subdivided into compartments by strips of sheet-rubber of varying degrees of elasticity. Although mathematically the marble may be anywhere inside the box, for consistency of the rate structure it must be kept within a certain compartment. As the divider strips are elastic to a degree, we may push the compartment walls somewhat out of shape, but if we go over into the next compartment we find that we must rate masonry higher than frame (other things equal), or perhaps rate an office higher than the carpentry shop in a similar building next door.

A geometric representation can be exact only when the number of schedule charges exceeds the number of the pre-determined rate levels by not more than three (otherwise more than three coordinates will be needed to express the position of the marble), but the analogy is mathematically valid regardless of how many charges and how few rate levels we assume. We cannot visualize an x-dimensional box where $x > 3$, but we still may manipulate in the abstract an n-component vector in the x-dimensional bounded set as easily as we manipulate the marble in a 3-dimensional box.

Though such an approach may depart from historically conventional approaches to the fire rating problem, it offers one tremendous advantage. By locking the schedule charges into a single vector and in turn locking that vector into a bounded set, complete mathematical expression in a single equation may be given to the entire pattern of class rate levels, the entire pattern of schedule charges, the actually existing field conditions, the mathematical axiom of non-negative charges and such engineering axioms as seem appropriate in a given case. It is only when all of the interlocking relationships existing within and between each and all of these several elements have been mathematically formulated in a single, readily-manipulated expression that significant mathematical bounds to judgment may be recognized. The simple scrutiny of individual schedule charges does not and cannot reveal their existence.

Basically, the whole problem would resolve itself into the extremely elementary problem of simultaneous equations if the inherent characteristics of fire risks would permit breakdown for statistical purposes into at least as many classes as there are charges in the schedule. We could then formulate a number of significant equations at least equal to the number of unknowns to be determined. Since we cannot change the inherent characteristics of the risks to be rated, we must turn to limiting inequalities for irremediable lack of determinative equations. We will find the inequalities to be perhaps more restrictive than is generally realized.

Mathematical Development

For simplicity of presentation, we make two restrictive assumptions:

1. All risks of all classes are equi-valued and carry the same percentage of insurance to value. As the effect upon the equations following of relaxing this restriction will be completely obvious, no further discussion seems necessary.

2. All charges of the schedule are additive.[10] The implications of this restriction are discussed in the Appendix. It can be relaxed in the interests of generality, but only at the cost of introducing mathematical complexities it is desired to avoid here. The historical examples given in the INTRODUCTION with specific reference to conversion of the Middle Department *U.G.S.* into its offspring, the *La. U.G.S.*, include application to multiplicative charges.

With these assumptions, we now write:

$$A_{11}P_1 + A_{12}P_2 + \cdots + A_{1j}P_j + \cdots + A_{1n}P_n = R_1 \tag{1.1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{i1}P_1 + A_{i2}P_2 + \cdots + A_{ij}P_j + \cdots + A_{in}P_n = R_i \tag{1.i}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$A_{m1}P_1 + A_{m2}P_2 + \cdots + A_{mj}P_j + \cdots + A_{mn}P_n = R_m \tag{1.m}$$

and we have:

R_i = The pre-determined target rate level for the ith class.

P_j = The schedule charge reflecting contribution to loss expectation of a specific feature of hazard, "Hazard j".

A_{ij} = A factor reflecting the distribution of Hazard j as it exists in the ith class. *E.g.*, if Hazard j is combustible wall construction A_{ij} will be the average percentage of combustible wall construction found by inspection to exist in the several risks of Class i. It follows that all A_{ij} will be non-negative.

In final formulation, equations (1) will be neither inconsistent nor redundant.[11] By completely conventional techniques, therefore, they may be solved for any chosen group of charges numbering m, in terms of the remaining (n-m) charges which serve as parameters. As the several charges may be numbered in any way we please, there is no loss of generality in choosing the first (n-m) charges as parameters. There will be obtained a new system of equations of the form:

$$P_1 w_{j1} + P_2 w_{j2} + \cdots + P_{(n-m)} w_{j(n-m)} + w_{j0} = P_j \tag{2}$$

where $(n\text{-}m) < j \leq n$.

To facilitate the transition to vector notation, we also formulate (n-m) additional equations using the tautology that $P_j = P_j$ where $j \leq (n\text{-}m)$. If for simplicity of notation we now let: $r = (n\text{-}m)$; $s = (n\text{-}m + 1)$; $t = (n\text{-}m + 2) \ldots$ we have:

$$P_1(1) + P_2(0) + \cdots + P_r(0) + 0 = P_1 \tag{2.1}$$
$$P_1(0) + P_2(1) + \cdots + P_r(0) + 0 = P_2 \tag{2.2}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_1(0) + P_2(0) + \cdots + P_r(1) + 0 = P_r \tag{2.r}$$
$$P_1(w_{s1}) + P_2(w_{s2}) + \cdots + P_r(w_{sr}) + w_{s0} = P_s \tag{2.s}$$
$$P_1(w_{t1}) + P_2(w_{t2}) + \cdots + P_r(w_{tr}) + w_{t0} = P_t \tag{2.t}$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$P_1(w_{n1}) + P_2(w_{n2}) + \cdots + P_r(w_{nr}) + w_{n0} = P_n \tag{2.n}$$

[10] If a basis rate of the *Analytic System* or similar schedule is multiplied separately by each of the individual percentage charges the result is a series of flat charges to be added into the final rate. These charges are thus in fact additive despite multiplicative appearance.

[11] See Appendix. Secs. 1 & 2.

Equations (2) may be immediately rewritten as:

$$
P_1 \begin{array}{c} W_1 \\ \begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ w_{s1} \\ w_{t1} \\ \cdots \\ w_{n1} \end{bmatrix} \end{array}
+ P_2 \begin{array}{c} W_2 \\ \begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \\ w_{s2} \\ w_{t2} \\ \cdots \\ w_{n2} \end{bmatrix} \end{array}
+ \cdots + P_r \begin{array}{c} W_r \\ \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ w_{sr} \\ w_{tr} \\ \cdots \\ w_{nr} \end{bmatrix} \end{array}
+ \begin{array}{c} W_0 \\ \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ w_{s0} \\ w_{t0} \\ \cdots \\ w_{n0} \end{bmatrix} \end{array}
= \begin{array}{c} X \\ \begin{bmatrix} P_1 \\ P_2 \\ \cdots \\ P_r \\ P_s \\ P_t \\ \cdots \\ P_n \end{bmatrix} \end{array}
\qquad (3)
$$

New let:

$P_j = \alpha_j P_1$ where $(1 < j \leqq r)$

and let:

$v_j = w_{j1} + \alpha_2 w_{j2} + \cdots + \alpha_r w_{jr}$ where $(r < j \leqq n)$ (4)

Substituting for $P_2 \cdots P_r$ in the left member of equation (3), and substituting "$(P_j \geqq 0)$" for "P_j" in the right member for all j, after certain manipulations and application of equation (4), we obtain:[12]

$$
P_1 \begin{array}{c} V_1 \\ \begin{bmatrix} 1 \\ \alpha_2 \\ \cdots \\ \alpha_r \\ v_s \\ v_t \\ \cdots \\ v_n \end{bmatrix} \end{array}
+ \begin{array}{c} W_0 \\ \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ w_{s0} \\ w_{t0} \\ \cdots \\ w_{n0} \end{bmatrix} \end{array}
= \begin{array}{c} (X \geqq \mathit{0})^{(13)} \\ \begin{bmatrix} P_1 \geqq 0 \\ P_2 \geqq 0 \\ \cdots \\ P_r \geqq 0 \\ P_s \geqq 0 \\ P_t \geqq 0 \\ \cdots \\ P_n \geqq 0 \end{bmatrix} \end{array}
\qquad (5)
$$

Equation (5) is not an "equation" at all, properly speaking. It is a system of n inequalities which it is convenient to express in the form of an equa-

[12] See Appendix. Sec. 3.

[13] The symbol "***0***" should not be confused with the numeral "0". The italicized "***0***" designates the "null vector", *i.e.* the vector each of whose components is the number "0".

tion.[14] It should be noted that the right member displays the complete pattern of all schedule charges in order from P_1 to P_n, and incorporates the mathematical axiom ($P_j \geqq 0$) for all j. The left member incorporates the parameter P_1 and the $(r-1)$ ratios $\alpha_2 \cdots \alpha_r$,[15] thereby expressing the r choices permitted the ratemaker. As the constants v_j and w_{j0} are derived through equations (2), (3) and (4) from the rate levels R_i and coefficients A_{ij} of equations (1), the left member also reflects the pre-determined class rate levels and the conditions of hazard actually existing in the field. We have not yet recognized the engineering axioms such as that: "wood burns more readily than concrete", or "the expected loss to a protected risk is less than to a similar risk unprotected", etc., etc., but first let us examine the equation as it now stands.

Equation (5) defines a vector set of a particular type.[16] The set so defined may be designated "S". From the derivation of equation (5) it follows that a given combination of charges ${}_\nu P_j$ will satisfy equations (1) (*i.e.* will produce the rate levels R_i) and will also satisfy the axiom $P_j \geqq 0$ if and only if that same combination satisfies equation (5). As the several charges in order from ${}_\nu P_1$ to ${}_\nu P_n$ are the components of a vector X_ν, it then follows that the combination of charges ${}_\nu P_j$ will produce the rate levels R_i and will satisfy $P_j \geqq 0$ *if* and *only if* X_ν belongs to S as defined by equation (5).

It is easily shown that S is completely bounded;[17] hence the vector X_ν will not belong to S for all ν. For any ν such that X_ν does not belong to S, the entire combination of n charges ${}_\nu P_j$ will, by the foregoing argument, be forbidden.

It is extremely important to recognize that it is the combination of charges displayed by the vector X_ν in such cases which is prohibited *as a combination* regardless of the fact that every individual charge ${}_\nu P_j$ may be valued within its own individual limits. It is in the existence of such forbidden combinations rather than in the limits to individual charges (though these latter exist) that the significant mathematical bounds to the fire ratemaker's judgment have their being. The purely mathematical bounds to judgment are equivalent to the mathematical bounds of the vector set S defined by equation (5).

Completely generalized treatment of the engineering axioms is difficult if not impossible. To illustrate, however, assume that on the *reductio ad absurdum* basis of "wood burns more readily than concrete", etc., it is established that, *e.g.:* $P_1 > P_2$ and $P_s > P_t$. Any combination of ${}_\nu P_j$ such that ${}_\nu P_1 \leqq {}_\nu P_2$ or ${}_\nu P_s \leqq {}_\nu P_t$ is immediately excluded regardless of all other considerations. There may, however, be further consequences.

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[14] As the parameters of equation (3) may be renumbered providing the vectors $W_1 \ldots W_r$ are correspondingly re-numbered, there is no loss of generality in selection of P_1 as the parameter of equation (5).

[15] Not only do $\alpha_2 \ldots \alpha_r$ appear directly in rows 2 to r, it should be remembered that v_j is a function of those same ratios by equation (4).

[16] See Appendix. Sec. 4.

[17] See Appendix. Sec. 5.

Given that $P_s > P_t$, then by rows s and t of equation (5):

$$P_1 v_s + w_{s0} > P_1 v_t + w_{t0}$$

from which:

$$P_1 > \frac{w_{t0} - w_{s0}}{v_s - v_t} \tag{6}$$

If the right member of inequality (6) is greater than zero, we have established a lower limit[18] for P_1 greater than given by $P_j \geqq 0$. The revised limit of the parameter will in turn affect the lower limits of all P_j as calculated by equation (5). This matter is pursued further in the Appendix, but the interlocking structure of the house of cards is already apparent, the more so when by application of equation (4) to v_s and v_t we may obtain from inequality (6):

$$P_1 > \frac{w_{t0} - w_{s0}}{(w_{s1} - w_{t1}) + \alpha_2 (w_{s2} - w_{t2}) + \cdots + \alpha_r (w_{sr} - w_{tr})} \tag{7}$$

Inequality (7) shows the limits of the parameter of equation (5) to be functions of the ratios $P_j/P_1 = \alpha_j$ where $(1 < j \leqq r)$. *The ratemaker's "free" choices are not mathematically independent.*

Hypothetical Example.

To attempt illustration of the foregoing theory by the use of any actual example would introduce detail so complex that principle would certainly be obscured. For one thing, the necessary recognition of multiplicative charges and credits would, as noted, require the use of mathematical functions considerably more involved than have been developed. Further, it would be necessary to explain in full detail the structure of any particular schedule referred to; and, finally, the resulting equations might well be virtually impossible of manual solution. Admittedly what follows has been over-simplified and is unrealistic. It is intended as a demonstration of basic principle, not as an example of operational techniques.

Two parenthetical observations should be made here. First, slide-rule accuracy is the best to be expected in reproducing some of the calculations, despite the fact that for certain purposes additional decimals have been retained in results as shown. (Significant figures have been lost at certain intermediate stages of the calculation.) Secondly, specific equations below are identified with general equations previously developed by retention of the numbering with addition of a lower case letter suffix; *e.g.* equation (3a.) will be the result of entering a specific set of data into the general equation (3).

Assume a schedule of seven additive, non-negative charges, $P_1 \cdots P_7$. The schedule is applicable to three classes whose pre-determined rate levels are: $R_1 = 0.400$; $R_2 = 0.550$; $R_3 = 0.420$. The coefficients A_{ij} of equations (1) are assumed to reflect only the proportion of risks in Class i which exhibit

[18] Inequality (6) will sometimes be reversed to give an upper rather than a lower limit.

Hazard j, and thus may be determined by simple count of risks.[19] The values of A_{ij} are assumed as shown. By equation (1) we now have:[20]

$$1.000\,P_1 + 0.250\,P_2 + 0\,P_3 + 0\,P_4 + 0.200\,P_5 + 1.000\,P_6 + 0\,P_7 = 0.400 \tag{1.1a}$$

$$0\,P_1 + 1.000\,P_2 + 0.500\,P_3 + 0.600\,P_4 + 0\,P_5 + 1.000\,P_6 + 0\,P_7 = 0.550 \tag{1.2a}$$

$$0.400\,P_1 + 0\,P_2 + 1.000\,P_3 + 0.300\,P_4 + 0\,P_5 + 1.000\,P_6 + 0\,P_7 = 0.420 \tag{1.3a}$$

It is immediately obvious that the P_7 terms should be dropped from all equations, as $A_{17} = A_{27} = A_{37} = 0$. This does not necessarily imply dropping the charge P_7 from the schedule unless the hazard reflected by P_7 is totally absent from all risks of all classes. It may and does happen, however, that a condition felt to be significantly hazardous will be found only in unusual risks too few in number to form a separate class. We may find that the values of A_{i7} are: $A_{17} = 0.00002$; $A_{27} = 0.00003$; $A_{37} = 0.00001$. For all practical purposes, these values become zero and the terms should be dropped, but the charge still may be retained for application to the vanishing percentage of atypical risks exhibiting the hazard.

After dropping the P_7 terms, equations (1.1a), (1.2a) and (1.3a) may be reduced to the following forms:

a. Choosing P_1, P_2 and P_3 as parameters:

$$P_1 \overset{{}_aW_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1.333 \\ -1.000 \\ -0.800 \end{bmatrix}} + P_2 \overset{{}_aW_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ -3.333 \\ -6.250 \\ 1.000 \end{bmatrix}} + P_3 \overset{{}_aW_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1.667 \\ 7.500 \\ -1.500 \end{bmatrix}} + \overset{{}_aW_0}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.433 \\ 0.550 \\ 0.290 \end{bmatrix}} = \overset{X}{\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}} \tag{3a}$$

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[19] Such counts are often made in practice as a routine preliminary to schedule revision. A sample of risks may be used if the class is large.

[20] *cf.* McIntosh. *Op. Cit.* P. 12 and footnote 3, P. 13. With an obvious change of notation and the addition of the terms in P_7, equations (1.1a), (1.2a) and (1.3a) will be recognized as equations (6.1a), (6.2a) and (6.3a) of the reference.

$$\begin{array}{ccccccc} & {}_aV_1 & & {}_aW_0 & & (X \geqq 0) & \\ P_1 & \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 \\ {}_av_4 \\ {}_av_5 \\ {}_av_6 \end{bmatrix} & + & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.433 \\ 0.550 \\ 0.290 \end{bmatrix} & = & \begin{bmatrix} P_1 \geqq 0 \\ P_2 \geqq 0 \\ P_3 \geqq 0 \\ P_4 \geqq 0 \\ P_5 \geqq 0 \\ P_6 \geqq 0 \end{bmatrix} & \end{array} \qquad 5(a)$$

where:

$$\left.\begin{aligned} {}_av_4 &= 1.333 - 3.333\alpha_2 + 1.667\alpha_3 \\ {}_av_5 &= -1.000 - 6.250\alpha_2 + 7.500\alpha_3 \\ {}_av_6 &= -0.800 + 1.000\alpha_2 - 1.500\alpha_3 \end{aligned}\right\} \qquad 4(a)$$

b. Choosing P_1, P_2 and P_4 as parameters:

$$\begin{array}{cccccccccc} & {}_bW_1 & & {}_bW_2 & & {}_bW_4 & & {}_bW_0 & & X \\ P_1 & \begin{bmatrix} 1 \\ 0 \\ -0.800 \\ 0 \\ -6.993 \\ 0.400 \end{bmatrix} & +P_2 & \begin{bmatrix} 0 \\ 1 \\ 2.000 \\ 0 \\ 8.762 \\ -2.000 \end{bmatrix} & +P_4 & \begin{bmatrix} 0 \\ 0 \\ 0.600 \\ 1 \\ 4.500 \\ -0.900 \end{bmatrix} & + & \begin{bmatrix} 0 \\ 0 \\ -0.260 \\ 0 \\ -1.400 \\ 0.680 \end{bmatrix} & = & \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix} \end{array} \qquad (3b)$$

$$\begin{array}{ccccccc} & {}_bV_1 & & {}_bW_0 & & (X \geqq 0) & \\ P_1 & \begin{bmatrix} 1 \\ \alpha_2 \\ {}_bv_3 \\ \alpha_4 \\ {}_bv_5 \\ {}_bv_6 \end{bmatrix} & + & \begin{bmatrix} 0 \\ 0 \\ -0.260 \\ 0 \\ -1.400 \\ 0.680 \end{bmatrix} & = & \begin{bmatrix} P_1 \geqq 0 \\ P_2 \geqq 0 \\ P_3 \geqq 0 \\ P_4 \geqq 0 \\ P_5 \geqq 0 \\ P_6 \geqq 0 \end{bmatrix} & \end{array} \qquad (5b)$$

where:

$$\left.\begin{aligned} {}_bv_3 &= -0.800 + 2.000\alpha_2 + 0.600\alpha_4 \\ {}_bv_5 &= -6.993 + 8.762\alpha_2 + 4.500\alpha_4 \\ {}_bv_6 &= 0.400 - 2.000\alpha_2 - 0.900\alpha_4 \end{aligned}\right\} \qquad (4b)$$

It is readily seen by inspection that upon setting all parameters of equations (3a) and (3b) equal to zero, the vectors ${}_aW_0$ and ${}_bW_0$ will be solutions to the respective equations. It is also seen that upon setting P_1 equal to zero,

${}_aW_0$ will be a solution to equation (5a), but the vector ${}_bW_0$ does *not* satisfy equation (5b). The 3rd and 5th components of ${}_bW_0$ are negative in violation of $P_j \geqq 0$ for all j.

Equations (5a) and (5b) define the same vector set, and we shall designate this set as S_{36}.[21]

The vector ${}_aW_0$ is a so-called "extreme point" of S_{36}.[22] The others may be found by reducing equations (1.1a), (1.2a) and (1.3a) to the form of equation (3) using in turn each of the 20 possible combinations of three parameter charges. Of the 20 vectors W_0 thereby resulting, twelve (including ${}_bW_0$), will be found to exhibit at least one negative component. The remaining eight (including ${}_aW_0$) exhibit only non-negative components and are the extreme points of S_{36}. The present importance is that each individual charge, P_j, will assume its absolute limiting values at one or more of these points.

Designating an extreme point as T_ν and letting ${}_aW_0 = T_1$, we have for the extreme points of S_{36}:

$$\begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \\ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.433 \\ 0.550 \\ 0.290 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \\ 0.193 \\ 0.756 \\ 2.000 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0.0880 \\ 0 \\ 0.140 \\ 0 \\ 0.378 \end{bmatrix} & \begin{bmatrix} 0 \\ 0.160 \\ 0.060 \\ 0 \\ 0 \\ 0.360 \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} T_5 & T_6 & T_7 & T_8 \\ \begin{bmatrix} 0 \\ 0.340 \\ 0.420 \\ 0 \\ 1.575 \\ 0 \end{bmatrix} & \begin{bmatrix} 0.363 \\ 0 \\ 0 \\ 0.917 \\ 0.188 \\ 0 \end{bmatrix} & \begin{bmatrix} 0.394 \\ 0.0250 \\ 0 \\ 0.875 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0.300 \\ 0.400 \\ 0.300 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

The absolute limits of the several charges P_j are thus seen to be (designating the point at which the upper limit is assumed):

$$(0 \leqq P_1 \leqq 0.394)(T_7);\ (0 \leqq P_2 \leqq 0.400)(T_8);\ (0 \leqq P_3 \leqq 0.420)(T_5);$$
$$(0 \leqq P_4 \leqq 0.917)(T_6);\ (0 \leqq P_5 \leqq 2.000)(T_2);\ (0 \leqq P_6 \leqq 0.378)(T_3).$$

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[21] The set is a 3-dimensional set of 6-dimensional vectors, hence the subscript "36". This notation is non-standard.

[22] See Appendix. Sec. 6.

These limits are obviously too broad to serve any but the academic purpose of showing that such limits do exist, but the vectors T_{ν} are extremely useful in certain other calculations[23] and the limits of P_j emerge.

Turning to equation (5b), however, we see that the ratemaker's judgment is not so unrestricted as the foregoing might indicate. In general terms, let it be estimated that the hazards reflected by P_1, P_2 and P_4, respectively, are about equally severe. We thus have: $P_1 \simeq P_2 \simeq P_4$; and $\alpha_2 = \alpha_4 = 1$. We assume no basis whatever for this estimate except pure judgment. Entering $\alpha_2 = \alpha_4 = 1$ into equations (4b):

$$\left.\begin{aligned} {}_b v_3 &= -0.800 + 2.000(1) + 0.600(1) = 1.800 \\ {}_b v_5 &= -6.993 + 8.672(1) + 4.500(1) = 6.269 \\ {}_b v_6 &= 0.400 - 2.000(1) - 0.900(1) = -2.500 \end{aligned}\right\} \qquad (4b.1)$$

whence we obtain:

$$\begin{array}{ccccccc} & {}_{b1}V_1 & & {}_bW_0 & & (X \geqq 0) & \\ P_1 & \begin{bmatrix} 1 \\ 1 \\ 1.800 \\ 1 \\ 6.269 \\ -2.500 \end{bmatrix} & + & \begin{bmatrix} 0 \\ 0 \\ -0.260 \\ 0 \\ -1.400 \\ 0.680 \end{bmatrix} & = & \begin{bmatrix} P_1 \geqq 0 \\ P_2 \geqq 0 \\ P_3 \geqq 0 \\ P_4 \geqq 0 \\ P_5 \geqq 0 \\ P_6 \geqq 0 \end{bmatrix} & (5b.1) \end{array}$$

From row 5 of equation (5b.1):

$6.269\ P_1 - 1.400 = (P_5 \geqq 0)$; whence: $P_1 \geqq 0.223$

and from row 6:

$-2.500\ P_1 + 0.680 = (P_6 \geqq 0)$; whence: $P_1 \leqq 0.272$

Entering these limits of the parameter into the equation, we find:

$$\begin{array}{cccc} & {}_bX_1 & & {}_bX_2 \\ \lim\limits_{P_1 \to 0.223} X = & \begin{bmatrix} 0.22 \\ 0.22 \\ 0.14 \\ 0.22 \\ 0 \\ 0.12 \end{bmatrix} & \textit{and: } \lim\limits_{P_1 \to 0.272} X = & \begin{bmatrix} 0.27 \\ 0.27 \\ 0.23 \\ 0.27 \\ 0.30 \\ 0 \end{bmatrix} \end{array}$$

The respective components of ${}_bX_1$ and ${}_bX_2$ are revised limits of the several

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[23] See Appendix. Sec. 6.

charges P_j arising from the estimate that $\alpha_2 = \alpha_4 = 1$, and are much narrower than the limits displayed above as components of the vectors T_ν. We may shrink them further.

Define P_1 as reflecting the hazard of full frame construction; P_5 as reflecting the hazard of a 25% frame attachment to a masonry building. The contribution of frame construction to loss expectation is not necessarily a linear function of perimeter or area percentage, but it is reasonable to assume, say, that $(0.10\,P_1 \leqq P_5 \leqq 0.40\,P_1)$. We now have from row 5 of equation (5b.1):

$$6.27\,P_1 - 1.40 = (0.10\,P_1 \leqq P_5 \leqq 0.40\,P_1)$$
$$(1.40/6.17) \leqq P_1 \leqq (1.40/5.87)$$
$$0.277 \leqq P_1 \leqq 0.239$$

Entering these parameter limits into equation (5b.1):

$$\lim_{P_1 \to 0.227} X = \begin{array}{c} {}_bX_3 \\ \begin{bmatrix} 0.23 \\ 0.23 \\ 0.15 \\ 0.23 \\ 0.02 \\ 0.11 \end{bmatrix} \end{array} \quad \textit{and:} \quad \lim_{P_1 \to 0.239} X = \begin{array}{c} {}_bX_4 \\ \begin{bmatrix} 0.24 \\ 0.24 \\ 0.17 \\ 0.24 \\ 0.10 \\ 0.08 \end{bmatrix} \end{array}$$

The limits to the several P_j as displayed in the components of ${}_bX_3$ and ${}_bX_4$ are elastic, obviously, since dependent upon the judgment that $\alpha_2 = \alpha_4 = 1$, and that $(0.10\,P_1 \leqq P_5 \leqq 0.40\,P_1)$, but they are extremely narrow. They could be stretched and still remain binding.

However, we may go still further. Let us define P_3 as reflecting the hazard of pig iron stocks; P_6 as reflecting the hazard of baled cotton. Now return to the limits of P_3 and P_6 as displayed in ${}_bX_1$ and ${}_bX_2$, noting that P_6 varies inversely with P_3. From ${}_bX_1$ we obtain a limiting ratio:

$$\frac{P_3}{P_6} = \frac{0.14}{0.12} = 1.2$$

which ratio increases as the parameter P_1 increases above its lower limit. Therefore, even though the P_3/P_6 ratio displayed in ${}_bX_2$ is indeterminate, a rigorous consequence of the judgment setting $P_1 \simeq P_2 \simeq P_4$ is that we must now set the occupancy charge for pig iron stocks at not less than 120% of the occupancy charge for baled cotton. "Judgment" or no "judgment" it might be advisable to re-evaluate the P_1, P_2, P_4 ratios. If we still have no mathematical indication of what the ratios α_2 and α_4 properly should be, we have a pretty clear mathematical indication of what they should *not* be. They should *not* be such that the ratemaker must subsequently charge a higher rate for pig iron than for baled cotton in order to break even in the overall.

Practical Application

Any direct practical application of the theory here proposed is presently impossible, but the author's own experience leads him to believe that conformity with the theory is implicit in the structure of any schedule producing pre-determined rate levels regardless of the operational techniques employed in schedule development. Some of the obstacles to direct application may be overcome in the future by electronic data processing. The first of these is obvious, the complexity of the calculations. Secondly, it will be recalled that the coefficients A_{ij} of equations (1) reflect the distribution of hazards actually exhibited by risks in the field. The raw data will be available on the rating inspection surveys, since inspections must be made regardless of whether the schedule itself is to be formulated by crystal gazing or by Mr. Einstein's Theory of Relativity. The transfer of such data from survey to punch card is, however, a manual process and, at present, a prohibitively expensive process.[24] If certain experiments now in progress with other goals in mind are ultimately successful, economically practicable solutions to the field data problem may emerge as by-products.

Data concerning values and amounts of insurance carried, obviously a major factor to be considered, also might someday become available through electronics.

The fact that as a general rule the number of schedule charges greatly exceeds the number of rating classes (particularly the number of classes even remotely credible) is not so formidable as it seems. The number of variables in the equations can be reduced by empirical means. To begin with, in any but the simplest schedules many of the charges reflect hazards found only in a very small proportion of risks. Though these charges must be retained in the schedule to rate the abnormal risk, they have no significant effect upon any class rate level because their coefficients approach zero. They should be dropped from the calculation and must be evaluated by comparative (not absolute) judgment. *E.g.*, the charge P_7 was dropped from equations (1.1a), (1.2a) and (1.3a) above. Having defined P_3 as pig iron and P_6 as baled cotton, if we now define P_7 as fireworks storage, we have in P_3 and P_6 a measure of sorts by which to judge P_7 and the value of P_6 will constitute a lower limit to P_7 unless someone cares to suggest that gunpowder is safer than cotton.

Furthermore, many of the remaining charges can be grouped at common values. This is illustrated by the numerous occupancy charges of the *Anayltic System* which are grouped into seven classes. This, of course, is judgment. It is precisely the type of judgment which must underly any rating method, namely the decision that thus-and-such a class shall be defined in exactly such-and-so a fashion.

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[24] At one stage of the *La. U.G.S.* operation, skeletonized data had to be transferred from 18,000 surveys to some 54,000 I.B.M. cards. The process might also be termed "endless", or so it seemed at the time.

Finally, the theory as presented is incomplete. Any direct practical operation will require extension to include multiplicative charges. Although a possible approach is suggested in the Appendix, there are problems involved for which no immediate solutions are offered. The author feels that a complete development might either parallel or ultimately converge upon Almer's theories of "factor analysis",[25] but this is pure conjecture.

CONCLUSION

There is no intent whatever to suggest that judgment has been, will be, can be or should be eliminated from fire schedule rating. Apart from all other considerations, it is completely obvious that no limits mathematically derived, as here, from a pattern of target class rate levels can be any more rigid than are those rate levels themselves, and the actuarial problems of fire loss credibility are far from solution.[26] But the fire ratemaker is permitted completely free exercise of judgment in constructing or revising his schedule only if he is willing to accept whatever class rate levels may result when the schedule is applied in the field. Where comparative class rate levels have been pre-set there are bounds beyond which the ratemaker's judgment must not carry him in thereafter evaluating the charges of the schedule. The *overall obsolute rate level* is easily adjusted by any of a number of simple techniques, but the ratemaker must make up his mind in advance whether to prejudge his *comparative class levels* or to pre-judge the values of his schedule charges. He cannot do both except by resorting to techniques which constitute the outright superposition of class rating methods upon the schedule rating process and which frequently lead to both theoretical absurdities and practical difficulties in field application.

It seems completely obvious that the class rate levels produced by application of any schedule under a given set of field conditions are mathematical functions of the several charges embodied in that schedule. This being so, the inverse relationships expressing the charges as functions of the class rate levels must exist, though we find these to be limiting upon rather than precisely determinative of the schedule charges. Equation (5) indicates the author's concept of the general shape of the relationships and equation (5) may be challenged, but the simple existence of such functions in some shape seems beyond question. If their existence in some shape is recognized, the proper role of judgment in fire schedule rating is seen in a perspective clearer than that sometimes employed in critical evaluation of the schedule rating process. The existence of mathematical limitations upon the exercise of judgment then becomes apparent and it becomes obvious that the more credible the classified fire loss experience, the more rigid such limitations will be.

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[25] B. Almer. *Risk Analysis in Theory and Practical Statistics.* T.XV I.C.A. Vol. 2. P. 314.

[26] *Cf.* Robert L. Hurley, *A Credibility Framework for Gauging Fire Classification Experience.* Reprinted in *Fire Insurance Ratemaking and Kindred Problems.* C.A.S. New York. (1960). P. 122.

APPENDIX

1. *Inconsistency of Equations (1)*

Equations (1) may be inconsistent for any or all of three reasons. First, the hazard analysis upon which the schedule structure is based may be in error. The ratemaker may have failed to reflect by separate charges significant differences between hazards mistakenly believed to be essentially identical in nature. The remedy is obviously to review the hazard analysis.

Secondly, random variation of classified loss experience less than fully credible may produce random variation in the pre-determined rate levels, R_i. In theory, the ratemaker would be justified in eliminating inconsistency from this source by arbitrary adjustment of R_i within the statistical confidence interval, though in present practice the confidence interval will not be known.[1]

Finally, the assumption unavoidable in schedule rating, that unanalyzable hazards (*e.g.* the morale hazard) will be uniformly distributed throughout all risks of all classes may have broken down in particular application.

In any case, consistency may be secured by empirical methods provided the methods used are appropriately reflected in the final form of the schedule. As a last resort, the offending equation(s) may be dropped and the class(es) involved be rated under separate schedule. This is an area where very definitely the judgment factor is paramount.

2. *Redundance of Equations (1)*

In practice, redundance of equations (1) will indicate serious error in hazard analysis. Either the ratemaker has failed to group two or more underwriting classes so similar that they should be consolidated for rating purposes even if remaining separate for underwriting, or he has failed to distinguish between classes of essentially dissimilar characteristics. Remembering that the coefficients A_{ij} reflect distribution of hazards in the field, anyone familiar with fire risks as they exist may estimate the likelihood that we will have $A_{ij} = cA_{ik}$, where c is any constant, for all i for any (j,k). The rest follows. As a practical matter, barring analytical error equations (1) will not be redundant, but the sceptic may bypass the question if he chooses. We have defined r by the equation: $r = (n - m)$. If we re-define r simply to be the number of parameters remaining in equations (2) and (3) after reduction of equations (1), we have $r \geqq (n - m)$ and the question of redundance in equations (1) becomes academic. The rest of the development still follows as presented.

3. *Derivation of Equation (5) from Equation (3)*

Equation (3) may be written in abbreviated notation as:

$$P_1W_1 + P_2W_2 + \cdots + P_rW_r + W_0 = X \tag{3A}$$

Letting $P_j = \alpha_j P_1$ where $1 < j \leqq r$; substituting:

$$P_1W_1 + \alpha_2P_1W_2 + \cdots + \alpha_rP_1W_r + W_0 = X$$

$$P_1(W_1 + \alpha_2P_1W_2 + \cdots + \alpha_rW_r) + W_0 = X$$

In full notation the parenthesis becomes:

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[1] *Cf.* Hurley. *Op. Cit.*

$$\begin{array}{ccccccccc}
W_1 & & \alpha_2 W_2 & & \alpha_r W_r & & V_1 & & V_1 \\
\begin{bmatrix} 1 \\ 0 \\ \cdots \\ 0 \\ w_{s1} \\ w_{t1} \\ \cdots \\ w_{n1} \end{bmatrix} & +\alpha_2 & \begin{bmatrix} 0 \\ 1 \\ \cdots \\ 0 \\ w_{s2} \\ w_{t2} \\ \cdots \\ w_{n2} \end{bmatrix} & +\cdots+\alpha_r & \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 1 \\ w_{sr} \\ w_{tr} \\ \cdots \\ w_{nr} \end{bmatrix} & = & \begin{bmatrix} 1 \\ \alpha_2 \\ \cdots \\ \alpha_r \\ w_{s1}+\alpha_2 w_{s2}+\cdots+\alpha_r w_{sr} \\ w_{t1}+\alpha_2 w_{t2}+\cdots+\alpha_r w_r \\ \cdots \\ w_{n1}+\alpha_2 w_{n2}+\cdots+\alpha_r w_{nr} \end{bmatrix} & = & \begin{bmatrix} 1 \\ \alpha_2 \\ \cdots \\ \alpha_r \\ v_s \\ v_t \\ \cdots \\ v_n \end{bmatrix}
\end{array}$$

The extreme right hand member follows by definition of v_j by equation (4):

$$v_j = w_{j1} + \alpha_2 w_{j2} + \cdots + \alpha_r w_{jr} \tag{4}$$

Substitution for the parenthesis gives immediately:

$$P_1V_1 + W_0 = X \tag{5X}$$

Substituting $(X \geqq \mathbf{0})$ for X in the right member of equation (5X):

$$P_1V_1 + W_0 = (X \geqq \mathbf{0}) \tag{5A}$$

which in full notation becomes equation (5).

4. *The Vector Set, S*

Equation (5) is, as has been noted, a system of n inequalities, and, therefore, defines the intersection of n half-spaces. Such an intersection defines a so-called "polyhedral, convex set."[2] In three dimensions, such sets may be geometrically represented by polyhedra, hence the term "polyhedral." A set is "convex" by definition if: Given that any two points are members of the set, then all points on the line segment joining the given points will also belong to the set.

For a polyhedral, convex set to be bounded, it is sufficient that the set not contain a ray.[3]

5. *To Prove that S is Bounded*

Equations (5) and (5A) define S, but equation (5X) is the completely general equation of a line. Whether or not the line defined by equation (5X) will intersect S will depend for all practical purposes upon the vector V_1,[4] which vector is a function of the ratios $\alpha_2 \cdots \alpha_r$, and may be conceived as the "slope" of the line. Assume V_1 to be such that the line does intersect S.

Returning to equations (1), choose the ith equation such that $A_{i1} > 0$,[5] and let all charges P_j except P_1 assume the lower limit of zero. Then:

$$A_{i1}P_1 + A_{i2}(0) + \cdots + A_{in}(0) = R_i \tag{1.i}$$

whence:

$$P_1 = R_i/A_{i1}$$

Since $A_{ij} \geqq 0$ for all (i,j), and also $P_j \geqq 0$ for all j, it is now obvious that if equation (1.i) is to be satisfied, then:

$$P_1 \leqq R_i/A_{i1}$$

Therefore, the intersection of the line defined by equation (5X) *for any V_1* will be not greater than is given by:

$$P_1V_1 + W_0 = X; \text{ where } (0 \leqq P_1 \leqq R_i/A_{i1}) \tag{5XS}$$

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[2] Kemeny, Mirkil, Snell and Thompson, *Finite Mathematical Structures*. Prentice-Hall, Inc. (1959). P. 337 *ff*.

[3] *Ibid*. P. 346.

[4] We may ignore as trivial the one-point intersection regardless of V_1 when W_0 belongs to S. The vector W_0 will invariably exhibit r zero components corresponding to the r parameter charges of equations (2) and (3). One-point intersection at W_0 implies that the schedule is cluttered with r charges each equal to zero. No ratemaker is that clumsy.

[5] It has been noted that if for any j, $A_{ij} = 0$ for all i, the jth term will be dropped from all equations, hence we must have $A_{i1} > 0$ for at least one i. The charge P_1 could not otherwise be retained as a parameter.

Having set limits to the parameter of equation (5X) we have now defined by equation (5XS) neither a line nor a ray, but only a completely general segment. Therefore, S cannot contain a ray, therefore S is bounded.

The fact that the actual segment of intersection may and in some cases will be shorter than given by equation (5XS)[6] is immaterial. The proof depends not upon the length of intersection, but upon the fact that the intersection is a segment and not a ray.

The proof as given appears valid only under the restrictive assumption of no multiplicative, charges, but see Section 7, following.

6. *The "Extreme Points" of S*

If despite the impossibility of visualizing a polyhedron of more than three dimensions we maintain the geometric analogy, the so-called "extreme points" of S may be conceived as the corners of the polyhedron.[7] As noted under *Hypothetical Example,* preceding, these points display the absolute limits of the several P_j as components of the vector X, and in some cases the upper limit so indicated will be significantly less than the least value of R_i/A_{ij} for any i.

As previously noted, the extreme points of S, which we designate as $T\nu$, may be found by reducing equations (1) to the form of equation (3) using in turn each of the $\binom{n}{r}$ possible combinations of r parameters among the n charges P_j. We will then obtain a set of $\binom{n}{r}$ vectors ${}_\nu W_0$. Discarding all ${}_\nu W_0$ in which any component P_j is negative, those vectors remaining will be the extreme points, $T\nu$, and since S is bounded, the points $T\nu$ will number at least $(r+1)$, *i.e.* the number of extreme points will be at least one more than the number of parameters in equations (2) and (3).[8] There are other methods to find the extreme points which are less tedious in application but which are difficult to present in general terms.

Apart from the display of limits to the several charges P_j, which is academic, the extreme points $T\nu$ have a peculiar utility. By equation (2) and (3) we have limited the ratemaker to r degrees of freedom, but we have left his judgment free in the exercise of any or all of them. Now, however, if S exhibits exactly $(r+1)$ extreme points, we may write:

$$a_1T_1 + a_2T_2 + \cdots + a_rT_r + a_{(r+1)}\,T_{(r+1)} = X \tag{8}$$

where:

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[6] *Cf.* under *Hypothetical Example,* preceding, the line defined by equation (5b.1) and the segment of intersection determined by the limits to P_1 as the parameter of that equation.

[7] See Kemeny *et al.*, *Op. Cit.* P. 345 for an exact definition.

[8] This is not immediately obvious. It arises from the fact that although we are manipulating an n-component vector in n-dimensional space, the set S is r-dimensional, and the number of extreme points must be at least one greater than the number of the dimension of the bounded set. By geometric analogy, the extreme points of a *one*-dimensional segment are the *two* end points; the extreme points of the simplest bounded 2-dimensional set are the *three* vertices of a triangle; the extreme points of the simplest bounded *3*-dimensional set are the *four* corners of a tetrahedron.

$$a_\nu \geqq 0 \text{ for all } \nu; \text{ and } \sum_{(\nu=1)}^{(r+1)} a_\nu = 1$$

We may rewrite equation (8) as:

$$a_1T_1 + a_2T_2 + \cdots + a_rT_r + (1 - \sum_{\nu=1}^{r} a_\nu) T_{(r+1)} = X; \text{ where } \sum_{\nu=1}^{r} a_\nu \leqq 1 \qquad (9)$$

The r degrees of freedom are now expressed by the coefficients $a_1 \cdots a_r$ of equation (9). From the restrictions imposed above upon a_ν, it is now completely obvious that these r degrees of freedom are not independent. As each degree of freedom is progressively exhausted, the bounds within which each subsequent choice must be exercised become progressively narrower. In the extreme case, let $a_\nu = 1$ and we have $X = T_\nu$, with no further freedom of choice whatever.

If S exhibits more than $(r + 1)$ extreme points, we still will find particular combinations of exactly $(r + 1)$ vectors T_ν such that any vector X of the entire set may be calculated by equation (9) with the same restrictions upon the coefficients a_ν. The *same* combination will not serve to calculate *all* X, but *some* combination of $(r + 1)$ vectors T_ν will serve to calculate *any* X in the set. Equation (9) is completely general provided only that S is bounded.[9]

7. *Apportionment Function and Multiplicative Charges*

There are two sets of functions which have been ignored for simplicity in the previous development, but which must be recognized in the interest of generality. The first, which may be called the "apportionment functions," reflect variation of the contribution to expectation with the extent of a given hazard in a given risk. The contribution of, *e.g.*, combustible walls to the expectation of a risk of mixed frame and masonry construction will be a function of that percentage of total wall perimeter[10] which is of frame construction; the hazard of flammable liquid storage is a function of the quantity stored.

The second set of functions might be termed the "contagion of hazard functions."[11] These functions reflect the fact that the contribution to expectation of a given hazard is not inherent to that hazard alone but is also a function of the environment. Put a pot-belly stove in the middle of the California desert and the worst to happen will be the singeing of incautious jackrabbits. Build a shack around the stove, and the stove becomes more hazardous. Now, put the same stove in a fireworks factory and ——?

Both the apportionment functions and the contagion of hazard functions

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[9] The number of combinations suitable for this purpose will not necessarily in general equal the number of all possible combinations of $(r + 1)$ vectors T_ν. See Kemeny *et al. Op. cit.* Ch. 5. Sec. 3 for further discussion of the concept of equation (9).

[10] Where the risk is comprised of separate but communicating sections of different wall construction, section area ratios are sometimes used rather than wall perimeter ratios.

[11] McIntosh. *Op. cit.* P. 11 and P. 29 *ff.* The author would welcome another term to avoid the confusing similarity between "contagion of hazard" as used here and the statistical term "contagion," referring to the apparent after effects of sampling. (*Cf.* Wm. Feller, *An Introduction to Probability Theory and Its Applications.* Vol. I. 2nd Ed. (3rd Printing) John Wiley & Sons, Inc. (1959). P. 112.)

may (and usually will) be non-linear, and may or may not be continuous. In the actual schedule, however, the former will appear either as linear approximations or as step functions in the form of specific values tabulated at selected intervals. The latter will appear in the schedule as step functions the tabulated values of which constitute the multiplicative charges. Multiplicative charges are not, properly speaking, "charges" at all. They are factors for application to the additive charges to reflect variation in the environment of the specific hazard for which the additive charge is made.[12] The simple additive charge P_j itself assumes "normal" conditions, *i.e.* an arbitrary standard environment for Hazard j, though the assumption may not be stated explicitly.

In completely general form, the terms of equations (1) will be:

$$\cdots + A_{ij}F_{ij}G_{ij}P_j + A_{ik}F_{ik}G_{ik}P_k + \cdots \qquad (10)$$

where:

A_{ij} = The proportion of risks in Class i which exhibit Hazard j.

F_{ij} = The average apportionment of Hazard j among those risks of Class i which exhibit Hazard j. If the severity of Hazard j is considered to be substantially independent of extent, then $F_{ij} = 1.0$. If $F_{ij} = 0$, then also $A_{ij} = 0$. Also $(0 \leqq F_{ij} \leqq 1.0)$.

G_{ij} = The average of the multiplicative charges applied to the additive charge P_j among those risks of Class i which exhibit Hazard j. The word "charge" here includes also multiplicative "credits". Also $(G_{ij} \geqq 0)$. (See footnote (12), preceding.)

If we now let:

$$Q_{ij} = F_{ij}G_{ij}P_j$$

the terms of (10) become:

$$\cdots + A_{ij}Q_{ij} + A_{ik}Q_{ik} + \cdots$$

and the original form of equations (1) is restored. The proof of bounds given in Section 5, above, is extended thereby to complete generality, and with it the entire development is likewise extended.

For practical purposes, the number of variables Q_{ij} becomes fantastic, but the problems can be shrunk back to reasonable proportions. The factors F_{ij} reflect weighted average values of a function $f_j(e_j)$, where e_j is the extent of Hazard j in a specific risk. The function $f_j(e_j)$ may be constant and equal to unity for some j. It is, however, a never-decreasing function. Therefore, not only will the factors F_{ij} be correlated for all i such that $F_{ij} > 0$, these factors may be placed *a priori* in order of increasing (or decreasing) values when average values of e_j have been determined for each class by physical inspection of risks.

The factors G_{ij} will reflect appropriately weighted average values of the products:

$$\cdots [{}_jg_k({}_jb_k)]\cdot[{}_jg_l({}_jb_l)]\cdots$$

[12] A percentage "credit" of, *e.g.*, 5% is obviously the exact equivalent of a factor of 0.95. The modified additive charge is not $(-0.05P_j)$ but is $(P_j - 0.05P_j)$.

where ${}_jb_k$ is the extent of Hazard k in the environment of Hazard j in a particular risk. If ${}_jb_k$ reflects the arbitrary standard environment of Hazard j, then ${}_jg_k({}_jb_k) = 1.0$. Otherwise, ${}_jg_k({}_jb_k) \geqq 0$. Only those Hazards k(l)(m) . . . are considered here which significantly affect Hazard j. *E.g.*, a stove does not affect the hazard of welding and *v.v.*, but either affects the hazard of spray painting and also *v.v.* The function ${}_jg_k({}_jb_k)$ may (if not constant) be a never-decreasing or a never-increasing function, but will be monotonic in either case. For some j (not all j), therefore, the factors G_{ij} may also be placed *a priori* in order of increasing (or decreasing) values when average values of ${}_jb_k$ have been determined.[13] In any case, the factors G_{ij} will be correlated for all i. Also the same Hazard k may affect several other hazards, so that for some j the factors G_{ij} will be correlated for several j.

Therefore, although recognition of apportionment factors and multiplicative charges increases the dimension of the vector set, S, and thereby introduces additional degrees of freedom, the ratemaker's choice is not unrestricted in exercising these additional degrees of freedom. The coefficients of any $\mathbf{P}_j$ are correlated for all i such that the coefficient is greater than zero, are for some j correlated with each other over several j and finally can in many cases be arranged *a priori* in order of values. The mathematical limits to judgment do not become so broad that all practical significance will be lost; else the preparation or major revision of a fire rating schedule to meet pre-determined class rate levels would not be so frustratingly tedious a task as it is proved to be by experience.

[13] It should be noted that for any given risk which exhibits both Hazard j and Hazard k, we will have $e_k = {}_jb_k$, although if Hazard j is absent and Hazard k is present, then $e_k > 0$ while ${}_jb_k = 0$. This establishes for some j and some k a further correlation of the respective coefficients of $\mathbf{P}_j$ and $\mathbf{P}_k$.