VALUATION OF THE DEATH BENEFITS PROVIDED BY THE WORKMEN'S COMPENSATION LAW OF N. Y.

by

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The problems raised by the subject of this paper are those which are of longstanding interest to the profession and go back to the very infancy of this society. In fact this paper is the latest of a series of the same title, the first, by Mr. W. W. Greene having appeared in Volume I of the Proceedings thirty-four years ago. In order to simplify the task so that nontechnical people could value cases as well as to insure uniformity the Workmen's Compensation Board has had prepared tables to value certain benefits under the N. Y. Workmen's Compensation Law. These tables have been as follows:

	Bulletin No.	*Date of Death*
N. Y. Dept. of Labor Bulletin	222	July 1, 1948
	207	July 1, 1939 to July 1, 1948
	190	July 1, 1937 to July 1, 1939
	120	July 1, 1922 to July 1, 1937
State Industrial Commission Bulletin	June 1917	
Olifiers' Tables	1915	

Several members of the Society have participated, chiefly in an advisory capacity, in the preparation of the present bulletin. These have been Miss Davis and Messrs. Carleton, Dorweiler, Graham, Johnson and Perryman.

The syllabus for the examination on Life Contingencies contains no references to the Proceedings later than Volume II and although the theory has not changed it is felt that our literature should be brought up to date at this opportune time because of current interest in the topic.

Chapter 232, Laws of 1948, effective July 1, 1948 amended the Workmen's Compensation Law of New York and made necessary the preparation of new tables for the valuation of death benefits. The purpose of this paper is to make available to the actuarial profession the formulae underlying these tables and to demonstrate to students a method of deriving these formulae.

The pertinent sections of the amendment are as follows:

> 16. 1-b. If there be a surviving wife (or dependent husband) and no child of the deceased under the age of eighteen years and no child of any *age dependent blind or crippled, and the death occurs on or after July first, nineteen hundred forty-eight, to such wife (or dependent husband) forty per centum of the average wages of the deceased during widowhood (or dependent widowerhood) with two years' compensation in one sum, upon re-marriage; and where the death occurred prior to July first,*

nineteen hundred forty-eight, to such wife (or dependent husband) thirty per centum of such wages during widowhood (or dependent widowerhood) with two years' compensation in one sum, upon remarriage.

Subdivision two of section sixteen of the Workmen's Compensation Law is hereby amended to read as follows:

16. 2. If there be a surviving wife (or dependent husband) and a surviving child or children of the deceased under the age of eighteen years or a surviving child or children of any age dependent blind or crippled, *and the death occurs on or after July first, nineteen hundred forty-eight, to such wife (or dependent husband) thirty per centum of the average wages of the deceased during widowhood (or dependent widowerhood) with two years' compensation in one sum, upon remarriage; and* the additional amount of *twenty* per centum of such wages for each such child until the age of eighten years or until the removal of the dependency of the blind or crippled child or children; in case of the subsequent death or remarriage of such surviving wife (or dependent husband) any surviving child of the deceased employee, at the time under eighteen years of age or dependent through mental or physical infirmity, shall have his compensation increased to *thirty* per centum of such wages, and the same shall be payable until he shall reach the age of 18 years or until such dependent blind or crippled condition shall have been removed; provided that the total amount payable shall in no case exceed sixty-six and two-thirds per centum of such wages. *Upon statutory termination of compensation payments to all such children, the compensation of the surviving wife (or dependent husband) shall be increased to forty per centum of such wages with two years' compensation, at such rate, in one sum, upon remarriage.*

If there be a surviving wife (or dependent husband) and any of the aforementioned surviving children, and the death occurred prior to July first, nineteen hundred forty-eight, to such wife (or dependent husband) thirty per centum of the average wages of the deceased during widowhood (or dependent widowerhood) with two years' compensation in one sum, upon remarriage; and the additional amount of ten per centum of such wages for each such child until eighteen years of age or until the removal of the dependency of the blind or crippled child or children; in case of the subsequent death or remarriage of such surviving wife (or dependent husband) any surviving child of the deceased shall have his compensation increased to fifteen per centum of such wages until he shall reach the age of eighteen years or until such dependent blind or crippled condition shall have been removed; provided that the total amount payable shall in no case exceed sixty-six and two-thirds per centum of such wages."

In valuing death benefits prior to this amendment one assumption made was that $p_{y_1} = p_{y_2} = \ldots = S = .99479364$, where p_{y_i} was the probability that the i_{th} youngest child would survive one year to age $y_i + 1$. By this device it was possible to calculate annuity values knowing only the number of children involved and the age of the oldest of these. Otherwise it would have been necessary to calculate annuities for each combination of ages. This assumption is still made.

In addition it has been necessary to make one other assumption. Prior to the amendment the benefit of the child depended on the status of the widow (whether she was alive and not remarried, or remarried, or deceased) but the widows benefit did not depend on the the status of the child. Now it does, for the widow receives 40% of the deceased's wages if she has no dependent children under eighteen and 30% otherwise. At the time of remarriage the widow receives a dowry of two years payments. This is interpreted to be two years of actual payments and not two years of payments being received at time of remarriage. For example, if a widow remarried and had a child or children, the youngest of whom, was 16½ she would (except for the remarriage) receive benefits at the rate of 30% for a year and a half, and at the rate of 40% thereafter. The remarriage benefit is then the present value of payments at the rate of 30% for the first year and a half and at the rate of 40% for the last half year. One proposed solution to the problem of how to value this remarriage endowment is to run the term of the endowment until the youngest child reaches age seventeen. This means that for valuation purposes we would pay the widow her dowry at a 30% rate if the youngest child was less than seventeen and at a 40% rate dowry if he was seventeen or greater. We have made the tacit assumption that the average age is seventeen of the youngest child of these widows who remarry with youngest child between sixteen and eighteen. The resulting inaccuracy is very small as may be seen by comparing it with the true expression.

The true value of the increasing 10% is

$$\tfrac{1}{2}\int_{16-y}^{18-y}\frac{20}{l_{x'}}(t-16+y)\,s^t\,v^t\,\frac{m_{x'+t}}{1-\tfrac{1}{2}q_{x'+t}}\,dt$$

If the youngest child is 16 at time of remarriage the widow gets none of the extra 10% and if the youngest child is 18 at time of remarriage the widow gets the extra 10% for 2 years. This expression

$$=\tfrac{1}{2}\int_{16-y}^{17-y} f(t)\,dt+\tfrac{1}{2}\int_{17-y}^{18-y} f(t)\,dt \quad \text{If we replace}$$

$$\tfrac{1}{2}\int_{17-y}^{18-y} f(t)\,dt \text{ by } \tfrac{1}{2}\int_{17-y}^{17-y} f(t)\,dt \quad \text{we have}$$

$$\int_{16-y}^{17-y} f(t)\,dt$$ as the expression to then be approximated. We have also neglected the small values for those cases where the remarriage takes place with youngest child $\leq$ 16 and this child dies before reaching age 18.

The tables affected by the amendment are as follows:

I Widow or Widower–Present Value of Compensation per $100 Annual Wages Payable Until Death or Remarriage.

$$40\ \bar{a}_{x'} + 80\ \bar{E}_{x'}$$

I-A Reduction on account of Youngest Child in Present Value of Widow's or Widower's Portion of Compensation per $100 of Annual Wages Payable (per Table I) Until Death or Remarriage.

$$10\ {}_{1}\bar{a}_{x'y_1:\overline{18-y_1}|} + 20\ {}_{1}\bar{E}_{x'y_1:\overline{17-y_1}|}$$

I-B Reduction on account of Second Youngest Child . . .

$$10\ ({}_{1}\bar{a}_{x'y_2:\overline{18-y_2}|} - {}_{2}\bar{a}_{x'y_2:\overline{18-y_2}|}) + 20\ ({}_{1}\bar{E}_{x'y_2:\overline{17-y_2}|} - {}_{2}\bar{E}_{x'y_2:\overline{17-y_2}|}$$

I-C Reduction on account of Third Youngest Child . . .

$$10\ ({}_{1}\bar{a}_{x'y_3:\overline{18-y_3}|} - 2\,{}_{2}\bar{a}_{x'y_3:\overline{18-y_3}|} + {}_{3}\bar{a}_{x'y_3:\overline{18-y_3}|}) + 20\ ({}_{1}\bar{E}_{x'y_3:\overline{18-y_3}|}$$
$$- 2\,{}_{2}\bar{E}_{x'y_3:\overline{18-y_3}|} + {}_{3}\bar{E}_{x'y_3:\overline{18-y_3}|})$$

I-D Reduction on account of Fourth Youngest Child . . .

$$10\ ({}_{1}\bar{a}_{x'y_4:\overline{18-y_4}|} - 3\ {}_{2}\bar{a}_{x'y_4:\overline{18-y_4}|} + 3\ {}_{3}\bar{a}_{x'y_4:\overline{18-y_4}|} - {}_{4}\bar{a}_{x'y_4:\overline{18-y_4}|}) + 20\ ({}_{1}\bar{E}_{x'y_4:\overline{17-y_4}|}$$
$$- 3\ {}_{2}\bar{E}_{x'y_4:\overline{17-y_4}|} + 3\ {}_{3}\bar{E}_{x'y_4:\overline{17-y_4}|} - {}_{4}\bar{E}_{x'y_4:\overline{17-y_4}|})$$

I-E Reduction on acount of Fifth Youngest Child . . .

$$10\ ({}_{1}\bar{a}_{x'y_5:\overline{18-y_5}|} - 4\ {}_{2}\bar{a}_{x'y_5:\overline{18-y_5}|} + 6\ {}_{3}\bar{a}_{x'y_5:\overline{18-y_5}|} - 4\ {}_{4}\bar{a}_{x'y_5:\overline{18-y_5}|} + {}_{5}\bar{a}_{x'y_5:\overline{18-y_5}|})$$
$$+ 20\ ({}_{1}\bar{E}_{x'y_5:\overline{17-y_5}|} - 4\ {}_{2}\bar{E}_{x'y_5:\overline{17-y_5}|} + 6\ {}_{3}\bar{E}_{x'y_5:\overline{17-y_5}|} - 4\ {}_{4}\bar{E}_{x'y_5:\overline{17-y_5}|}$$
$$+ {}_{5}\bar{E}_{x'y_5:\overline{17-y_5}|})$$

IV Youngest Child–Present Value Per $100 Annual Wages Payable Until Age 18

$$30\ {}_{1}\bar{a}_{y_1:\overline{18-y_1}|} - 10\ {}_{1}\bar{a}_{x'y_1:\overline{18-y_1}|}$$

V Second Youngest Child

$$30\ {}_{1}\bar{a}_{y_2:\overline{18-y_2}|} - 10\,{}_{1}\bar{a}_{x'y_2:\overline{18-y_2}|} - 3\tfrac{1}{3}\ {}_{2}\bar{a}_{x'y_2:\overline{18-y_2}|}$$

VI Third Youngest Child

$$30\ {}_{1}\bar{a}_{y_3:\overline{18-y_3}|} - 23\tfrac{1}{3}\ {}_{3}\bar{a}_{y_3:\overline{18-y_3}|} - 10\ {}_{1}\bar{a}_{x'y_3:\overline{18-y_3}|} - 6\tfrac{2}{3}\ {}_{2}\bar{a}_{x'y_3:\overline{18-y_3}|} + 10\ {}_{3}\bar{a}_{x'y_3:\overline{18-y_3}|}$$

VII 4th Youngest Child–Present Value Per $100 Annual Wages Payable Until Age 18

$$30\ {}_{1}\bar{a}_{y_4:\overline{18-y_4}|} - 70\ {}_{3}\bar{a}_{y_4:\overline{18-y_4}|} + 40\ {}_{4}\bar{a}_{y_4:\overline{18-y_4}|} - 10\ {}_{1}\bar{a}_{x'y_4:\overline{18-y_4}|} - 10\ {}_{2}\bar{a}_{y_4:\overline{18-y_4}|}$$
$$+ 30\ {}_{3}\bar{a}_{x'y_4:\overline{18-y_4}|} - 10\ {}_{4}\bar{a}_{x'y_4:\overline{18-y_4}|}$$

VIII 5th Youngest Child–Present Value Per \$100 Annual Wages Payable Until Age 18

$30\ {}_1\bar{a}_{y_5:\overline{18-y_5}|} - 140\ {}_3\bar{a}_{y_5:\overline{18-y_5}|} + 160\ {}_4\bar{a}_{y_5:\overline{18-y_5}|} - 50\ {}_5\bar{a}_{y_5:\overline{18-y_5}|} - 10\ {}_1\bar{a}_{x'y_5:\overline{18-y_5}|}$

$- 13\tfrac{1}{3}\ {}_2\bar{a}_{x'y_5:\overline{18-y_5}|} + 60\ {}_3\bar{a}_{x'y_5:\overline{18-y_5}|} - 40\ {}_4\bar{a}_{x'y_5:\overline{18-y_5}|} + 3\tfrac{1}{3}\ {}_5\bar{a}_{x'y_5:\overline{18-y_5}|}$

IX Children at 30%–Present Value Per \$100 Annual Wages Payable Until Age 18

1st & 2nd Child $30\ {}_1\bar{a}_{y:\overline{18-y}|}$

3rd Child $30\ {}_1\bar{a}_{y:\overline{18-y}|} - 23\tfrac{1}{3}\ {}_3\bar{a}_{y:\overline{18-y}|}$

4th Child $30\ {}_1\bar{a}_{y:\overline{18-y}|} - 70\ {}_3\bar{a}_{y:\overline{18-y}|} + 40\ {}_4\bar{a}_{y:\overline{18-y}|}$

5th Child $30\ {}_1\bar{a}_{y:\overline{18-y}|} - 140\ {}_3\bar{a}_{y:\overline{18-y}|} + 160\ {}_4\bar{a}_{y:\overline{18-y}|} - 50\ {}_5\bar{a}_{y:\overline{18-y}|}$

6th Child $30\ {}_1\bar{a}_{y:\overline{18-y}|} - 233\tfrac{1}{3}\ {}_3\bar{a}_{y:\overline{18-y}|} + 400\ {}_4\bar{a}_{y:\overline{18-y}|} - 250\ {}_5\bar{a}_{y:\overline{18-y}|}$

$+ 53\tfrac{1}{3}\ {}_6\bar{a}_{y:\overline{18-y}|}$

7th Child $30\ {}_1\bar{a}_{y:\overline{18-y}|} - 350\ {}_3\bar{a}_{y:\overline{18-y}|} + 800\ {}_4\bar{a}_{y:\overline{18-y}|} - 750\ {}_5\bar{a}_{y:\overline{18-y}|}$

$+ 320\ {}_6\bar{a}_{y:\overline{18-y}|} - 50\ {}_7\bar{a}_{y:\overline{18-y}|}$

IX-A Brothers, Sisters and Grandchildren at 25%–Present Value per \$100 Annual Wages Payable Until Age 18

1st & 2nd Child $25\ {}_1\bar{a}_{y:\overline{18-y}|}$

3rd Child $25\ {}_1\bar{a}_{y:\overline{18-y}|} - 8\tfrac{1}{3}\ {}_3\bar{a}_{y:\overline{18-y}|}$

4th Child $25\ {}_1\bar{a}_{y:\overline{18-y}|} - 25\ {}_3\bar{a}_{y:\overline{18-y}|}$

5th Child $25\ {}_1\bar{a}_{y:\overline{18-y}|} - 50\ {}_3\bar{a}_{y:\overline{18-y}|} + 25\ {}_6\bar{a}_{y:\overline{18-y}|}$

6th Child $25\ {}_1\bar{a}_{y:\overline{18-y}|} - 83\tfrac{1}{3}\ {}_3\bar{a}_{y:\overline{18-y}|} + 125\ {}_5\bar{a}_{y:\overline{18-y}|} - 66\tfrac{2}{3}\ {}_6\bar{a}_{y:\overline{18-y}|}$

7th Child $25\ {}_1\bar{a}_{y:\overline{18-y}|} - 125\ {}_3\bar{a}_{y:\overline{18-y}|} + 375\ {}_5\bar{a}_{y:\overline{18-y}|} - 400\ {}_6\bar{a}_{y:\overline{18-y}|}$

$+ 125\ {}_7\bar{a}_{y:\overline{18-y}|}$

X Parent or Grandparent–Present Value Per \$100 Annual Wages Payable Until Age 18

$40\ \bar{a}_w$

XV Suspension of Payments to Widow (X) and no Children

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|}$$

XVI Suspension of Widow's Portion of Payments to Widow (X) with One Child

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|} - 10\ \bar{a}_{x'y_1:\overline{t}|} - 20\ \bar{E}_{x'y_1:\overline{t}|}$$

XVII Suspension of Widow's Portion of Payments to Widow (X) with Two Children

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|} - 10\ (2\ {}_1\bar{a}_{x':\overline{t}|} - {}_2\bar{a}_{x':\overline{t}|}) - 20\ (2\ {}_1\bar{E}_{x':\overline{t}|} - {}_2\bar{E}_{x':\overline{t}|})$$

XVIII Suspension of Widow's Portion of Payments to Widow (X) with Three Children

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|} - 10\ (3\ {}_1\bar{a}_{x':\overline{t}|} - 3\ {}_2\bar{a}_{x':\overline{t}|} + {}_3\bar{a}_{x':\overline{t}|})$$
$$- 20\ (3\ {}_1\bar{E}_{x':\overline{t}|} - 3\ {}_2\bar{E}_{x':\overline{t}|} + {}_3\bar{E}_{x':\overline{t}|})$$

XIX Suspension of Widow's Portion of Payments to Widow (X) with Four Children

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|} - 10\ (4\ {}_1\bar{a}_{x':\overline{t}|} - 6\ {}_2\bar{a}_{x':\overline{t}|} + 4\ {}_3\bar{a}_{x':\overline{t}|} - {}_4\bar{a}_{x':\overline{t}|})$$
$$- 20\ (4\ {}_1\bar{E}_{x':\overline{t}|} - 6\ {}_2\bar{E}_{x':\overline{t}|} + 4\ {}_3\bar{E}_{x':\overline{t}|} - {}_4\bar{E}_{x':\overline{t}|})$$

XX Suspension of Widow's Portion of Payments to Widow (X) with Five Children

$$40\ \bar{a}_{x':\overline{t}|} + 80\ \bar{E}_{x':\overline{t}|}$$
$$- 10\ (5\ {}_1\bar{a}_{x':\overline{t}|} - 10\ {}_2\bar{a}_{x':\overline{t}|} + 10\ {}_3\bar{a}_{x':\overline{t}|} - 5\ {}_4\bar{a}_{x':\overline{t}|} + {}_5\bar{a}_{x':\overline{t}|})$$
$$- 20\ (5\ {}_1\bar{E}_{x':\overline{t}|} - 10\ {}_2\bar{E}_{x':\overline{t}|} + 10\ {}_3\bar{E}_{x':\overline{t}|} - 5\ {}_4\bar{E}_{x':\overline{t}|} + {}_5\bar{E}_{x':\overline{t}|})$$

In deriving these formulae, the chief difficulty lies in valuing the limitation due to the maximum benefit which is 66⅔% of the deceased's wages. A rather neat expression can be developed for this.

Let ϕ be the reduction from the value of the benefit on account of the maximum limitation.

${}_w\phi\ (y)$ = reduction for w children considering the mortality of the children y

${}_w\phi\ (x'y)$ = reduction for w children considering the mortality of the widow x' and the children y

$\phi_w\ (y)$ = reduction for the wth youngest child

$\phi_w\ (x'y)$ = reduction for the wth youngest child

p_y = probability that a child aged y will survive t years to age y + t

For purposes of these tables ${}_tp_y$ is a constant for y < 18

$${}^mC_r = \frac{m!}{r!\,(n-r)!} = \text{number of combination of } m \text{ things taken } r \text{ at a time.}$$

The probability that exactly r children survive out of m is then

$$P = {}^mC_r\,p^r\,(1-p)^{m-r}$$

$$= {}^mC_r\,p^r\,(1 - {}^{m-r}C_1\,p + {}^{m-r}C_2\,p^2 - {}^{m-r}C_3\,p^3 \ldots\ldots (-1)^{m-r}p^{m-r})$$

$$= {}^mC_r\,\left\{p^r - {}^{m-r}C_1\,p^{r+1} + {}^{m-r}C_2\,p^{r+2} \ldots\ldots (-1)^{m-r}\,p^m\right\}$$

For a group of r children surviving the probability is p^r and the r survivors may be selected in ${}^mC_r$ ways from the group of m children. All of the remaining m-r children fail to survive.

Since ${}^mC_r\ {}^{m-r}C_t = {}^mC_{r+t}\ {}^{r+t}C_t$

$$P = {}^mC_r\,p^r - {}^mC_r\,{}^{m-r}C_1\,p^{r+1} + {}^mC_r\,{}^{m-r}C_2\,p^{r+2} + \ldots\ldots (-1)^{m-r}\,{}^mC_r\,p^m$$

$$= {}^mC_r\,p^r - {}^mC_{r+1}\,{}^{r+1}C_1\,p^{r+1} + {}^mC_{r+2}\,{}^{r+2}C_2\,p^{r+2} + \ldots (-1)^{m-r}\,{}^mC_r\,p^m$$

Let ${}^mC_{r+t}\,p^{r+t} = Z^{r+t}$

and $m \to \infty$

$$P = Z^r - {}^{r+1}C_1\,Z^{r+1} + {}^{r+2}C_2\,Z^{r+2} \ldots\ldots\ldots$$

$$= Z^r\,(1 - {}^{r+1}C_1\,Z + {}^{r+2}C_2\,Z^2 - {}^{r+3}C_3\,Z^3 \ldots)$$

$$= Z^r\,(1-Z)^{-\overline{r+1}} = \frac{Z^r}{(1+Z)^{r+1}}$$

so that $\dfrac{Z^r}{(1+Z)^{r+1}}$

represents the probability of exactly r survivors out of m for all combinations of r survivors. By converting probabilities to annuities and defining

$Z^{r+t} = {}^mC_{r+t}\ {}_{r+t}\bar{a}$ we have $\dfrac{Z^r}{(1+Z)^{r+1}}$ = present value of

annuities payable while there are exactly r survivors of m.

To derive ϕ proceed as follows:
Let *$J* be the benefit to each of m persons and let *$L* be the maximum benefit to all. The youngest beneficiary will then draw *$J* as will the next youngest and the third youngest etc. provided $\$\,rJ < \$\,L$. The $(r+1)^{st}$ youngest will receive $\$(L-rJ)$ and the $\$(r+1)J$ benefits will be reduced $(\$r+1)J-L$ because of the operation of the maximum. The $(r+2)^{nd}$ youngest will receive nothing and thus the total benefits are reduced *$J because of him. For* y_{r+1} and y_{r+2} the reduction has become $\$(r+2)J-L$
The value of ϕ is then

$$\phi = \left\{(r+1)J - L\right\}\frac{Z^{r+1}}{(1+Z)^{r+2}} + \left\{(r+2)J - L\right\}\frac{Z^{r+2}}{(1+Z)^{r+3}} + \left\{(r+3)J - L\right\}\frac{Z^{r+3}}{(1+Z)^{r+4}} + \dots$$

This is easily summed by breaking it into two series.

$$\phi = \left\{(r+1)J - L\right\}\frac{Z^{r+1}}{(1+Z)^{r+2}}\left[1 + \frac{Z}{1+Z} + \frac{Z^2}{(1+Z)^2} + \frac{Z^3}{(1+Z)^3} + \dots\right]$$

$$+ \frac{J\,Z^{r+2}}{(1+Z)^{r+3}}\left[1 + \frac{2Z}{1+Z} + \frac{3Z^2}{(1+Z)^2} + \frac{4Z^3}{(1+Z)^3} + \dots\dots\right]$$

$$= \left\{(r+1)J - L\right\}\frac{Z^{r+1}}{(1+Z)^{r+2}}\left(1 - \frac{Z}{1+Z}\right)^{-1} + \frac{J\,Z^{r+2}}{(1+Z)^{3}}\left(1 - \frac{Z}{1+Z}\right)^{-2}$$

$$= \left\{(r+1)J - L\right\}\frac{Z^{r+1}}{(1+Z)^{r+1}} + \frac{J\,Z^{r+2}}{(1+Z)^{r+1}}$$

$$= \left[\left\{(r+1)J - L\right\}Z^{r+1} + J\,Z^{r+2}\right](1+Z)^{-\overline{r+1}}$$

As an example consider Formula IX:

	Benefit	Reduction
1st Youngest Child	\$30	\$ 0
2nd Youngest Child	30	0
3rd Youngest Child	30	23⅓
4th Youngest Child	30	53⅓
etc.		

L = 66⅔ J = 30

r = 2

$\phi = (23\frac{1}{3}\, Z^3 + 30\, Z^4)(1 + Z)^{-3}$

From this ϕ function the reduction may be obtained for the formulae tabulated above. Consideration of the maximum does not enter into the derivation of the first six tables. Here the widows benefit is reduced \$10 while she has a dependent child < 18 alive; so that

$$\phi\,(y) = \frac{10\,Z}{1+Z} + \frac{10\,Z^2}{(1+Z)^2} + \frac{10\,Z^3}{(1+Z)^3} + \cdots\cdots$$

$$= \frac{10\,Z}{1+Z} = 10\,Z - 10\,Z^2 + 10\,Z^3 \ldots\ldots$$

Z^r for $r > m$ is undefined, therefore

$_1\phi\,(y) = 10\,Z = 10\,{}_1\bar{a}$

$_2\phi\,(y) = 10\,Z - 10\,Z^2 = 10\,(2\,{}_1\bar{a} - {}_2\bar{a})$

$_3\phi\,(y) = 10\,(Z - Z^2 + Z^3) = 10\,(3\,{}_1\bar{a} - 3\,{}_2\bar{a} + {}_3\bar{a})$

$_4\phi\,(y) = 10\,(Z - Z^2 + Z^3 - Z^4) = 10\,(4\,{}_1\bar{a} - 6\,{}_2\bar{a} + 4\,{}_3\bar{a} - {}_4\bar{a})$

$_5\phi\,(y) = 10\,(Z - Z^2 + Z^3 - Z^4 + Z^5) = 10\,(5\,{}_1\bar{a} - 10\,{}_2\bar{a} + 10\,{}_3\bar{a} - 5\,{}_4\bar{a} + {}_5\bar{a})$

Since $\phi_w(y) = {}_w\phi\,(y) + {}_{w-1}\phi(y)$

$\phi_1(y) = 10\,{}_1\bar{a}$

$\phi_2(y) = 10\,({}_1\bar{a} - {}_2\bar{a})$

$\phi_3(y) = 10\,({}_1\bar{a} - 2\,{}_2\bar{a} + {}_3\bar{a})$

$\phi_4(y) = 10\,({}_1\bar{a} - 3\,{}_2\bar{a} + 3\,{}_3\bar{a} - {}_4\bar{a})$

$\phi_5(y) = 10\,({}_1\bar{a} - 4\,{}_2\bar{a} + 6\,{}_3\bar{a} - 4\,{}_4\bar{a} + {}_5\bar{a})$

If we should define Θ to be for the remarriage benefit what ϕ is for the annuity benefit,

$\Theta_1(y) = 20\,{}_1\bar{E}$

$\Theta_2(y) = 20\,({}_1\bar{E} - {}_2\bar{E})$

$$\Theta_3(y) = 20\,({}_1\bar{E} - 2\,{}_2\bar{E} + {}_3\bar{E})$$

$$\Theta_4(y) = 20\,({}_1\bar{E} - 3\,{}_2\bar{E} + 3\,{}_3\bar{E} - {}_4\bar{E})$$

$$\Theta_5(y) = 20\,({}_1\bar{E} - 4\,{}_2\bar{E} + 6\,{}_3\bar{E} - 4\,{}_4\bar{E} + {}_5\bar{E})$$

The first set of formulae are

$$\text{I A} = \phi_1(y) + \Theta_1(y)$$

$$\text{I B} = \phi_2(y) + \Theta_2(y)$$

$$\text{I C} = \phi_3(y) + \Theta_3(y)$$

$$\text{I D} = \phi_4(y) + \Theta_4(y)$$

$$\text{I E} = \phi_5(y) + \Theta_5(y)$$

At first it seems incongruous that we can subtract annuities of varying terms but this is permissible as an example will illustrate. Suppose we start with m children and consider the terms involving r survivors which are ${}_r\bar{a}$ and ${}_r\bar{E}$. In ${}_r\Theta$ and ${}_r\phi$ we have ${}^mC_r$ of these terms and in ${}_{r-1}\phi$ and ${}_{r-1}\Theta$ we have ${}^{m-1}C_r$ of these. The addition of y_r to the group has increased the number of ${}_r\bar{a}$ and ${}_r\bar{E}$ terms by $({}^mC_r - {}^{m-1}C_r)$ terms and these are the ones where y_r is the last survivor. Therefore these terms run for $18-y_r$ years and it is these that are counted in ϕ_r and Θ_r.

In the second set of formulae we can use the ϕ function developed above.

	Benefit (No Widow)	Reduction
1st Youngest Child	\$30	0
2nd Youngest Child	30	0
3rd Youngest Child	30	23⅓
4th Youngest Child	30	53⅓
etc.		

	Benefit (With Widow)	Reduction
1st Youngest Child	\$20	
2nd Youngest Child	20	3⅓
3rd Youngest Child	20	23⅓
4th Youngest Child	20	43⅓
etc.		

In the first instance $J = \$30$, $r = 2$, $L = 66\frac{2}{3}$ so that,

$$\phi\,(y) = (23\tfrac{1}{3}\, Z^3 + 30\, Z^4)\,(1 + Z)^{-3}$$
$$= 23\tfrac{1}{3}\, Z^3 - 40\, Z^4 + 50\, Z^5 \ldots\ldots$$

$${}_1\phi\,(y) = 0$$

$${}_2\phi\,(y) = 0$$

$${}_3\phi\,(y) = 23\tfrac{1}{3} Z^3 = 23\tfrac{1}{3}\, {}_3\bar{a}$$

$${}_4\phi\,(y) = 23\tfrac{1}{3}\, Z^3 - 40\, Z^4 = 93\tfrac{1}{3}\, {}_3\bar{a} - 40\, {}_4\bar{a}$$

$${}_5\phi\,(y) = 23\tfrac{1}{3}\, Z^3 - 40\, Z^4 + 50\, Z^5 = 233\tfrac{1}{3}\, {}_3\bar{a} - 200\, {}_4\bar{a} + 50\, {}_5\bar{a}$$

and

$$\phi_1(y) = 0$$

$$\phi_2(y) = 0$$

$$\phi_3(y) = 23\tfrac{1}{3}\, {}_3\bar{a}$$

$$\phi_4(y) = 70\, {}_3\bar{a} - 40{}_4\bar{a}$$

$$\phi_5(y) = 140\, {}_3\bar{a} - 160\, {}_4\bar{a} + 50\, {}_5\bar{a}$$

In the 2nd instance $J = \$20$, $r = 1$, $L = 66\frac{2}{3}$

$$\phi\,(x{\cdot}y) = (3\tfrac{1}{3}\, Z^2 + 20\, Z^3)\,(1 + Z)^{-2}$$
$$= 3\tfrac{1}{3}\, Z^2 + 13\tfrac{1}{3}\, Z^3 - 30\, Z^4 + 46\tfrac{2}{3}\, Z^5$$

$${}_1\phi\,(x{\cdot}y) = 0$$

$${}_2\phi\,(x{\cdot}y) = 3\tfrac{1}{3}\, Z^2 = 3\tfrac{1}{3}\, {}_2\bar{a}$$

$${}_3\phi\,(x{\cdot}y) = 3\tfrac{1}{3}\, Z^2 + 13\tfrac{1}{3}\, Z^3 = 10\, {}_2\bar{a} + 13\tfrac{1}{3}\, {}_3\bar{a}$$

$${}_4\phi\,(x{\cdot}y) = 3\tfrac{1}{3}\, Z^2 + 13\tfrac{1}{3}\, Z^3 - 30\, Z^4$$
$$= 20\, {}_2\bar{a} + 53\tfrac{1}{3}\, {}_3\bar{a} - 30\, {}_4\bar{a}$$

$$_5\phi\ (x'y) = 3\tfrac{1}{3}\ Z^2 + 13\tfrac{1}{3}\ Z^3 - 30\ Z^4 + 46\tfrac{2}{3}\ Z^5$$
$$= 33\tfrac{1}{3}\ {}_2\bar{a} + 133\tfrac{1}{3}\ {}_3\bar{a} - 150\ {}_4\bar{a} + 46\tfrac{2}{3}\ {}_5\bar{a}$$

There we have another seeming incongruity in subtracting terms involving *(y)* from terms involving *(x'y)*. Since the ϕ *(y)* terms are independent of x' in deducting those we do so whether or not the widow is alive. If we now deduct again for the case of the widow alive we would be compounding the deduction hence it is necessary to subtract previous deductions of the form $\phi_w(y)$

$$\phi_1(x'y) = 0$$

$$\phi_2(x'y) = 3\tfrac{1}{3}\ {}_2\bar{a}$$

$$\phi_3(x'y) = 3\tfrac{1}{3}\ {}_2\bar{a} - 10\ {}_3\bar{a}$$

$$\phi_4(x'y) = 10\ {}_2\bar{a} - 30\ {}_3\bar{a} + 10\ {}_4\bar{a}$$

$$\phi_5(x'y) = 13\tfrac{1}{3}\ {}_2\bar{a} - 60\ {}_3\bar{a} + 40\ {}_4\bar{a} - 3\tfrac{1}{3}\ {}_5\bar{a}$$

Formula IV $= 30\ \bar{a}_{y_1 : \overline{18\text{-}y1|}} - 10\ {}_1\bar{a}_{x'y : \overline{18\text{-}y1|}}$

$$\text{V} = 30\ \bar{a}_{y2 : \overline{18\text{-}y2|}} - 10\ {}_1\bar{a}_{x'y2 : \overline{18\text{-}y2|}} - \phi_2(y) - \phi_2(x'y)$$

$$\text{IV} = 30\ \bar{a}_{y3 : \overline{18\text{-}y3|}} - 10\ {}_1\bar{a}_{x'y3 : \overline{18\text{-}y3|}} - \phi_3(y) - \phi_3(x'y)$$

$$\text{VII} = 30\ \bar{a}_{y4 : \overline{18\text{-}y4|}} - 10\ {}_1\bar{a}_{x'y4 : \overline{18\text{-}y4|}} - \phi_4(y) - \phi_4(x'y)$$

$$\text{VIII} = 30\ \bar{a}_{y5 : \overline{18\text{-}y5|}} - 10\ {}_1\bar{a}_{x'y5 : \overline{18\text{-}y5|}} - \phi_5(y) - \phi_5(x'y)$$

For Formula IX the ϕ function is

$$\phi = (23\tfrac{1}{3}\ Z^3 + 30\ Z^4)\ (1 + Z)^{-3}$$

$$\phi\ (y) = 23\tfrac{1}{3}\ Z^3 - 40\ Z^4 + 50\ Z^5 - 53\tfrac{1}{3}\ Z^6 + 50\ Z^7 \ldots\ldots$$

$$_1\phi\ (y) = 0$$

$$_2\phi\ (y) = 0$$

$$_3\phi\ (y) = 23\tfrac{1}{3}\ Z^3 = 23\tfrac{1}{3}\ {}_3\bar{a}$$

$${}_4\phi\ (y) = 23\tfrac{1}{3}\ Z^3 - 40\ Z^4 = 93\tfrac{1}{3}\ {}_3\bar{a} - 40\ {}_4\bar{a}$$

$${}_5\phi\ (y) = 23\tfrac{1}{3}\ Z^3 - 40\ Z^4 + 50\ Z^5$$
$$= 233\tfrac{1}{3}\ {}_3\bar{a} - 600\ {}_4\bar{a} + 50\ {}_5\bar{a}$$

$${}_6\phi\ (y) = 23\tfrac{1}{3}\ Z^3 - 40\ Z^4 + 50\ Z^5 - 53\tfrac{1}{3}\ Z^6$$
$$= 466\tfrac{2}{3}\ {}_3\bar{a} - 600\ {}_4\bar{a} + 300\ {}_5\bar{a} - 53\tfrac{1}{3}\ {}_6\bar{a}$$

$${}_7\phi\ (y) = 23\tfrac{1}{3}\ Z^3 - 40\ Z^4 + 50\ Z^5 - 53\tfrac{1}{3}\ Z^6 + 50\ Z^7$$
$$= 716\tfrac{2}{3}\ {}_3\bar{a} - 1400\ {}_4\bar{a} + 1050\ {}_5\bar{a} - 373\tfrac{1}{3}\ {}_6\bar{a} + 50\ {}_7\bar{a}$$

and

$$\phi_1(y) = 0$$

$$\phi_2(y) = 0$$

$$\phi_3(y) = 23\tfrac{1}{3}\ {}_3\bar{a}$$

$$\phi_4(y) = 70\ {}_3\bar{a} - 40\ {}_4\bar{a}$$

$$\phi_5(y) = 140\ {}_3\bar{a} - 160\ {}_4\bar{a} + 50\ {}_5\bar{a}$$

$$\phi_6(y) = 233\tfrac{1}{3}\ {}_3\bar{a} - 400\ {}_4\bar{a} + 250\ {}_5\bar{a} - 53\tfrac{1}{3}\ {}_6\bar{a}$$

$$\phi_7(y) = 350\ {}_3\bar{a} - 800\ {}_4\bar{a} + 750\ {}_5\bar{a} - 320\ {}_6\bar{a} + 50\ {}_7\bar{a}$$

For Formula IX-A

$$\phi = (8\tfrac{1}{3}\ Z^3 + 25\ Z^4)\ (1 + Z)^{-3}$$
$$= 8\tfrac{1}{3}\ Z^3 - 25\ Z^5 + 66\tfrac{2}{3}\ Z^6 - 125\ Z^7 + \ldots\ldots$$

$${}_1\phi\ (y) = 0$$

$${}_2\phi\ (y) = 0$$

$${}_3\phi\ (y) = 8\tfrac{1}{3}\ Z^3 = 8\tfrac{1}{3}\ {}_3\bar{a}$$

$${}_4\phi\ (y) = 8\tfrac{1}{3}\ Z^3 = 33\tfrac{1}{3}\ {}_3\bar{a}$$

$$_5\phi\,(y) = 8\tfrac{1}{3}\,Z^3 - 25\,Z^5 +$$

$$_6\phi\,(y) = 8\tfrac{1}{3}\,Z^3 - 25\,Z^5 + 66\tfrac{2}{3}\,Z^6$$

$$_7\phi\,(y) = 8\tfrac{1}{3}\,Z^3 - 25\,Z^5 + 66\tfrac{2}{3}\,Z^6 - 125\,Z^7$$

$$\phi_1(y) = 0$$

$$\phi_2(y) = 0$$

$$\phi_3(y) = 8\tfrac{1}{3}\;{}_3\bar{a}$$

$$\phi_4(y) = 25\;{}_3\bar{a}$$

$$\phi_5(y) = 50\;{}_3\bar{a} - 25\;{}_5\bar{a}$$

$$\phi_6(y) = 83\tfrac{1}{3}\;{}_3\bar{a} - 125\;{}_5\bar{a} + 66\tfrac{2}{3}\;{}_6\bar{a}$$

$$\phi_7(y) = 125\;{}_3\bar{a} - 375\;{}_5\bar{a} + 400\;{}_6\bar{a} - 125\;{}_7\bar{a}$$

Formulae X and XI follow from the definitions of the benefits they value. Formulae XVI, XVII, XVIII, XIX, XX are similar to Formula I combined with Formulae I-A, I-B, I-C, I-D and I-E. The derivation is the same except that the term is common for all rather than varying according to the age of each child.

It is believed that this paper is sufficiently self-contained that the reader will be enabled to understand the latest application of life contingencies in the Casualty Actuarial field. An index of commutation and other symbols is appended for the use of those who may wish to develop these formulae from first principles.

INDEX OF SYMBOLS AND COMMUTATIONS USED IN FORMULAE FOR RECOMPUTED WORKMEN'S COMPENSATION TABLES

$l_{x'}$ The number alive and unmarried at precise age x' who according to the Danish Survivorship Annuitants' Table of Mortality and according to the Remarriage Tables, terminating at age 65, of the Dutch Royal Insurance Institution are the unmarried survivors of those alive and unmarried at age $x'-1$.

v The present value of $1 due one year hence at 3% compound interest.

$m_{x'}$ The number remarrying between ages x' and $x'-1$ and alive at age $x'+1$.

$$\bar{C}_{x'} = v^{x'+\frac{1}{2}} \left(\frac{m_{x'}}{1 - \frac{1}{2} q_{x'}} \right)$$

$$\bar{M}_{x'} = \sum_{x'=x'}^{x'=\infty} \bar{C}_{x'}$$

$$D_{x'} = l_{x'} v^{x'}$$

$$\bar{D}_{x'} = \frac{D_{x'} + D_{x'+1}}{2}$$

S = .99479364, the probability of a child age 17 or under living one year.

$${}_wD_{x'} = S_{w-1}^{x'} D_{x'}$$

$${}_w\bar{C}_{x'} = S^{x'+\frac{1}{2}} {}_{w-1}\bar{C}_{x'}$$

$${}_w\bar{D}_{x'} = \frac{{}_wD_{x'} + {}_wD_{x'+1}}{2}$$

$${}_w\bar{N}_{x'} = \sum_{x'=x'}^{x'=\infty} {}_w\bar{D}_{x'}$$

$${}_w\bar{M}_{x'} = \sum_{x'=x'}^{x'=\infty} {}_w\bar{C}_{x'}$$

$\bar{a}_{x'}$ The value of \$1 per year payable continuously commencing at age x and continuing as long as x' lives $= \dfrac{\bar{N}_{x'}}{D_{x'}}$

$\bar{a}_{y\,:\,\overline{18-y}|}$ The value of \$1 per year payable continuously commencing at age y and continuing to age 18.

${}_w\bar{a}_{x'y_w\,:\,\overline{18-y_w}|}$ The value of \$1 per year payable continuously during the joint lives of a widow age x' at date of valuation and w children of which the oldest is age y_w for $18\text{-}y_w$ years.

$$= \frac{{}_w\overline{N}_{x'} - {}_w\overline{N}_{x'+18-y_w}}{{}_wD_{x'}}$$

$\bar{E}_{x'}$ The value of \$1 payable in a lump sum upon remarriage of a widow age x' at date of valuation.

$$\frac{\overline{M}_{x'}}{D_{x'}}$$

${}_w\bar{E}_{x'y_w:\,\overline{17-y_w}|}$ The value of \$1 payable in a lump sum upon the remarriage of a widow, age x' at date of valuation, with w children, of which the oldest is age y_w within *18-y_w* years, if all w children are alive at time of marriage.

$$= \frac{{}_w\overline{M}_{x'} - {}_w\overline{M}_{x'+17-y_w}}{{}_wD_{x'}}$$