ON GRADUATING EXCESS PURE PREMIUM RATIOS

BY

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The objective in graduating data is to obtain their smooth rearrangement according to some pattern which there is reason to believe would fit the data if their volume were increased indefinitely. As some knowledge of the general characteristics of the data to be graduated is requisite in selecting the pattern to be used, it is desirable to make a preliminary survey of the material.

Nature of Data

The data discussed in this paper concern the excess pure premium ratio, which, with respect to an individual risk, may be defined as the ratio of the risk's losses in excess of a specific selected loss ratio to the total losses of the risk. For a group of risks the excess pure premium ratio for a given selected loss ratio r, is the ratio of the aggregate of the losses in excess of the loss ratio r in each risk to the aggregate total losses of the group. This may be expressed more precisely in mathematical form by the equation

$$y = \frac{\Sigma\,(L - r\,x)}{\Sigma\,L}, \text{ where}$$

y denotes the excess pure premium ratio for losses in excess of the loss ratio r.

L denotes the actual losses of the risk.

x denotes the risk premium.

r denotes the selected loss ratio, the losses above which are to be considered excess losses.

Σ denotes summation of the values for each risk in the group.

This equation is not general for it is necessary to place a restriction on the formula so that only the positive values of the term $(L - r\,x)$ are to be used. The selected loss ratio may be expressed either as an ordinary loss ratio r, that is as an index of the premium x which is taken as the base unity, or as an index of the expected loss ratio E. If the latter form is denoted by r' then r becomes $r'\,E$ in the formula.

Let the relation of the three variables; the excess pure premium ratio y, the selected loss ratio r, and the risk premium x be denoted in general by

$$y = F(r, x)$$

This general equation in rectangular coordinates (y, r, x) represents a surface. However, certain restrictions apply to the variables so that only a part of the surface is included in this study. A review of previous discussions of the subject and a study of exhibits showing these variables, together with reflection on the definitions of the terms will bring out these necessary restrictions:

x is positive, varying from 0 to ∞
r is positive, varying from 0 to ∞
y is positive and varies from 1 to 0.

If it is assumed that the experience of the risk under proper classification and on a correct premium level will approach the expected as the risk becomes indefinitely large* then still further limitations may be placed on these variables.

The data used in this paper pertain to Compensation Insurance exclusively. At the present time this is the only line having a large volume of experience available in a form that may be used readily for graduating excess pure premium ratios. As the characteristics of the excess pure premium ratios for various lines of Casualty Insurance are similar, the generalizations deduced from these data may be applied in varying degree to other lines.

Representation of Data in Three Dimensions

Assume that the variables y, r and x have been arranged in the definite order given in Table Ia, p. 148, or in Table I, p. 21, Vol. XX, P.C.A.S. In a system of rectangular coordinates take the risk premium x along the horizontal axis to the right, the selected loss ratio r along the horizontal axis toward the observer, and the excess pure premium ratio y along the vertical axis upward. The surface $y = F(r, x)$ and the coordinate planes YOX, YOR and

* This assumption is equivalent to the assumption in probabilities that the actual result will approximate the theoretically expected as the number of trials is increased indefinitely.

ROX will form a solid somewhat as indicated in Fig. I. If the figure is extended indefinitely to the right, a section parallel to the *YOR* plane approaches a right triangle. If the figure is extended indefinitely toward the observer, the height remains constant and equal to unity, but the width at the base decreases continually so that a section parallel to the *YOX* plane approaches a vertical straight line of unit length as the ultimate limit. If the figure is extended along the *ROX* plane when r and x are increased indefinitely, the height decreases and may be viewed ultimately as a mere film on the *ROX* plane.

Consider a section parallel to the *YOR* plane; its intersection with the surface consists of a curve which may be represented by the function $y = f(r)$. The curve starts at the point $(0, 1)$, decreases slowly as r is increased when the section is taken just to the left of the *YOR* plane, and decreases more rapidly at the beginning when the section is taken farther to the right. Ultimately the curve approaches the straight line $y + r/E = 1$, where E is the expected loss ratio, when the section is taken at the extreme right for indefinitely large risks.

Consider a section parallel to the *YOX* plane; its intersection with the surface is a curve, $y = f(x)$, which starts at the point $(0, 1)$ and decreases to the right, approaching an asymptote as x becomes indefinitely large. When $r < E$, the asymptote is the intersection line formed by the section and the plane represented by $y + r/E = 1$. When $r > E$, the asymptote is the line of intersection of the section with the *ROX* plane. The farther the section is taken from the *YOX* plane, i.e., the larger the r under consideration, the more rapidly the curve descends at the beginning. As r becomes extremely large, the curve approaches a vertical line of unit length as its limit.

Consider a section parallel to the *ROX* plane; its intersection with the surface represents the relation between the selected loss ratio r and the size of risk premium x for a fixed excess pure premium ratio y. It may be noted that the intersection is a curve which is asymptotic to the line where the section cuts the *YOR* plane at one end, and also asymptotic to the line where the section cuts the plane represented by $y + r/E = 1$ at the other.

The surface $y = F(r, x)$, intersects the *YOR* plane at $y = 1$ and the *YOX* plane at $y = 1$. In the region where $r < E$ the surface becomes asymptotic to the plane, $y + r/E = 1$, as x becomes indefinitely large. The surface also becomes asymptotic to the YOR plane as x decreases and r increases indefinitely. When both x and r are indefinitely increased the surface becomes asymptotic to the ROX plane. It should be noted that while the general shape of the surface in Fig. I is much affected by the relative size of the units chosen for y, r and x, the characteristics mentioned above are retained under any selected relativity of units.

Selection of Pattern

It is not practicable to use a general function representing such a complex surface as a pattern for graduating the excess pure premium ratios. It is practicable to use equations representing the curves formed by the intersection of the surface with planes parallel to one of the vertical coordinate planes as patterns. There is little interest in the relation of r and x for a given value of y, consequently the curves for sections parallel to the ROX plane will receive no further consideration. Primary interest exists in these relations:

1. The relation of y and r for a constant x, or the relation of the excess pure premium ratio and the selected loss ratio for a group of risks having approximately the same premium.
2. The relation of y and x for a constant r, or the relation of the excess pure premium ratio and the individual risk premium for a given selected loss ratio.

Relation of y and r, x constant

The characteristics for curves representing the first relation are those possessed by curves formed by the intersections of the surface $y = F(r, x)$ and planes parallel to the YOR plane. These characteristics may be recognized to some extent by using as a pattern equations in y and r, $y = f(r)$, which pass through the point $(0, 1)$ and become asymptotic to $y = 0$ as r is increased indefinitely. Among curves fulfilling these conditions are those represented by $y = 10^{-ar-br^2}$, and $y = c^{-r^n}$. These equations might be used as patterns and their constants determined for each size of risk so as to produce good fits.

The excess pure premium ratios were determined by premium size groups of risks. Any irregularity in the experience of a group which affects any of the excess pure premium ratios for a given selected loss ratio r, will also affect the excess pure premium ratios for all selected loss ratios which are less than r. As a result, an abnormality in the experience will affect the whole curve, or a large portion of it. Deviations of this sort cannot be overcome by smoothing the data. The graduations for various sizes of x would result in curves which viewed laterally formed elements of a surface which still had troughs and ridges very much as in the original data. It would be expected that curves formed by the intersections of lateral sections with such a projected surface would require considerable smoothing to eliminate these troughs and ridges. For this reason no effort has been made here to consider the curves represented by $y = f(r)$ but rather to direct attention to the curves formed by the intersections of the surface with sections parallel to the YOX plane.

Relation of y and x, r constant

The characteristics of curves under the second relation are those possessed by curves formed by the intersection of planes parallel to the YOX plane and the surface, $y = F(r, x)$. These characteristics would be recognized if the graduation used as a pattern an equation, $y = f(x)$, whose graph passed through the point $(0, 1)$, then decreased and as x increased indefinitely approached a definite asymptote dependent on r. An equation of the graph possessing these characteristics is $y = a + b/c^{x^n}$, where a, b, c and n are constants to be determined so that the formula fits the data for the particular section corresponding to a specific selected loss ratio r. The condition imposed on a and b in the formula and the application of the normal equations, are discussed in Appendix I. As an illustration, the graduation of the excess pure premium ratios corresponding to the selected loss ratio .50 is given in Table III.

The ungraduated excess pure premium ratios shown in Table Ia were determined after the premium level had been adjusted to produce the expected loss ratio not only for the experiences as a whole but for each size of risk group. In Table Ib are shown the graduated values as determined by the method explained in Appendix I.

Generally the results shown in the Table Ib indicate a fair fit. On inspection it will be noted that the adjusted values for each selected loss ratio in the $4,000-5,000 premium size group are under the original values in each instance. This might indicate that while the equations used as the pattern in graduating may be made to fit fairly well for the range of risks over $5,000, there may be some doubt as to whether the formula has sufficient flexibility to fit the whole range of risks including those under $5,000. To test the flexibility of the formula, the method of graduation was applied to the range of risks extending in size from $10 to over $16,000, shown in Table V, p. 173, Vol. XIII, P.C.A.S. The results shown below indicate a reasonable fit. However, there is a wide spread in the original ratios, which are based on scant data, and any general smoothing would seem likely to succeed in bringing the adjusted values reasonably within the extremes. It may be shown that if the formula were extended clear to the zero point, it would not fit small hypothetical risks, for example $1 premium or less.

Lower Limit Risk Group	Actual Pure Prem. Ratio	Graduated Ratio	Lower Limit Risk Group	Actual Pure Prem. Ratio	Graduated Ratio
(1)	(2)	(3)	(1)	(2)	(3)
$ 10	.945	.929	$ 500	.652	.637
25	.884	.896	700	.541	.582
50	.834	.867	1,000	.540	.533
75	.852	.843	1,500	.436	.456
100	.821	.816	2,500	.414	.383
150	.756	.784	4,000	.223	.259
200	.704	.747	8,000	.169	.143
300	.720	.707	16,000	.028	.029
400	.752	.674			

Graphs of Excess Pure Premium Ratios—Unlimited Per Case Losses

Graphs for the graduated excess pure premium ratios for various selected loss ratios in Table Ib have been drawn and are shown in Chart I. To the right, beyond the range of the actual data the curves have been extended showing how they approach their asymptotes. To the left below the smallest premiums in the data the curves have been extended in broken lines through selected points determined from the formula.

It has been found convenient to use three-cycle semi-logarithm paper in representing the premium field ($1,000-1,000,000) under consideration. With this ruling of the paper it is possible to devote two-thirds of the chart to that part, approximately one-tenth of the premium field, which is of primary importance. It is possible to use four-cycle paper to extend the field down to $100 premium risks, or to use five-cycle paper to go down to the $10 premium risks, though the latter would be questionable. It would not be advisable to use the semi-logarithm paper if the field were to be extended to $1 risks, and obviously it would be impossible to use this paper if the field were extended to zero. In interpreting semi-logarithm charts it must be recognized that slopes and curves do not have the same meaning as in charts having uniform scales. The ogive form of the "10M" curve, and the similar form which the selected loss ratio curves assume if extended far enough to the left are due entirely to the use of the semi-logarithm paper with its constantly changing horizontal scale, and not to any property inherent in the curves.

Effect of Per Case Limit

The line marked "10M" separates Chart I into a left side in which a further limitation of primary losses to $10,000 per case can have no possible effect on the excess pure premium ratios because the per case limit is greater than the per aggregate limit implicit in the given selected loss ratio, and into a right side in which the restriction of cases to $10,000 may result in reducing the primary losses and consequently in increasing the excess losses. There are regions also to the right of the "10M" line where the $10,000 limit will have no effect. In the upper right of the chart, for example, correctly classified risks having premiums of $500,000 or more on a proper premium level would develop, almost to a mathematical certainty, loss ratios of say at least .15, even if individual cases were limited to $10,000. The amount that would be excluded from primary losses under the per case limit would in all probability already have been excluded under the small aggregate limit imposed by the low selected loss ratio. The upper right region excluded cannot be demarcated definitely for in going upward or to the right the effect of the per case limit becomes very small gradually, ultimately becoming infinitesimal.

The effect produced by imposing a maximum per case limit in addition to a per loss ratio limit may be determined largely from theoretical considerations. A study and interpretation of Fig. I, Table I, Chart I, and the definitions will show these deductions to be reasonable.

(a) Imposing a per case limit in addition to a per loss ratio limit will have no effect on the excess pure premium ratio if the per case limit is greater that the imposed per aggregate limit implicit in the selected loss ratio and the risk size. This condition prevails in the region to the left of the "10M" line in Chart I.

(b) For a given selected loss ratio which is greater than the expected, the per case limit will begin to be effective when the risk premium reaches the point where the selected loss ratio curve crosses the "10M" line in Chart I. The effect at first is small but gradually increases until the full value of .042 (see Table IV) has been attained.

(c) For selected loss ratios less than the expected, the per case limit becomes effective gradually after the selected loss ratio curve has intersected the "10M" curve, reaches a maximum some time later and then decreases until the effect disappears entirely in extremely large risks. To this general relation there are two exceptions, when the selected loss ratio is very small and also when the selected loss ratio lies between the expected loss ratio (.598) and one which is .042 $\times$ .598 or .025 less than the expected.

When the selected loss ratio is so small that the risk must become so large before the selected loss ratio curve crosses the "10M" line that virtually every risk, even with losses limited on a per case basis, will develop a loss ratio not less than the given loss ratio r, then the presence of a maximum limit per case will produce no effect on the excess pure premium ratios.

When the selected loss ratio lies between the expected and one which is .025 less than the expected, the effect on indefinitely large risks will approach and ultimately equal the difference between the selected loss ratio and one which is .025 less than the expected when this difference is expressed in terms of the expected loss ratio.

Effect Expressed with Use of Symbols

These relations may be expressed more precisely with the aid of symbols;

where l denotes the per case limit on losses, \$10,000
r denotes the selected loss ratio
x denotes the risk premium
E denotes the expected loss ratio, (.598)
y denotes the excess pure premium ratio
e denotes the increment on y due to superimposing a per case limit l on a per loss ratio limit r.

then for $r > 0$,	$e = 0$ if $x < l/r$
$r > E$,	e begins at point where $x = l/r$, increases to .042, for $x = \infty$
$E > r > (E - .025)$	e begins at point where $x = l/r$, increases to $[.025 - (E - r)]/E$ for $x = \infty$
$(E - .025) > r$,	e begins at point where $x = l/r$, increases first and then decreases to 0, for $x = \infty$

The equations of the asymptotes of the curves $y = f(x)$ for the various selected loss ratios with unlimited losses and with limited losses are as follows:

	Equations of Asymptotes	
Value of r	*Unlimited Losses*	*Limited Losses*
$r > E$	$y = 0$	$y = .042$
$E > r > (E - .025)$	$y = (E - r)/E$	$y = .042$
$r < (E - .025)$	$y = (E - r)/E$	$y = (E - r)/E$

Graphs of Excess Pure Premium Ratios—Limited Per Case Losses

The graphs for selected loss ratios with losses limited on a per case basis might be constructed directly from actual excess pure premium ratios calculated by omitting the excess per case losses in obtaining the numerator of the ratio but using unlimited losses for the denominator. The excess pure premium ratios could then be graduated by some process similar to that used for the excess pure premium ratios with unlimited losses in Appendix I. This procedure would produce different adjusted pure premium ratios for each per case limit even for the low selected loss ratios and small risks where the limit could not possibly affect the result. The method followed here consists of using the graduated curves for the unlimited losses as basic and then adapting them graphically to conform to the known requirement of having the departure begin at the point where the "10M" line crosses the curves, of approaching the new asymptote at the extreme right of the chart when the risk becomes indefinitely large, and of passing through some of the intermediate points determined from a comparison of the excess pure premium ratios calculated first with unlimited losses and then with losses limited on a per case basis.

The effect of the per case limit is given in Table IVa which shows the remainders when the actual excess pure premium ratios with limited per case losses are subtracted from the corresponding actual excess pure premium ratios without per case limits on losses. The portion of the total losses excluded from primary losses with both per loss ratio and per case limits on losses is equal to the excess pure premium ratio calculated with excess losses unlimited plus .042, the value of the New York losses eliminated by the excess per case limit of $10,000. The net increase of the non-primary or excess losses in excess of a given selected loss ratio, combined with a $10,000 per case limit, over the non-primary losses without per case limits is .042 minus the values shown in Table IVa. As might be expected on account of the small volume of experience, the per case limits affected the various premium size groups differently and in not a single group was the .042 average derived from all New York losses combined reproduced in Table IVa. To eliminate these variations, all the differences in Table IVa were expressed as indexes of the left hand column, then multiplied by .042. These results are shown in Table IVb. To obtain the net increase in non-primary losses the values in Table IVb must be subtracted from .042. These differences are shown in Table IVc. If the values in Table IVc are first smoothed and expressed in a new Table IVd, then with proper interpolations this Table IVd may be used to determine the effect of the per case limit for intermediate points. Using the value of certain intermediate points from IVd along with the known relations at the point where the selected loss ratio curve crosses the "10M" line, and knowledge of the asymptotes for curves with per case limit losses, the necessary adaptations to the Chart I curves, were made graphically. The new curves are shown in Chart II, which is a reproduction of the New York Board chart.

In actual practice the relations of y, r and x are usually shown in charts in which the excess pure premium ratio y is plotted against the selected loss ratio r for specific risk sizes x. Chart I and Chart II may be readily transformed into new charts having these relations by taking the vertical line corresponding to a definite risk size x, and plotting the intersections of this line with the selected loss ratio curves onto a new rectangular chart in which the ordinates represent excess pure premium ratios and the abscissas represent the selected loss ratio. The points pertaining to a definite risk size are then joined and the connecting curve is designated by the risk premium size. Such a transformation of Chart I has been made and is shown as Chart III. A similar transformation of Chart II is shown as Chart IV.

In summary it is apparent that the selection of the equation used is arbitrary. The advantages that may be credited to it are its relative simplicity, its not too restricted flexibility, and its adaptability to the conditions at the very beginning, the zero point, and at the extreme end, the indefinitely large risk. The disadvantages that may be charged against it are that it lacks extended flexibility, that it is necessary to give special arbitrary treatment to zero values and that it is not well adapted to apply to actual experience, but really requires prior adjustment of experience to the expected loss ratio basis.

It is apparent too, that the procedure is somewhat hybrid using first algebraic methods to graduate the excess pure premium ratio for unlimited loss experience and then superimposing graphic methods to depict the deviations caused by the per case limit on losses. The choice of this procedure arose out of a desire to consider the excess pure premium ratios for selected excess loss ratios with various per case limits in terms of the basic. It will be recalled that to the left of the point where the per case limit equals the aggregate loss limit on the selected loss ratio curve, no effect results from placing an additional per case limit on the losses, and it would seem desirable to leave that portion of the ungraduated curve the same irrespective of any later effect due to the use of per case limits on losses.

Finally, there may be serious question whether at this stage of our knowledge such a refinement of graduated pure premium ratios is warranted. Practically the same result could have been accomplished by a simple graphical method, particularly where large aggregates of experience are involved. Admittedly, at this time the more fundamental considerations of whether the excess pure premium ratios should be based on all industry combined or should vary by industry, whether they should be based on actual experience or on adjusted experience, are of greater significance than the refinement brought about by any graduating process. Recognizing the relative importance of these problems, it seems, nevertheless, that the study of the problem of graduation of excess pure premium ratios by the Actuarial Committee of the New York Compensation Insurance Rating Board, in which the method described in this paper was developed, has been worth while if not for the direct results produced in greater refinements, then for the development of a more intimate knowledge of the behavior of excess pure premium ratios.

APPENDIX I

Graduating Excess Pure Premium Rates by Method of Least Squares

(1) Let $y = a + b/c^{x^n}$

where $y =$ excess pure premium ratio
$x =$ risk premium in thousands
$r =$ selected loss ratio
$a, b,$ and c are constants, to be determined for each r
$a = (E - r)/E$
$b = 1 - a = r/E$
$E =$ expected loss ratio, .598

Then $(y-a)/b = 1/c^{x^n}$

$x^n \log c = -\log[(y-a)/b]$

$n \log x + \log\log c = \log\{-\log[(y-a)/b]\}$

(2) $n \log x + B - A = 0$

where $B = \log\log c$
$A = \log\{-\log[(y-a)/b]\}$

When formula (2) is applied to the pure premium ratios corresponding to a selected loss ratio r the values for $\log x$ and A may be determined for each of the fourteen risk premium groups in Table Ia. The problem is to determine in accordance with the method of least squares, n and B and then c so that equation (1) will represent the best fit for the fourteen points.

The normal equations for n and B in (2) are:

$$n \Sigma \log x + \Sigma B - \Sigma A = 0$$
$$n \Sigma (\log x)^2 + B \Sigma \log x - \Sigma A \log x = 0.$$

It is worth while when applying the method of least squares repeatedly to a set of data as in Table Ia to calculate an auxiliary table as an aid in solving the normal equations. Table II has been made for this purpose. As an illustration the process of graduating the excess pure premium ratios for the selected loss ratio .50 in Table Ia will be shown in detail.

$$a = (.598 - .50)/.598 = .1639$$
$$b = .50/.598 = .8361$$
$$y = .1639 + .8631/c^{x^n}$$
$$n \log x + B - A = 0$$
$$\text{where } B = \log \log c$$
$$A = \log \{-\log [(y - .1639)/.8361]\}$$

The development of the normal equations requires the preliminary calculation of coefficients and constants which may be made most conveniently in some tabular form. In Table III, Columns 1-10, which are self explanatory, these calculations have been made. From Table III and the auxiliary Table II the normal equations may be written as:

$$18.35278\, n + 14\, B + 1.99797 = 0$$
$$26.72278\, n + 18.35278\, B + 1.85452 = 0$$

The solution of the normal equations may be obtained more readily by passing directly to the derived equation for B given below Table II and substituting therein values taken from columns 8 and 10 of Table III and from columns 4, 6 and 9 of Table II.

Solving

$$B = -.518998$$
$$\log c = .30269$$
$$c = 2.0077$$
$$n = .28704$$

Substituting these values equation (3) becomes:

$$y = .1639 + .8361/2.0077^{x^{.28704}}$$

The adjusted y's are calculated in Table III, Columns 11-16, and entered on line $r = .50$ in Table Ib. If a similar procedure is followed for each value of r given in Table Ia; then the adjusted values in Table Ib will be determined.

There is one difficulty inherent in the formula that arises

$$\text{when } y = 0, \qquad \text{for } r > E$$
$$\text{and } y = a \qquad \text{for } r < E$$

In this case infinite values are introduced into Table III, and consequently into the normal equations giving some terms such great weight that the resulting curve becomes a straight line—its asymptote. This invalidates the procedure making other recourses necessary. To circumvent this difficulty two courses may be followed. The zero value point may be omitted entirely, which amounts to giving no weight whatever to the experience, or a small arbitrary value may be used. This value should be small enough so that its effect is to depress the resulting curve below the curve that would be obtained if the point were omitted entirely. When there are zero values for several consecutive risk size groups, only those corresponding to the *lower* premium size groups need be given arbitrary values, the others being omitted.

If the premium level of each premium size group had not been adjusted so as to produce the expected loss ratio E for the group, then such additional difficulties would arise in groups having redundant premiums where the excess pure premium ratio might become less than a—the ordinate of the asymptote—that the formula would become useless.

Relation of Parameters a, b, c and n

The selection of the formula $y = a + b/c^{x^n}$ as a pattern creates an interest in the interrelations of the parameters a, b, c and n. The relation of a and b as connected with r and E which has already been stated arises from the conditions which require the graph to pass through $(0, 1)$ at the left and to be asymptotic to $y = (E-r)/E$ if $r < E$ and asymptotic to $y = 0$ if $r > E$. These relations are fairly evident.

The relations of c and n to each other and the other parameters are not so clear. As x and y are taken positive and as $y < 1$, it follows that c must be greater than one. It appears from empirical relations that c is approximately at a minimum when $r = E$ and increases as r moves away from E in either direction. The behavior of n is more obscure. It may be shown that for $r > E$ the values of n should be equal and possibly this relation holds for $r < E$. This means that a constant n should produce the best fit for the surface over the region where $r > E$. An interpretation of the varying n's may be given as meaning that each n produces the best fit according to the standards of the method of least squares for the excess pure premium ratios corresponding to the specific value selected for r and the particular groupings of risks as used here. Any rearrangement of the premium size groupings would in general produce a different set of n's.

The table following shows the values of c and n corresponding to the various selected risk ratios used in the graduation.

r	a	b	c	n
.10	.8328	.1672	4.4104	.43904
.20	.6656	.3344	3.1096	.35887
.30	.4983	.5017	2.4572	.34931
.40	.3311	.6689	2.0878	.34099
.50	.1639	.8361	2.0077	.28704
.60	0	1.0000	1.9179	.24892
.70	0	1.0000	1.9921	.29182
.80	0	1.0000	2.0741	.32321
.90	0	1.0000	2.1001	.36388
1.00	0	1.0000	2.1514	.39212
1.25	0	1.0000	2.2152	.46276
1.50	0	1.0000	2.5142	.46944
2.00	0	1.0000	3.1729	.47853
3.00	0	1.0000	4.9151	.46631

$a = (.598 - r)/.598$, $b = 1 - a$

If $r = .598$, $a = 0$, and $b = 1$

FIGURE I

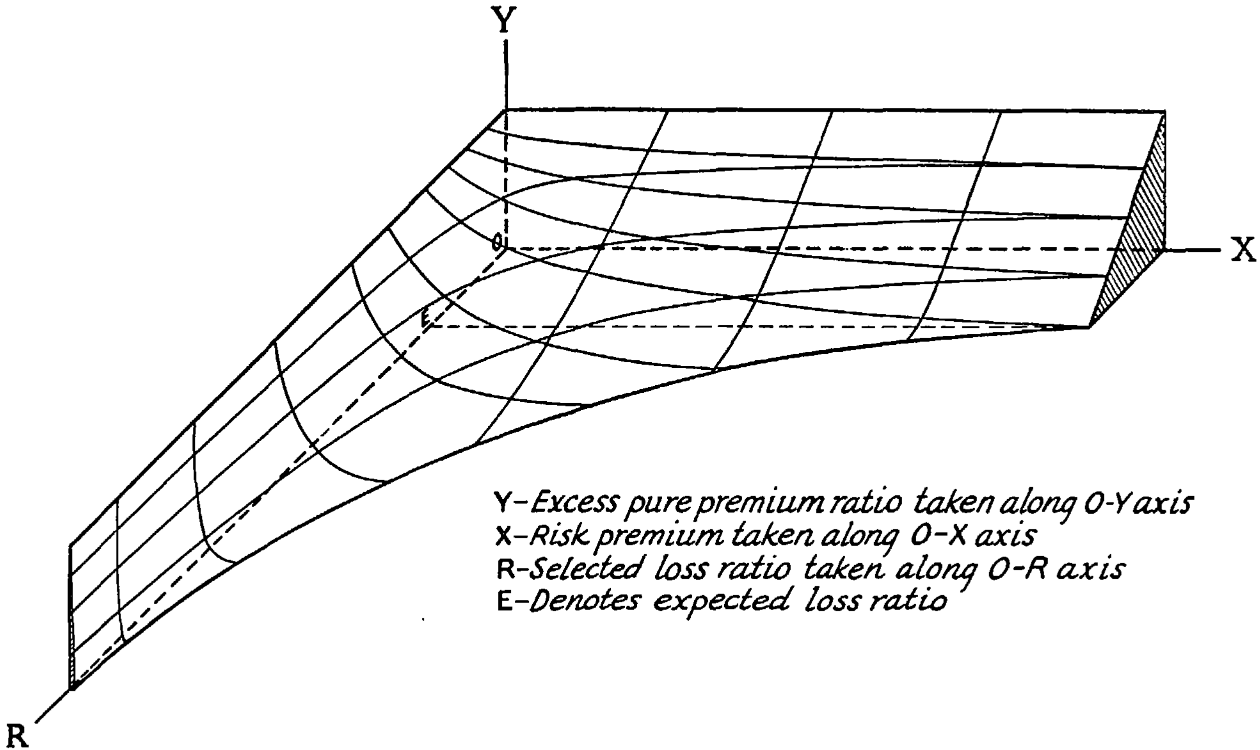
Y
X
R
O
E
Y-Excess pure premium ratio taken along O-Y axis
X-Risk premium taken along O-X axis
R-Selected loss ratio taken along O-R axis
E-Denotes expected loss ratio

TABLE Ia

Actual Pure Premium Ratios for Excess Limits per Loss Ratio

Table showing pure premium ratios—ratios of losses in excess of selected risk loss ratios to total losses—for various selected risk loss ratios by size of risk groups. New York Compensation Experience. All Industry Combined: Policy Years 1936-1937 for risks under $10,000 premium. Policy Years 1934-1937 for risks over $10,000 premium.

a — Lower Limits of Premium Size Groups in $1,000 Units
b — Average Risk Premium of Groups in $1,000 Units
c — Number of Risks in Groups

Selected Risk Loss Ratios														
a	4.0	5.0	7.0	8.0	9.0	10.0	15.0	20.0	25.0	30.0	40.0	50.0	75.0	100.0
b	4.6	6.1	7.5	8.1	9.8	11.8	18.0	22.1	25.9	34.9	48.6	60.8	85.5	127.4
c	940	1025	328	260	222	916	365	205	126	118	68	67	21	15
00%	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10	.845	.843	.838	.840	.834	.834	.833	.833	.833	.833	.833	.833	.833	.833
20	.720	.709	.694	.700	.688	.684	.677	.671	.673	.676	.671	.668	.667	.666
30	.619	.595	.574	.567	.559	.553	.537	.527	.527	.534	.524	.511	.513	.499
40	.537	.502	.481	.466	.449	.447	.424	.404	.400	.406	.401	.369	.364	.335
50	.470	.425	.407	.385	.356	.365	.332	.310	.297	.299	.302	.249	.232	.204
60	.416	.361	.349	.321	.284	.301	.259	.226	.221	.220	.224	.162	.128	.111
70	.369	.307	.300	.266	.225	.249	.201	.166	.159	.157	.163	.099	.073	.052
80	.331	.263	.262	.221	.181	.208	.156	.122	.112	.124	.113	.060	.036	.029
90	.299	.228	.233	.186	.147	.174	.124	.091	.080	.086	.074	.032	.014	.015
100	.270	.199	.205	.156	.119	.146	.099	.069	.056	.071	.050	.013	.003	.006
125	.211	.142	.152	.100	.070	.092	.058	.040	.024	.046	.015	.000	.000	.000
150	.168	.101	.111	.066	.042	.058	.033	.028	.008	.033	.003	.000	.000	.000
200	.110	.053	.064	.037	.017	.025	.013	.008	.001	.021	.000	.000	.000	.000

TABLE Ib

Graduated Pure Premium Ratios for Excess Limits per Loss Ratio

Table showing the data in Table Ia after graduation by method outlined in Appendix I.

a: Lower Limits of Premium Size Groups in $1,000 Units
b: Average Risk Premiums of Groups in $1,000 Units
c: Number of Risks in Groups

Selected Risk Loss Ratios	a b c	4.0 4.6 940	5.0 6.1 1025	7.0 7.5 328	8.0 8.1 260	9.0 9.8 222	10.0 11.8 916	15.0 18.0 365	20.0 22.1 205	25.0 25.9 126	30.0 34.9 118	40.0 48.6 68	50.0 60.8 67	75.0 85.5 21	100.0 127.4 15
00%		1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
10		.842	.839	.837	.837	.836	.835	.834	.833	.833	.833	.833	.833	.833	.833
20		.713	.704	.698	.696	.691	.687	.679	.676	.674	.671	.669	.668	.667	.666
30		.607	.591	.580	.576	.567	.558	.541	.534	.529	.521	.514	.510	.505	.502
40		.525	.503	.486	.480	.466	.452	.424	.412	.403	.388	.373	.365	.354	.345
50		.448	.424	.405	.399	.383	.367	.333	.318	.306	.285	.264	.251	.233	.215
60		.386	.361	.341	.334	.317	.300	.262	.245	.231	.207	.180	.164	.139	.113
70		.341	.312	.289	.281	.262	.242	.201	.183	.168	.143	.118	.102	.080	.059
80		.303	.271	.246	.239	.218	.198	.156	.138	.124	.100	.077	.064	.046	.030
90		.275	.240	.213	.205	.183	.161	.119	.102	.089	.067	.047	.037	.024	.013
100		.249	.212	.184	.176	.154	.133	.092	.076	.064	.046	.030	.022	.013	.006
125		.200	.160	.132	.124	.102	.082	.048	.036	.028	.016	.008	.005	.002	.001
150		.152	.117	.093	.086	.068	.053	.028	.019	.014	.008	.003	.002	.001	.000
200		.091	.065	.048	.043	.032	.023	.010	.006	.004	.002	.001	.000	.000	.000

CHART I.

EXCESS PER LOSS RATIO PURE PREMIUM RATIOS FOR SELECTED LOSS RATIOS (S.L.R.)

Data from Table I-b.

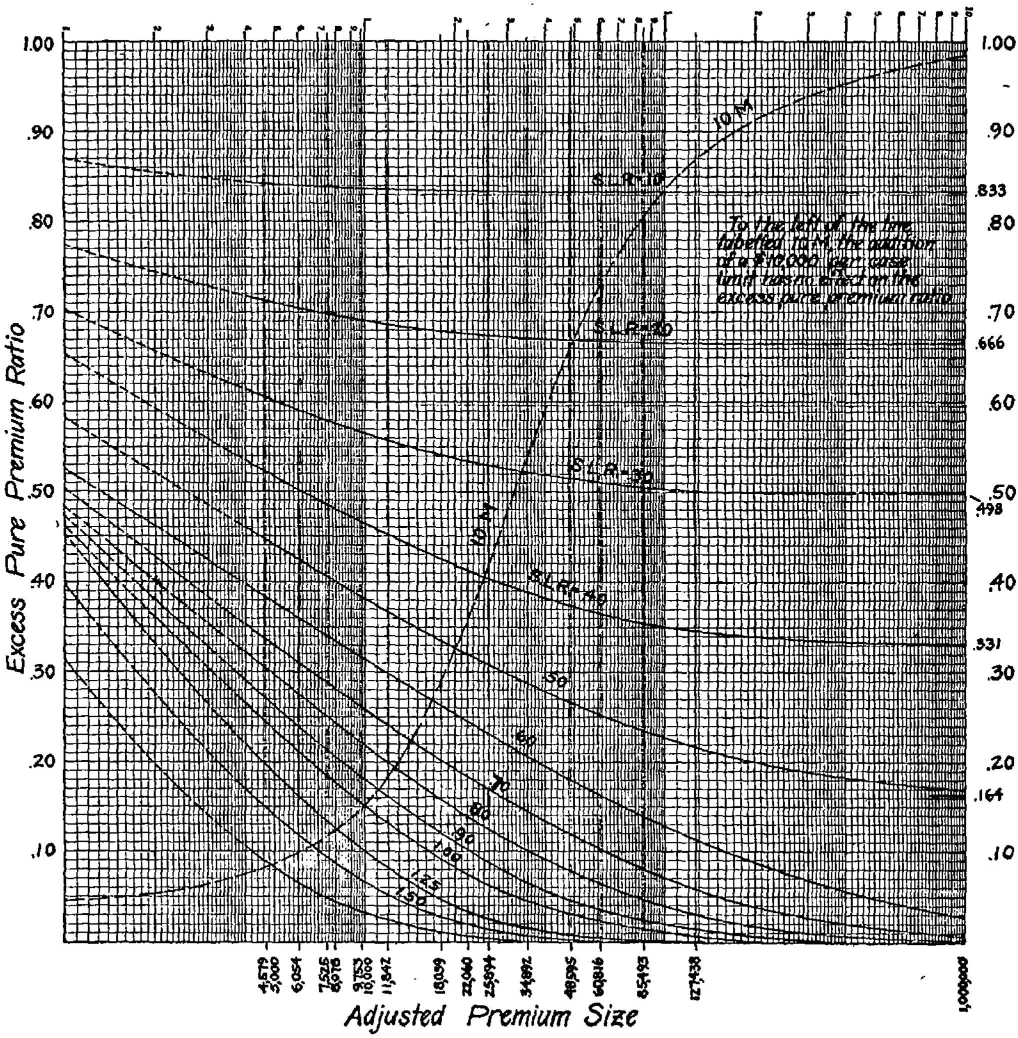

CHART II.

EXCESS PER LOSS RATIO PURE PREMIUM RATIOS
FOR SELECTED LOSS RATIOS (S.L.R.)

Graduated Ratios Derived from Actual Values (Col. a) of C.I.R.B. Exhibit B, 1/17/1941. $10,000 Loss Limitation Increment Added Graphically. All Industries. Risks over $4,000. Losses Limited to $10,000 per case.

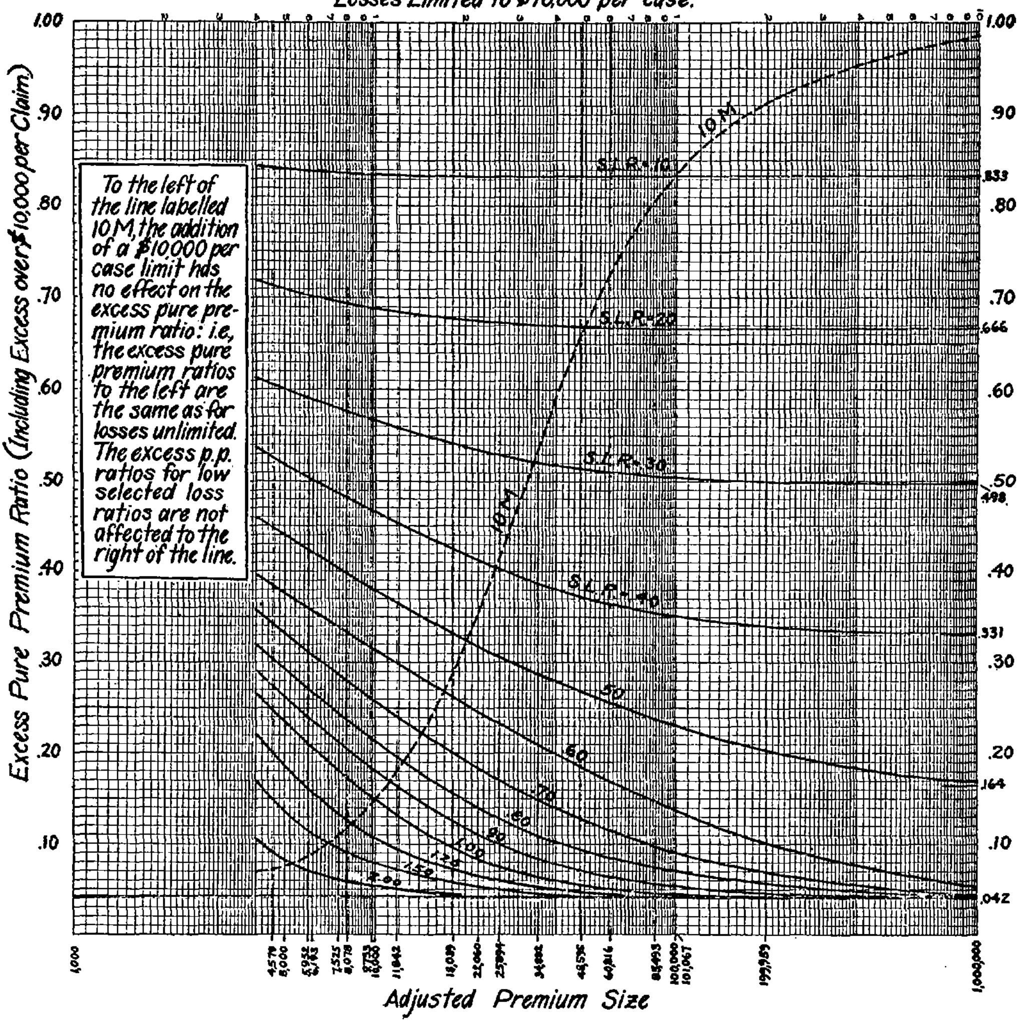

CHART III

Based on Chart I.

Ratio to Total Losses of Losses in Excess of Various Selected Loss Ratios per Risk. Individual Losses Unlimited

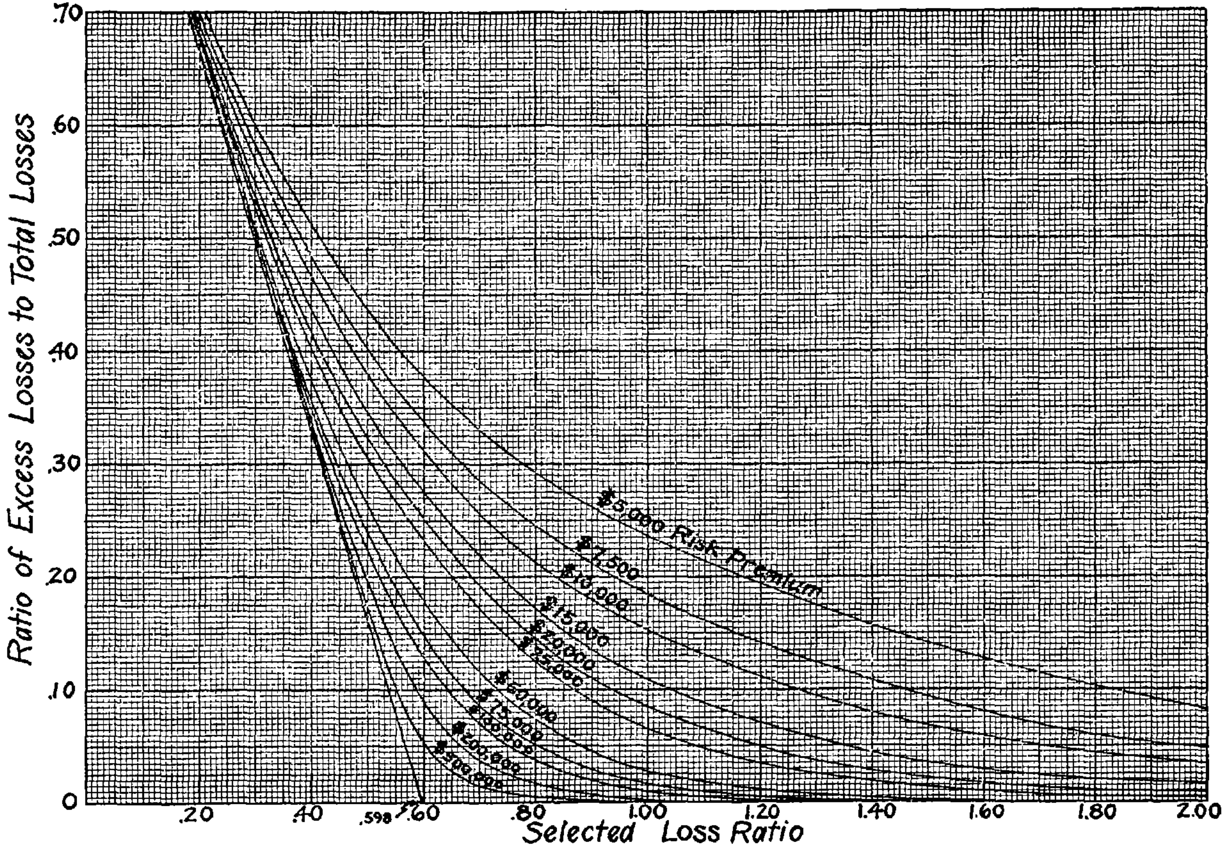

CHART IV

Based on Chart II.

Ratio to Total Losses of Losses in Excess of $10,000 Limit per Claim and Limited Losses in Excess of Various Selected Loss Ratios per Risk.

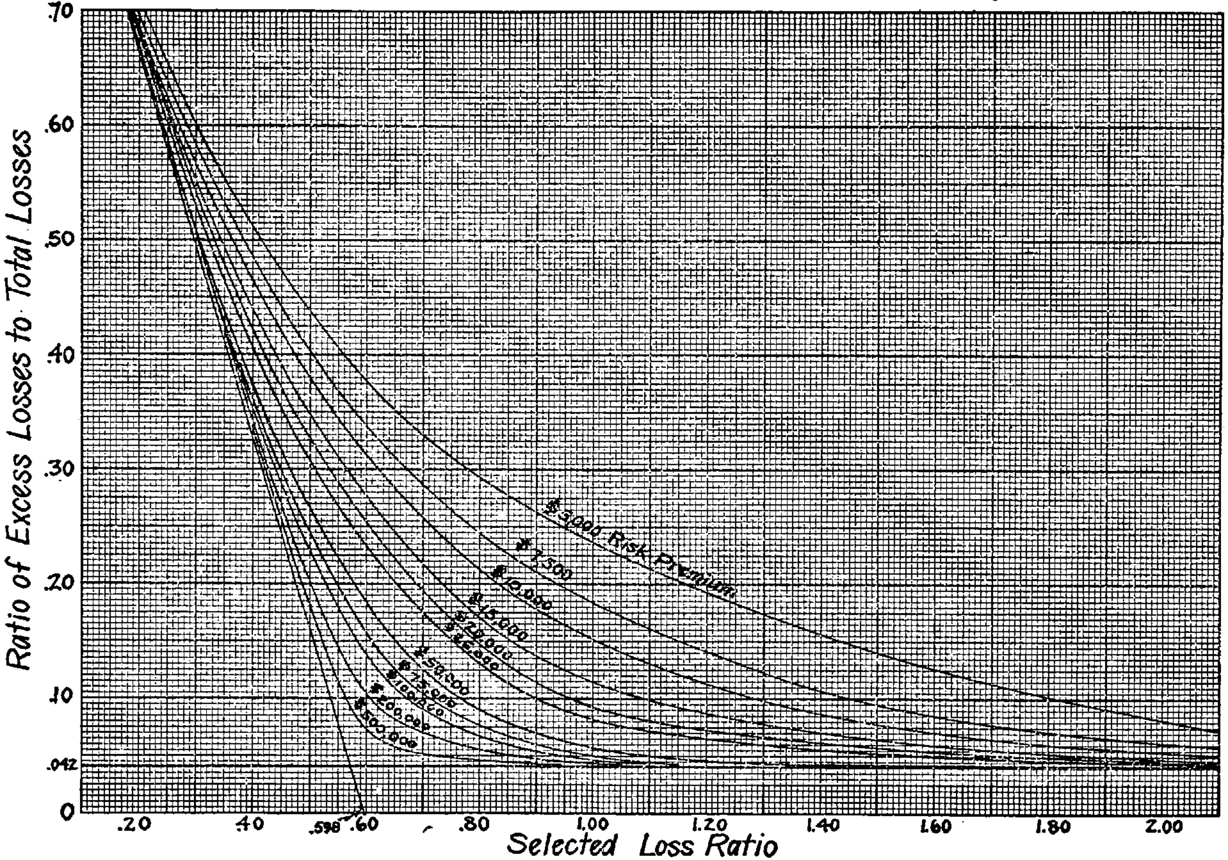

TABLE II

Auxiliary Table for Normal Equations

N	x	log x	Σ log x	$(\log x)^2$	$\Sigma(\log x)^2$	$\frac{N}{\Sigma \log x}$	$\frac{\Sigma \log x}{\Sigma(\log x)^2}$	(7)-(8)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	4.579	.66077	.66077	.43662	.43662	1.5134	1.5134	-
2	6.054	.78204	1.44281	.61159	1.04821	1.38618	1.37645	.00973
3	7.525	.87651	2.31932	.76827	1.81648	1.29348	1.27682	.01666
4	8.078	.90730	3.22662	.82319	2.63967	1.23969	1.22236	.01733
5	9.753	.98914	4.21576	.97840	3.61807	1.18603	1.16520	.02083
6	11.842	1.07343	5.28919	1.15225	4.77032	1.13439	1.10877	.02562
7	18.039	1.25621	6.54540	1.57806	6.34838	1.06945	1.03103	.03842
8	22.060	1.34361	7.88901	1.80529	8.15367	1.01407	.967541	.04653
9	25.894	1.41320	9.30221	1.99713	10.15080	.967512	.916402	.051110
10	34.892	1.54273	10.84494	2.38002	12.53082	.922089	.865461	.056628
11	48.595	1.68659	12.53153	2.84459	15.37541	.877786	.815037	.062749
12	60.816	1.78402	14.31555	3.18273	18.55814	.838249	.771389	.066860
13	85.493	1.93193	16.24748	3.73235	22.29049	.800124	.728897	.071227
14	127.438	2.10530	18.35278	4.43229	26.72278	.762827	.686784	.076043

Normal Equations may be written

$$n + NB/\Sigma \log x - \Sigma A/\Sigma \log x = 0$$

$$n + B\Sigma \log x/\Sigma(\log x)^2 - \Sigma(A \log x)/\Sigma(\log x)^2 = 0$$

$$\left[N/\Sigma \log x - \Sigma \log x/\Sigma(\log x)^2\right] B = \Sigma A/\Sigma \log x - \Sigma(A \log x)/\Sigma(\log x)^2$$

where $B = \log \log c$

$$A = \log\left\{- \log\left[(y - a)/b\right]\right\}$$

$$\Sigma B = NB$$

TABLE III

Calculation sheet for fitting, by method of least squares, the formula, $y = .1639 + .8361/c^{n^x}$, to the excess pure premium ratios for selected loss ratio .50 in Table Ia. Adjusted excess pure premium ratios in Column 16, Table III are entered for selected loss ratio .50 in Table Ib.

x	y	(2) - .1639	(3)/.8361	log(4) + 10	-(5)+10	log (6) + 10	(7)-10	log x	(8)×(9)	.28704 × (9)	antilog (11)	.30269 ×(12)	-(13) +10	antilog [(14)-10]	.1639 + .8361×(15)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
4.579	.470	.3061	.36610	9.56360	.43640	9.63988	-.36012	.66077	-.23796	.18967	1.5476	.46844	9.53156	.34006	.448
6.054	.425	.2611	.31228	9.49454	.50546	9.70369	-.29631	.78204	-.23173	.22448	1.6768	.50755	9.49245	.31078	.424
7.525	.407	.2431	.29075	9.46352	.53648	9.72955	-.27045	.87651	-.23705	.25159	1.7848	.54024	9.45976	.28824	.405
8.078	.385	.2211	.26444	9.42233	.57767	9.76168	-.23832	.90730	-.21623	.26043	1.8215	.55135	9.44865	.28096	.399
9.753	.356	.1921	.22976	9.36127	.63873	9.80532	-.19468	.98914	-.19257	.28392	1.9227	.58198	9.41802	.26183	.383
11.842	.365	.2011	.24052	9.38115	.61885	9.79159	-.20841	1.07343	-.22371	.30812	2.0329	.61534	9.38466	.24247	.367
18.039	.332	.1681	.20105	9.30330	.69670	9.84305	-.15695	1.25621	-.19716	.36058	2.2939	.69434	9.30566	.20214	.333
22.060	.310	.1461	.17474	9.24239	.75761	9.87945	-.12055	1.34361	-.16197	.38567	2.4304	.73566	9.26434	.18380	.318
25.894	.297	.1331	.15919	9.20192	.79808	9.90205	-.09795	1.41320	-.13842	.40564	2.5447	.77026	9.22974	.16972	.306
34.892	.299	.1351	.16158	9.20839	.79161	9.89851	-.10149	1.54273	-.15657	.44283	2.7722	.83912	9.16088	.14484	.285
48.595	.302	.1381	.16517	9.21793	.78207	9.89325	-.10675	1.68659	-.18004	.48412	3.0487	.92281	9.07719	.11945	.264
60.816	.249	.0851	.10178	9.00766	.99234	9.99666	-.00334	1.78402	-.00596	.51209	3.2515	.98420	9.01580	.10371	.251
85.493	.232	.0681	.08145	8.91089	1.08911	10.03707	+.03707	1.93193	+.07162	.55454	3.5854	1.08526	8.91474	.08218	.233
127.438	.204	.0401	.04796	8.68088	1.31912	10.12028	+.12028	2.10530	+.25323	.60431	4.0208	1.21706	8.78294	.06067	.215
					10.54023		-1.99797		-1.85452						

T A B L E I V.

Part IVa - shows the values of $y-\bar{y}$ derived from New York Board Data. Unit = .001
y = excess pure premium ratio, losses unlimited.
$\bar{y}$ = excess pure premium ratio, losses limited to $10,000 per case.

Part IVb - shows the values in IVa expressed as indexes of value in "00" column and then multiplied by 42. Unit = .001

	Aver. Prem. X	Selected Risk Loss Ratio expressed in terms of Expected Loss Ratio																			
		.00	.15	.30	.45	.60	.75	.90	1.05	1.20	1.35	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	3.30	5.00
IVa	4.5	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	44	32
	5.5	20	24	20	20	20	20	19	20	20	19	15	20	20	20	19	19	20	20	20	9
	6.5	22	23	23	22	22	22	22	22	23	22	21	22	22	22	23	22	22	23	19	5
	7.5	36	36	36	36	35	36	36	36	36	36	36	36	36	36	36	36	35	35	34	12
	8.5	28	29	28	28	28	28	28	28	28	28	29	28	29	29	28	28	26	25	21	-
	9.5	11	11	10	11	10	11	11	9	11	11	11	10	11	11	10	10	9	9	6	-
	12.1	47	47	47	48	47	47	47	47	47	47	47	47	45	42	39	36	32	27	17	4
	17.1	57	57	57	56	57	57	57	56	55	54	52	57	54	40	37	32	29	25	13	1
	22.3	36	35	35	36	36	36	36	35	33	29	26	22	20	18	18	18	18	17	7	-
	27.3	29	29	29	29	28	29	29	28	24	21	17	14	12	10	6	5	4	3	-	-
	33.8	43	43	43	44	43	43	43	41	36	31	26	19	16	13	9	6	4	4	3	4
	45.0	57	57	57	57	58	57	57	54	48	44	36	30	26	18	12	9	4	4	-	-
	61.6	60	60	59	60	60	58	56	50	43	29	14	6	4	1	-	-	-	-	-	-
	85.7	46	45	46	46	45	40	32	24	19	9	5	4	-	-	-	-	-	-	-	-
	119.6	61	61	61	61	61	61	51	44	20	17	3	-	-	-	-	-	-	-	-	-
	204.1	20	20	21	20	21	20	20	7	-	-	-	-	-	-	-	-	-	-	-	-
	Aver.	42																			
IVb	4.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	32
	5.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	19
	6.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	36	10
	7.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	41	41	40	14
	8.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	39	37	32	-
	9.5	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	42	34	34	23	-
	12.1	42	42	42	42	42	42	42	42	42	42	42	42	40	38	35	32	26	22	12	1
	17.1	42	42	42	42	42	42	42	42	42	42	42	42	40	29	27	24	21	18	10	1
	22.3	42	42	42	42	42	42	42	41	39	34	30	26	23	21	21	21	21	20	8	-
	27.3	42	42	42	42	42	42	42	41	35	30	25	20	17	14	9	7	6	4	-	-
	33.8	42	42	42	42	42	42	42	40	35	30	25	19	16	13	9	6	4	4	3	4
	45.0	42	42	42	42	42	42	42	40	35	32	27	22	19	13	9	7	3	3	-	-
	61.6	42	42	42	42	42	41	39	35	30	20	10	4	3	1	-	-	-	-	-	-
	85.7	42	42	42	42	42	37	29	22	17	8	5	4	-	-	-	-	-	-	-	-
	119.6	42	42	42	42	42	42	35	30	14	12	2	-	-	-	-	-	-	-	-	-
	204.1	42	42	42	42	42	42	42	15	-	-	-	-	-	-	-	-	-	-	-	-

T A B L E I V
(continued)

IVc - shows remainders when values in IVb are subtracted from 42. Unit = .001

IVd - shows the values in IVc after smoothing. This represents the additional loss in the excess portion when $10,000 per case limit is added to the per loss ratio limit. Unit = .001

	Aver. Prem. X	Selected Risk Loss Ratio expressed in terms of Expected Loss Ratio																			
		.00	.15	.30	.45	.60	.75	.90	1.05	1.20	1.35	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	3.30	5.00
IVc	4.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
	5.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	23
	6.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	33
	7.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	28
	8.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3	5	10	42
	9.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	8	8	19	42
	12.1	0	0	0	0	0	0	0	0	0	0	0	0	2	4	7	10	16	20	30	41
	17.1	0	0	0	0	0	0	0	0	0	0	0	0	2	13	15	18	21	23	32	41
	22.3	0	0	0	0	0	0	0	1	3	8	12	16	19	21	21	21	21	22	34	42
	27.3	0	0	0	0	0	0	0	1	7	12	17	22	25	28	33	35	36	38	42	42
	33.8	0	0	0	0	0	0	0	2	7	12	17	23	26	29	33	36	38	38	39	38
	45.0	0	0	0	0	0	0	0	2	7	10	15	20	23	29	33	35	39	39	42	42
	61.6	0	0	0	0	0	1	3	7	12	22	32	38	39	41	42	42	42	42	42	42
	85.7	0	0	0	0	0	5	13	20	25	34	37	38	42	42	42	42	42	42	42	42
	119.6	0	0	0	0	0	0	7	12	28	30	40	42	42	42	42	42	42	42	42	42
	204.1	0	0	0	0	0	0	0	27	42	42	42	42	42	42	42	42	42	42	42	42
IVd	4.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10
	5.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	20
	6.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	6	28
	7.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	3	15	35
	8.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	6	18	38
	9.5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	3	8	16	22	40
	12.1	0	0	0	0	0	0	0	0	0	0	0	0	1	3	7	11	16	22	32	41
	17.1	0	0	0	0	0	0	0	0	0	1	2	3	4	9	18	25	28	33	40	42
	22.3	0	0	0	0	0	0	0	1	3	7	12	17	22	26	30	34	35	40	41	42
	27.3	0	0	0	0	0	0	1	2	6	10	17	22	26	30	34	38	40	41	42	42
	33.8	0	0	0	0	0	0	1	3	7	14	20	26	30	35	37	40	41	42	42	42
	45.0	0	0	0	0	0	0	1	3	8	16	24	30	35	38	40	41	42	42	42	42
	61.6	0	0	0	0	0	1	3	9	15	22	30	37	40	42	42	42	42	42	42	42
	85.7	0	0	0	0	0	2	8	15	22	30	35	40	42	42	42	42	42	42	42	42
	119.6	0	0	0	0	0	1	6	15	25	36	40	42	42	42	42	42	42	42	42	42
	204.1	0	0	0	0	0	0	2	18	30	40	42	42	42	42	42	42	42	42	42	42