TABLES ADAPTED FOR MACHINE COMPUTATION

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In actuarial and particularly in casualty actuarial work the occasion often arises when it is necessary to make a more or less isolated calculation for which full tables are not available covering the particular function involved. For example, we may have to determine the present value of $12.00 a week for 200 weeks at 3½% per annum compound interest, and it may be necessary to do this with considerable accuracy; for instance, in order to comply with some statutory or other legal requirement. If we do not have available a table of the present values of such weekly annuities certain, we have to make the calculation from first principles or from the appropriate formula.

Thus in the example cited if we assume there are 52 weeks to the year the required value is

$$12 \times 52\, a_{\overline{n}|}^{(52)} \text{ or } 624 \frac{1 - v^n}{j_{(52)}} \text{ at } 3\tfrac{1}{2}\%$$

where $n = \dfrac{200}{52}$, $v = (1 + i)^{-1}$ and $j_{(52)} = 52 \{ (1 + i)^{\frac{1}{52}} - 1 \}$

To calculate v^n logarithms must be resorted to (unless a troublesome series development is used) and tables of these to more than 7 places are not very usual in offices and even if available are unhandy to use, involving considerable interpolations. Seven place tables do not always give sufficient accuracy. Then as regards $j_{(52)}$ probably no tables are available and again we have to fall back on logarithms (or else sum a series) and for any moderate accuracy extended logarithm tables must be used.

On the other hand, let us remember that efficient calculating machines are in everyday use in modern offices. In making calculations of the type considered above, not much assistance can be had from a calculating machine that nevertheless can add, subtract, multiply and divide almost instantaneously. Why is this? The answer is of course that the usual logarithm tables are not adapted to the special requirements and limitations of the calculating machines and basic tables so arranged as to be usable on the machines are not at hand. As I will show it is easily possible to compile suitable tables with the aid of which logarithmic computations can be made rapidly. I have had such tables prepared and the purpose of this paper is to publish them with necessary instructions for their use.

I will assume there is available a calculating machine that will multiply a 10 figure number by a 10 figure number giving the result to 20 significant figures (though 10 will be sufficient for our purposes): the machine will also divide a number of 20 figures (10 will be sufficient) by another 10 figure number giving the quotient to 10 places. The tables given are for this capacity but of course can be used with a machine of 8×8 capacity in which case the final result will naturally be accurate to a less number of significant figures. Let us consider in detail a calculation requiring the use of logarithms, say for example

$$12.3456789^{9.87654321}.$$

This involves three steps

(i) the determination of the logarithm of 12.3456789
(ii) its multiplication by 9.87654321
(iii) the determination of the antilogarithm of the product

The second step is easily done on the machine. As for the other steps it would require impractically large tables to give logarithms and antilogarithms to 9 or 10 places by mere inspection or even with the aid of simple interpolations. However by factorizing the number whose logarithm is required we can reduce the size of the necessary tables to a manageable size and the factorizing can be effected quickly with the aid of the machine. Similarly for antilogarithms we get the answer in the form of factors which are easily multiplied together on the machine.

What I have done is to provide tables of logarithms to 10 places of

(a) a series of numbers (each of 3 figures) from 1.00 to 10.00 such that the ratio of any number to its predecessor is not greater than 1.02235 (150 numbers in this series) (see Table III)
(b) numbers from 1.00000 to 1.02235 by intervals of .00015 (150 numbers) (see Table IV)
(c) numbers from 1.000000 to 1.000149 by intervals of .000001 (150 numbers) (see Table V)

plus (d) a simple rule involving one multiplication to find the logarithm to 10 places of any number between 1.000000 and 1.000001 (see Table VI).

Any number is readily reduced by the machine to four factors whose logarithms are given by the three tables and the rule respectively. The addition of the four logarithms of the factors give the logarithm of the number. (As in the case of ordinary logarithm tables, what is given by my tables is the mantissa of the logarithm; the characteristic is as usual to be supplied by inspection—the readers of this paper are familiar with this procedure and it is not necessary for me to elaborate on it.) Thus to take (at last!) our example, to find the logarithm of 1.23456789 we divide this by 1.22, the largest number in series (a) which is not greater than 1.23456789; the quotient is 1.01194. . . . This is as far as we need proceed on the first division for we can see the largest number in series (b) which is not greater than this quotient is 1.01185. Now 1.22×1.01185 equals 1.2344570 and dividing this into 1.23456789 we get 1.000089829 (to 10 significant figures) which can be resolved at sight into $1.000089 \times 1.000000829$. So $1.23456789 = 1.22 \times 1.01185 \times 1.000089 \times 1.000000829$ (to 10 significant figures).

The tables give directly the logarithms of the first three factors: as to the fourth, its logarithm is $.000000829 \times .434294$ or .0000003600 (to 10 places).

log 1.22	.0863598307
log 1.01185	.0051161360
log 1.000089	.0000386505
log 1.000000829	.0000003600
log 1.23456789	.0915149772

So log 12.3456789 = 1.0915149772.

This is the first step and it takes much longer to describe than to do—with the tables in front of the operator the first two factors are picked out in a few seconds and the last two in a few more. The addition of the logarithm takes but a few more: very little need be written down.

Now multiplying the logarithm just found by 9.87654321 we get 10.7803948367. We must now find the antilogarithm of this; the process is just the reverse of finding a logarithm. We see that 6.00 is the number in series (a) whose logarithm is the nearest below .7803948367; and subtracting therefore the logarithm of 6.00 from this we get .0022435863 from which we subtract the largest possible logarithm out of series (b), namely that of 1.00510, and the remainder is .0000343133: from this we subtract the largest possible logarithm out of series (c), namely that of 1.000079, and the remainder is .0000000054. The antilogarithm of this is 1 plus .0000000054 $\times$ 2.30259 or 1.000000012 (to 10 significant figures). So the required antilogarithm of .7803948367 is 6.00 $\times$ 1.00510 $\times$ 1.000079 $\times$ 1.000000012. By inspection the product of the last two factors is 1.000079012 and by multiplication the product of the first two is 6.0306 and therefore the antilogarithm is

$$6.0306 \times 1.000079012 \text{ or } 6.031076490.$$

Therefore the antilogarithm of 10.7803948367 is 60,310,764,900. This result is, of course, not reliable to the last significant figure. In fact, using more extended logarithm tables, I find that log 1.23456789 = .9815149771700 . . . and the final answer is 60,310,764,882.44 . . . so that the result from our tables is wrong by two units in the tenth significant place.

The above is an illustration of Tables III to VI described below. These tables form a compact logarithm table and can be used for any purpose for which such a table is required. As for Tables I and II, these are special compound interest tables. Table I gives the logarithm of $1 + i$ to 12 decimal places for 64 rates of interest from ⅛% to 10%. This table enables us to avoid the calculation of $\log (1 + i)$ for each problem and also gives enough decimal places so that the logarithm of $(1 + i)^n$ may be calculated accurately to 10 places when n is large say 50 or 100. Table II is a table of values of $j_{(r)}$ for rates of interest from ¼% to 7½%. This is necessary for calculating the values of annuities certain payable semi-annually, quarterly, monthly, weekly or continuously; and besides saving the calculation of the value of $j_{(r)}$ from the other tables gives it more accurately. Instructions for the use of these tables are given next, followed by the tables themselves, after which are various illustrations covering some of the purposes to which the table can be put.

I trust that these tables will be of service to the actuarial profession. Tables I, III and IV were derived from existing tables (chiefly the 20 place Logarithmetica Britannica) with precautions to ensure accuracy. Table V was calculated specially as was also

Table II. The idea of obtaining logarithms by factorizing is of course not new: it is as old as logarithms themselves. Last century Peter Gray published tables to 24 places based on this method. It is interesting to note that A. J. Thompson in the introduction to his 20 place Logarithmetica Britannica first gave the "classical" method of obtaining logarithms and antilogarithms by central difference interpolation but later added auxiliary tables based on the factorization method with the comment that, as contrasted with interpolation methods, factorization methods "are at least as short for finding logarithms, and distinctly shorter for finding antilogarithms." This is my experience also.

I should state here, for completeness, that no life contingencies are involved in any of the tables or examples of this paper and that all annuities mentioned are annuities certain: also that all logarithms dealt with are "common" logarithms, that is to base 10.

Description of Tables and Instructions for Use

Table I

Logarithms of $(1 + i)$ for rates of interest from ⅛% to 6% by intervals of ⅛% and from 6% to 10% by intervals of ¼%.

These logarithms are given to 12 decimal places.

No particular comments required here except

(i) if, as in problems involving present values, $v^n = (1 + i)^{-n}$ is required, say for example $(1.05)^{-20}$, we can either (a) write

$$\log v = -\log(1 + i) = -\log 1.05 = -.021189299070$$
$$= -1 + .978810700930 = \bar{1}.978810700930$$

and then multiply by 20, thus

$$-20 + 19.576214018600 = \bar{1}.576214018600$$

or (b) multiply $\log(1 + i) = .021189299070$ by 20 getting

$$\log(1 + i)^n = \log 1.05^{20} = 0.423785981400$$

and then $\log v^n = -0.423785981400 = \bar{1}.576214018600$.

(ii) for values of i not in the table, e.g., $3\frac{5}{16}\%$, we can either (a) calculate $\log 1.03\frac{5}{16} = \log 1.033125$ from Tables III, IV, V and VI, or (b) interpolate in Table I as indicated in Appendix I—the latter method will usually be quicker and always be accurate to more decimal places.

TABLE II

Values of $j_{(r)}$ for rates of interest proceeding from ¼% to 7½% by intervals of ¼%, for values of r = 2, 4, 12, 52, 52.1775 and ∞. These values are given to 12 decimal places so that at least 10 significant figures are available.

$j_{(r)} = r\{(1+i)^{\frac{1}{r}} - 1\}$ is the nominal rate of interest convertible r times a year equivalent to the effective rate of interest i. The amount and present value of an annuity of 1 per annum for n years payable annually and r times a year are

	Payable annually	Payable r times a year
amount	$s_{\overline{n}\rvert} = \dfrac{(1+i)^n - 1}{i}$	$s^{(r)}_{\overline{n}\rvert} = \dfrac{(1+i)^n - 1}{j_{(r)}}$
present value	$a_{\overline{n}\rvert} = \dfrac{1 - v^n}{i}$	$a^{(r)}_{\overline{n}\rvert} = \dfrac{1 - v^n}{j_{(r)}}$

$j_{(2)}$, $j_{(4)}$, $j_{(12)}$ are to be used for annuities payable semi-annually, quarterly, and monthly respectively: $j_{(52)}$ is to be used for annuities payable 52 times a year, that is weekly if a year is regarded as consisting of 52 weeks: $j_{(52.1775)}$ is to be used for weekly annuities if it is assumed that a year contains 52.1775 weeks on the average:* $j_{(\infty)} = \delta$ is to be used for annuities payable continuously. The procedure to be used if $j_{(r)}$ is required for a rate of interest or a value of r will not be given in the Table will be found in Appendix I.

Note that this Table can be used in conjunction with ordinary annuity tables: if for instance $a^{(4)}_{\overline{35}\rvert}$ is required, this is equal to $a_{\overline{35}\rvert}$ (the value of which can be taken from any table of annuities certain) multiplied by $\dfrac{i}{j_{(4)}}$ (where $j_{(4)}$ is taken from Table II).

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* The value 52.1775 is arrived at as follows — a year contains 52 weeks plus one day in ordinary years and plus two days in leap years. In a period of 400 years there are 97 leap years (one every four years except in even century years like 1900 where the 19 is *not* divisible by four). Thus in 400 years there are 497 extra days (over the 52 weeks per year). Now 497 is conveniently divisible by 7 so there are 71 extra weeks in 400 years. So the average number of weeks in a year is $52\frac{71}{400}$ or 52.1775. It is convenient to use this figure with a terminating decimal rather than, say, the average number of weeks obtained from considering a year as consisting of 52 weeks plus 1¼ days on the average, for this gives as the average number of weeks $52\frac{5}{28}$ or $52.1\dot{7}85714\dot{2}$, a recurring decimal.

TABLES III, IV, V AND VI

These together form a condensed logarithm table—and their use will be clear from the following examples.

(a) To find the logarithm of 105. We first find the log of 1.05, that is the number with the same significant figures but with the decimal point between the first two significant figures. Out of Table III pick the number nearest to but not exceeding 1.05—this is 1.04. Dividing this on the machine into 1.05 we get a quotient which is always between 1.00000 and 1.02235: we carry the division only far enough to determine the number in Table IV that is the nearest below the quotient. In our case $\frac{1.05}{1.04} = 1.009615\ldots$ and the nearest number below this in Table IV is 1.00960. Now multiplying on the machine 1.04 by 1.00960 we get 1.0499840 which divided into our number 1.05 gives 1.000015238 which quotient will always be between 1.000000 and 1.000150. Now taking from Table V the number nearest below this, that is 1.000015, we factorize by inspection 1.000015238 into $1.000015 \times 1.000000238$. Thus $1.05 = 1.04 \times 1.00960 \times 1.000015 \times 1.000000238$ (to 10 significant figures) and the logarithms of the first three factors we take from Tables III, IV and V respectively. Table VI, which is not strictly a Table but an instruction, tells us that the log of 1.000000238 is $.000000238 \times .434294$ to 10 places or .0000001034, performing the multiplication on the machine. Now adding the logs of the four factors we get

log 1.04	.0170333393
log 1.00960	.0041493419
log 1.000015	.0000065144
log 1.000000338	.0000001034
log 1.05	.0211892990

(compare this with Table I which gives
log 1.05 = .021189299070)

and therefore log 105 = 2.0211892990.

(b) To find the antilogarithm of $\bar{2}.6$. We first find the antilogarithm of .6, by reversing the process of finding logarithms. From Table III we pick out the logarithm next less than .6—this is .5932860670, the logarithm of 3.92. We subtract this from .6 obtaining .0067139330 (which will always be between .00000 and .00960). From Table IV we pick out the logarithm next less than this remainder, this will be .0066585439 the logarithm of 1.01545; subtract this and obtain .0000553891 (which will always be less than .00006514). From Table V we pick out the logarithm next less than this remainder; this will be .0000551519, the logarithm of 1.000127. Subtract this. The balance, namely .0000002372, we divide by .434294 or multiply by 2.30259, as per Table VI, performing the operation on the machine, and the result (to 9 decimal places) added to 1 is the antilogarithm of the balance, in our case antilogarithm $.0000002372 = 1.000000546$ (to ten significant figures). Thus antilogarithm $.6 = 3.92 \times 1.01545 \times 1.000127 \times 1.000000546$. We multiply the last two factors together by inspection, thus 1.000127546, and the first two on the machine, getting 3.980564; then $3.980564 \times 1.000127546 = 3.981071705 =$ antilogarithm .6. So antilogarithm $\bar{2}.6 = .03981071705$ (the correct result is .0398107170553...).

TABLE I

Logarithms of $(1 + i)$ to 12 decimal places for values of i proceeding from ⅛% by ⅛% to 6% and by ¼% to 10%

%	i	$\log(1+i)$	%	i	$\log(1+i)$
⅛	.00125	.00054 25290 92	4⅛	.04125	.01755 50144 15−
¼	.00250	.00108 43812 92	4¼	.04250	.01807 60636 46
⅜	.00375	.00162 55582 87	4⅜	.04375	.01859 64884 92
½	.00500	.00216 60617 57	4½	.04500	.01911 62904 47
⅝	.00625	.00270 58933 76	4⅝	.04625	.01963 54710 01
¾	.00750	.00324 50548 13	4¾	.04750	.02015 40316 38
⅞	.00875	.00378 35477 30	4⅞	.04875	.02067 19738 37
1	.01000	.00432 13737 83	5	.05000	.02118 92990 70
1⅛	.01125	.00485 85346 20	5⅛	.05125	.02170 60088 06
1¼	.01250	.00539 50318 87	5¼	.05250	.02222 21045 08
1⅜	.01375	.00593 08672 19	5⅜	.05375	.02273 75876 33
1½	.01500	.00646 60422 49	5½	.05500	.02325 24596 34
1⅝	.01625	.00700 05586 02	5⅝	.05625	.02376 67219 58
1¾	.01750	.00753 44178 97	5¾	.05750	.02428 03760 47
1⅞	.01875	.00806 76217 48	5⅞	.05875	.02479 34233 39
2	.02000	.00860 01717 62	6	.06000	.02530 58652 65−
2⅛	.02125	.00913 20695 40	6¼	.06250	.02632 89387 22
2¼	.02250	.00966 33166 79	6½	.06500	.02734 96077 75−
2⅜	.02375	.01019 39147 68	6¾	.06750	.02836 78836 97
2½	.02500	.01072 38653 92	7	.07000	.02938 37776 85+
2⅝	.02625	.01125 31701 27	7¼	.07250	.03039 73008 57
2¾	.02750	.01178 18305 48	7½	.07500	.03140 84642 52
2⅞	.02875	.01230 98482 20	7¾	.07750	.03241 72788 33
3	.03000	.01283 72247 05+	8	.08000	.03342 37554 87
3⅛	.03125	.01336 39615 58	8¼	.08250	.03442 79050 25+
3¼	.03250	.01389 00603 28	8½	.08500	.03542 97381 85−
3⅜	.03375	.01441 55225 61	8¾	.08750	.03642 92656 27
3½	.03500	.01494 03497 93	9	.09000	.03742 64979 41
3⅝	.03625	.01546 45435 58	9¼	.09250	.03842 14456 42
3¾	.03750	.01598 81053 84	9½	.09500	.03941 41191 76
3⅞	.03875	.01651 10367 92	9¾	.09750	.04040 45289 14
4	.04000	.01703 33392 99	10	.10000	.04139 26851 58

TABLE II

Nominal rates of interest $j_{(r)}$ convertible r times a year equivalent to effective rate i

For i up to 7½% and r = 2, 4, 12, 52, 52.1775 and ∞.

%	i	$j_{(2)}$	$j_{(4)}$	$j_{(12)}$
¼	.0025	.00249 84394 50+	.00249 76596 62	.00249 71399 84
½	.0050	.00499 37655 76	.00499 06522 50+	.00498 85781 37
¾	.0075	.00748 59899 88	.00747 89980 62	.00747 43416 14
1	.0100	.00997 51242 24	.00996 27172 57	.00995 44573 72
1¼	.0125	.01246 11797 50−	.01244 18298 59	.01242 89521 76
1½	.0150	.01494 41679 61	.01491 63557 52	.01489 78525 97
1¾	.0175	.01742 41001 83	.01738 63146 91	.01736 11850 16
2	.0200	.01990 09876 72	.01985 17262 93	.01981 89756 23
2¼	.0225	.02237 48416 16	.02231 26100 45−	.02227 12504 24
2½	.0250	.02484 56731 32	.02476 89853 03	.02471 80352 38
2¾	.0275	.02731 34932 71	.02722 08712 92	.02715 93557 01
3	.0300	.02977 83130 18	.02966 82871 11	.02959 52372 68
3¼	.0325	.03224 01432 90	.03211 12517 29	.03202 57052 12
3½	.0350	.03469 89949 38	.03454 97839 91	.03445 07846 29
3¾	.0375	.03715 48787 46	.03698 39026 15−	.03687 05004 39
4	.0400	.03960 78054 37	.03941 36261 96	.03928 48773 86
4¼	.0425	.04205 77856 66	.04183 89732 06	.04169 39400 42
4½	.0450	.04450 48300 26	.04425 99619 97	.04409 77128 05+
4¾	.0475	.04694 89490 46	.04667 66107 96	.04649 62199 06
5	.0500	.04939 01531 92	.04908 89377 16	.04888 94854 04
5¼	.0525	.05182 84528 68	.05149 69607 48	.05127 75331 94
5½	.0550	.05426 38584 17	.05390 06977 65−	.05366 03870 05−
5¾	.0575	.05669 63801 20	.05630 01665 26	.05603 80704 00
6	.0600	.05912 60281 97	.05869 53846 75−	.05841 06067 84
6¼	.0625	.06155 28128 09	.06108 63697 38	.06077 80193 97
6½	.0650	.06397 67440 55+	.06347 31391 31	.06314 03313 22
6¾	.0675	.06639 78319 77	.06585 57101 57	.06549 75654 83
7	.0700	.06881 60865 58	.06823 41000 07	.06784 97446 49
7¼	.0725	.07123 15177 21	.07060 83257 62	.07019 68914 32
7½	.0750	.07364 41353 33	.07297 84043 94	.07253 90282 92

TABLE II (Continued)

Nominal rates of interest $j_{(r)}$ convertible r times a year equivalent to effective rate i

For i up to 7½% and r = 2, 4, 12, 52, 52.1775 and ∞.

%	i	$j_{(52)}$	$j_{(52.1775)}$	$j_{(\infty)}=\delta$
¼	.0025	.00249 69401 46	.00249 69399 42	.00249 68801 99
½	.0050	.00498 77807 07	.00498 77798 93	.00498 75415 11
¾	.0075	.00747 25517 01	.00747 25498 75−	.00747 20148 39
1	.0100	.00995 12829 24	.00995 12796 85+	.00995 03308 53
1¼	.0125	.01242 40039 53	.01242 39989 05−	.01242 25199 99
1½	.0150	.01489 07441 47	.01489 07368 95−	.01488 86124 94
1¾	.0175	.01735 15326 49	.01735 15228 02	.01734 86383 35−
2	.0200	.01980 63983 91	.01980 63855 60	.01980 26272 96
2¼	.0225	.02225 53700 91	.02225 53538 92	.02225 06089 35−
2½	.0250	.02469 84762 60	.02469 84563 10	.02469 26125 90
2¾	.0275	.02713 57452 02	.02713 57211 20	.02712 86673 88
3	.0300	.02956 72050 14	.02956 71764 24	.02955 88022 42
3¼	.0325	.03199 28835 93	.03199 28501 20	.03198 30458 53
3½	.0350	.03441 28086 33	.03441 27699 04	.03440 14267 17
3¾	.0375	.03682 70076 28	.03682 69632 76	.03681 39731 23
4	.0400	.03923 55078 76	.03923 54575 34	.03922 07131 53
4¼	.0425	.04163 83364 80	.04163 82797 84	.04162 16746 91
4½	.0450	.04403 55203 49	.04403 54569 38	.04401 68854 17
4¾	.0475	.04642 70862 01	.04642 70157 16	.04640 63728 14
5	.0500	.04881 30605 62	.04881 29826 47	.04879 01641 69
5¼	.0525	.05119 34697 72	.05119 33840 74	.05116 82865 74
5½	.0550	.05356 83399 84	.05356 82461 53	.05354 07669 28
5¾	.0575	.05593 76971 67	.05593 75948 53	.05590 76319 38
6	.0600	.05830 15671 07	.05830 14559 64	.05826 89081 24
6¼	.0625	.06065 99754 08	.06065 98550 94	.06062 46218 16
6½	.0650	.06301 29474 96	.06301 28176 68	.06297 47991 61
6¾	.0675	.06536 35086 18	.06536 03689 39	.06531 94661 21
7	.0700	.06770 26838 46	.06770 25339 80	.06765 86484 74
7¼	.0725	.07003 94980 78	.07003 93376 90	.06999 23718 20
7½	.0750	.07237 09760 38	.07237 08047 97	.07232 06615 80

TABLE III

Logarithms of Numbers from 1.00 to 10.00

N	log N	N	log N
1.00	.00000 00000	1.80	.25527 25051
1.02	.00860 01718	1.83	.26245 10897
1.04	.01703 33393	1.86	.26951 29442
1.06	.02530 58653	1.89	.27646 18042
1.08	.03342 37555—	1.92	.28330 12287
1.10	.04139 26852	1.95	.29003 46114
1.12	.04921 80227	1.98	.29666 51903
1.14	.05690 48513	2.01	.30319 60574
1.16	.06445 79892	2.04	.30963 01674
1.18	.07188 20073	2.07	.31597 03455—
1.20	.07918 12460	2.10	.32221 92947
1.22	.08635 98307	2.13	.32837 96034
1.24	.09342 16852	2.16	.33445 37512
1.26	.10037 05451	2.19	.34044 41148
1.28	.10720 99696	2.22	.34635 29745—
1.30	.11394 33523	2.25	.35218 25181
1.32	.12057 39312	2.30	.36172 78360
1.34	.12710 47984	2.35	.37106 78623
1.36	.13353 89084	2.40	.38021 12417
1.38	.13987 90864	2.45	.38916 60844
1.40	.14612 80357	2.50	.39794 00087
1.42	.15228 83444	2.55	.40654 01804
1.44	.15836 24921	2.60	.41497 33480
1.47	.16731 73347	2.65	.42324 58739
1.50	.17609 12591	2.70	.43136 37642
1.53	.18469 14308	2.75	.43933 26938
1.56	.19312 45984	2.80	.44715 80313
1.59	.20139 71243	2.85	.45484 48600
1.62	.20951 50145+	2.90	.46239 79979
1.65	.21748 39442	2.95	.46982 20160
1.68	.22530 92817	3.00	.47712 12547
1.71	.23299 61103	3.05	.48429 98393
1.74	.24054 92483	3.10	.49136 16938
1.77	.24797 32664	3.15	.49831 05538

TABLE III (Continued)

Logarithms of Numbers from 1.00 to 10.00

N	log N	N	log N
3.20	.50514 99783	5.68	.75434 83357
3.25	.51188 33610	5.76	.76042 24834
3.30	.51851 39399	5.88	.76937 73261
3.35	.52504 48070	6.00	.77815 12504
3.40	.53147 89170	6.12	.78675 14221
3.45	.53781 90951	6.24	.79518 45897
3.50	.54406 80444	6.36	.80345 71156
3.55	.55022 83531	6.48	.81157 50059
3.60	.55630 25008	6.60	.81954 39355+
3.68	.56584 78187	6.72	.82736 92731
3.76	.57518 78449	6.84	.83505 61017
3.84	.58433 12244	6.96	.84260 92396
3.92	.59328 60670	7.08	.85003 32577
4.00	.60205 99913	7.20	.85733 24964
4.08	.61066 01631	7.32	.86451 10811
4.16	.61909 33306	7.44	.87157 29355+
4.24	.62736 58566	7.56	.87852 17955+
4.32	.63548 37468	7.68	.88536 12200
4.40	.64345 26765—	7.80	.89209 46027
4.48	.65127 80140	7.92	.89872 51816
4.56	.65896 48427	8.04	.90525 60487
4.64	.66651 79806	8.16	.91169 01588
4.72	.67394 19986	8.28	.91803 03368
4.80	.68124 12374	8.40	.92427 92861
4.88	.68841 98220	8.52	.93043 95948
4.96	.69548 16765—	8.64	.93651 37425—
5.04	.70243 05364	8.76	.94250 41062
5.12	.70926 99610	8.88	.94841 29658
5.20	.71600 33436	9.00	.95424 25094
5.28	.72263 39225+	9.20	.96378 78273
5.36	.72916 47897	9.40	.97312 78536
5.44	.73559 88997	9.60	.98227 12330
5.52	.74193 90777	9.80	.99122 60757
5.60	.74818 80270	10.00	1.00000 00000

TABLE IV

Logarithms of Numbers from 1.00000 to 1.02235

N	log N	N	log N
1.00000	.00000 00000	1.00570	.00246 84501
1.00015	.00006 51393	1.00585	.00253 32203
1.00030	.00013 02688	1.00600	.00259 79807
1.00045	.00019 53886	1.00615	.00266 27315+
1.00060	.00026 04985+	1.00630	.00272 74727
1.00075	.00032 55988	1.00645	.00279 22042
1.00090	.00039 06892	1.00660	.00285 69261
1.00105	.00045 57700	1.00675	.00292 16383
1.00120	.00052 08409	1.00690	.00298 63409
1.00135	.00058 59022	1.00705	.00305 10338
1.00150	.00065 09536	1.00720	.00311 57171
1.00165	.00071 59954	1.00735	.00318 03908
1.00180	.00078 10274	1.00750	.00324 50548
1.00195	.00084 60496	1.00765	.00330 97092
1.00210	.00091 10621	1.00780	.00337 43540
1.00225	.00097 60649	1.00795	.00343 89892
1.00240	.00104 10580	1.00810	.00350 36147
1.00255	.00110 60413	1.00825	.00356 82307
1.00270	.00117 10149	1.00840	.00363 28370
1.00285	.00123 59788	1.00855	.00369 74337
1.00300	.00130 09330	1.00870	.00376 20208
1.00315	.00136 58775−	1.00885	.00382 65983
1.00330	.00143 08122	1.00900	.00389 11662
1.00345	.00149 57373	1.00915	.00395 57245+
1.00360	.00156 06526	1.00930	.00402 02733
1.00375	.00162 55583	1.00945	.00408 48124
1.00390	.00169 04542	1.00960	.00414 93419
1.00405	.00175 53405−	1.00975	.00421 38618
1.00420	.00182 02170	1.00990	.00427 83722
1.00435	.00188 50839	1.01005	.00434 28730
1.00450	.00194 99411	1.01020	.00440 73642
1.00465	.00201 47886	1.01035	.00447 18458
1.00480	.00207 96264	1.01050	.00453 63179
1.00495	.00214 44545+	1.01065	.00460 07803
1.00510	.00220 92730	1.01080	.00466 52332
1.00525	.00227 40818	1.01095	.00472 96766
1.00540	.00233 88809	1.01110	.00479 41104
1.00555	.00240 36703	1.01125	.00485 85346

TABLE IV (Continued)

Logarithms of Numbers from 1.00000 to 1.02235

N	log N	N	log N
1.01140	.00492 29493	1.01695	.00729 96007
1.01155	.00498 73544	1.01710	.00736 36543
1.01170	.00505 17500−	1.01725	.00742 76985+
1.01185	.00511 61360	1.01740	.00749 17333
1.01200	.00518 05125+	1.01755	.00755 57586
1.01215	.00524 48794	1.01770	.00761 97745+
1.01230	.00530 92368	1.01785	.00768 37810
1.01245	.00537 35847	1.01800	.00774 77780
1.01260	.00543 79231	1.01815	.00781 17656
1.01275	.00550 22519	1.01830	.00787 57438
1.01290	.00556 65711	1.01845	.00793 97125+
1.01305	.00563 08809	1.01860	.00800 36718
1.01320	.00569 51811	1.01875	.00806 76217
1.01335	.00575 94718	1.01890	.00813 15622
1.01350	.00582 37530	1.01905	.00819 54933
1.01365	.00588 80247	1.01920	.00825 94150−
1.01380	.00595 22869	1.01935	.00832 33273
1.01395	.00601 65396	1.01950	.00838 72301
1.01410	.00608 07827	1.01965	.00845 11236
1.01425	.00614 50164	1.01980	.00851 50076
1.01440	.00620 92405+	1.01995	.00857 88823
1.01455	.00627 34552	1.02010	.00864 27476
1.01470	.00633 76604	1.02025	.00870 66034
1.01485	.00640 18561	1.02040	.00877 04499
1.01500	.00646 60422	1.02055	.00883 42870
1.01515	.00653 02190	1.02070	.00889 81148
1.01530	.00659 43862	1.02085	.00896 19331
1.01545	.00665 85439	1.02100	.00902 57421
1.01560	.00672 26922	1.02115	.00908 95417
1.01575	.00678 68310	1.02130	.00915 33319
1.01590	.00685 09603	1.02145	.00921 71128
1.01605	.00691 50802	1.02160	.00928 08843
1.01620	.00697 91906	1.02175	.00934 46464
1.01635	.00704 32915+	1.02190	.00940 83992
1.01650	.00710 73830	1.02205	.00947 21426
1.01665	.00717 14650−	1.02220	.00953 58766
1.01680	.00723 55375+	1.02235	.00959 96013

TABLE V

Logarithms of Numbers from 1.000000 to 1.000149

N	log N	N	log N
1.000000	.00000 00000	1.000038	.00001 65029
1.000001	.00000 04343	1.000039	.00001 69372
1.000002	.00000 08686	1.000040	.00001 73714
1.000003	.00000 13029	1.000041	.00001 78057
1.000004	.00000 17372	1.000042	.00001 82400
1.000005	.00000 21715−	1.000043	.00001 86743
1.000006	.00000 26058	1.000044	.00001 91085+
1.000007	.00000 30401	1.000045	.00001 95428
1.000008	.00000 34743	1.000046	.00001 99771
1.000009	.00000 39086	1.000047	.00002 04114
1.000010	.00000 43429	1.000048	.00002 08456
1.000011	.00000 47772	1.000049	.00002 12799
1.000012	.00000 52115+	1.000050	.00002 17142
1.000013	.00000 56458	1.000051	.00002 21485−
1.000014	.00000 60801	1.000052	.00002 25827
1.000015	.00000 65144	1.000053	.00002 30170
1.000016	.00000 69487	1.000054	.00002 34513
1.000017	.00000 73829	1.000055	.00002 38855+
1.000018	.00000 78172	1.000056	.00002 43198
1.000019	.00000 82515+	1.000057	.00002 47541
1.000020	.00000 86858	1.000058	.00002 51883
1.000021	.00000 91201	1.000059	.00002 56226
1.000022	.00000 95544	1.000060	.00002 60569
1.000023	.00000 99887	1.000061	.00002 64912
1.000024	.00001 04229	1.000062	.00002 69254
1.000025	.00001 08572	1.000063	.00002 73597
1.000026	.00001 12915+	1.000064	.00002 77940
1.000027	.00001 17258	1.000065	.00002 82282
1.000028	.00001 21601	1.000066	.00002 86625−
1.000029	.00001 25944	1.000067	.00002 90968
1.000030	.00001 30286	1.000068	.00002 95310
1.000031	.00001 34629	1.000069	.00002 99653
1.000032	.00001 38972	1.000070	.00003 03995+
1.000033	.00001 43315−	1.000071	.00003 08338
1.000034	.00001 47658	1.000072	.00003 12681
1.000035	.00001 52000	1.000073	.00003 17023
1.000036	.00001 56343	1.000074	.00003 21366
1.000037	.00001 60686	1.000075	.00003 25709

TABLE V (Continued)

Logarithms of Numbers from 1.000000 to 1.000149

N	log N	N	log N
1.000076	.00003 30051	1.000113	.00004 90725+
1.000077	.00003 34394	1.000114	.00004 95067
1.000078	.00003 38736	1.000115	.00004 99410
1.000079	.00003 43079	1.000116	.00005 03752
1.000080	.00003 47422	1.000117	.00005 08095−
1.000081	.00003 51764	1.000118	.00005 12437
1.000082	.00003 56107	1.000119	.00005 16780
1.000083	.00003 60449	1.000120	.00005 21122
1.000084	.00003 64792	1.000121	.00005 25465−
1.000085	.00003 69135−	1.000122	.00005 29807
1.000086	.00003 73477	1.000123	.00005 34149
1.000087	.00003 77820	1.000124	.00005 38492
1.000088	.00003 82162	1.000125	.00005 42834
1.000089	.00003 86505−	1.000126	.00005 47177
1.000090	.00003 90847	1.000127	.00005 51519
1.000091	.00003 95190	1.000128	.00005 55861
1.000092	.00003 99533	1.000129	.00005 60204
1.000093	.00004 03875+	1.000130	.00005 64546
1.000094	.00004 08218	1.000131	.00005 68889
1.000095	.00004 12560	1.000132	.00005 73231
1.000096	.00004 16903	1.000133	.00005 77573
1.000097	.00004 21245+	1.000134	.00005 81916
1.000098	.00004 25588	1.000135	.00005 86258
1.000099	.00004 29930	1.000136	.00005 90600
1.000100	.00004 34273	1.000137	.00005 94943
1.000101	.00004 38615+	1.000138	.00005 99285+
1.000102	.00004 42958	1.000139	.00006 03627
1.000103	.00004 47300	1.000140	.00006 07970
1.000104	.00004 51643	1.000141	.00006 12312
1.000105	.00004 55985+	1.000142	.00006 16654
1.000106	.00004 60328	1.000143	.00006 20997
1.000107	.00004 64670	1.000144	.00006 25339
1.000108	.00004 69013	1.000145	.00006 29681
1.000109	.00004 73355+	1.000146	.00006 34024
1.000110	.00004 77698	1.000147	.00006 38366
1.000111	.00004 82040	1.000148	.00006 42708
1.000112	.00004 86383	1.000149	.00006 47051

Table VI

Logarithms of numbers from 1.000000 to 1.000001

To find the logarithm of a number between 1 and 1.000001 multiply the decimal portion by .434294 and the product is the logarithm to 10 decimal places.

$$\text{Example: } \log 1.000000421 = .000000421 \times .434294$$
$$= .0000001828.$$

To find the antilogarithm of a number between 0 and .0000004343 multiply the number by 2.30259 and the product, to 9 decimal places, added to 1 is the antilogarithm.

$$\text{Example: antilog } .0000001828 = 1 + .0000001828 \times 2.30259$$
$$= 1.000000421.$$

Examples of the Use of the Tables

(1) $\frac{355}{113}$ is an approximation to the value of π the true value of which is 3.141592654...

Find the error in using the approximation in π^{19}.

$\frac{355}{113} = 3.141592920 \ldots$ (call this p for short). We have to calculate $p^{19} - \pi^{19} = 3.141592920^{19} - 3.141592654^{19}$.

Factorizing according to the instructions for Tables III, IV, V and VI, we find

$$p = 3.141592920 = 3.10 \times 1.01335 \times 1.000066 \times 1.000000187$$
$$\pi = 3.141592654 = 3.10 \times 1.01335 \times 1.000066 \times 1.000000103$$

So	.4913616938		.4913616938
	.0057594718		.0057594718
	.0000286625		.0000286625
	.0000000812		.0000000447
log p =	.4971499093	log π =	.4971498728
19 log p =	9.4458482767	19 log π =	9.4458475832

from which, proceeding according to the instructions, we find

$$\text{antilog } .4458482767 = 2.75 \times 1.01500 \times 1.000114 \times 1.000000078 = 2.791568420$$

$$\text{antilog } .4458475832 = 2.75 \times 1.01500 \times 1.000112 \times 1.000000481 = 2.791563963$$

So $p^{19} = 2{,}791{,}568{,}420.$
$\pi^{19} = 2{,}791{,}563{,}963.$

Difference = 4,457.

(The correct difference, taking π to more decimal places than given above and using fifteen place logarithms, is 4,503.80 ...).

(2) Find the amount of 1625.14 accumulated at 3¾% per annum compound interest for 400 weeks, assuming 52.1775 weeks to the year.

400 weeks = 7.666139620 years (to 10 significant figures) so we have to calculate $1625.14 \times 1.0375^{7.666139620}$.

From Table I log 1.0375 = .015988105384 and multiplying by 7.666139630 we get, to 10 places, .1225670481, the antilogarithm of which we must find. Proceeding as per the instructions we get

	.1225670481
log 1.32	.1205739312
	.0019931169
log 1.00450	.0019499411
	.0000431758
log 1.000099	.0000429930
	.0000001828 = log 1.000000421.

Then the product of the last two factors is 1.000099421 and $1.32 \times 1.00450 = 1.32594$. Multiplying these two together we get 1.326071826 which, finally, has to be multiplied by 1625.14 giving 2155.0524.

(3) Find the present value of 1625.14, at 3¾% per annum compound interest, due 400 weeks hence, assuming 52.1775 weeks to the year.

400 weeks = 7.666139620 so we have to calculate

$$1625.14\, v^{7.666139620} \qquad v = 1.0375^{-1}.$$

As in example (2) we find

$$7.666139620 \times \log 1.0375 = .1225670481$$

and so we have to find the antilogarithm of

$$-.1225670481 = \bar{1}.8774329519.$$

Proceeding as usual we find

$$\begin{aligned}\text{antilog } .8774329519 &= 7.44 \times 1.01350 \times 1.000083 \times 1.000000503\\ &= 7.54044 \times 1.000083503\\ &= 7.541069649\end{aligned}$$

Thus $v^{7.666139620} = .7541069649$

which multiplied by 1625.14 gives the final answer of 1225.5294.

(4) Find the amount of an annuity certain of 12.83 a week accumulated at 3¾% per annum compound interest for 400 weeks (52.1775 weeks to the year).

The amount of an annuity certain of 1 per annum payable r times a year for n years is

$$s_{\overline{n}|}^{(r)} = \frac{(1+i)^n - 1}{j_{(r)}}.$$

In this example $n = 400/52.1775 = 7.666139620$

$r = 52.1775$ and $j_{(r)} = .036826963276$

(per Table II).

The value of $(1+i)^n$, per example (2), is 1.326071826

$$\text{so} \quad s_{\overline{n}|}^{(r)} = \frac{.326071826}{.036826963276} = 8.8541606$$

which must be multiplied by the *annual* annuity payment, namely 12.83×52.1775 or 669.437325: so the final answer is

$$669.437325 \times 8.8541600 = 5927.3052.$$

(5) Find the present value of an annuity certain of 12.83 a week, at 3¾% per annum compound interest, payable for 400 weeks (52.1775 weeks to the year).

The present value of an annuity certain of 1 per annum payable r times a year for n years is

$$a_{\overline{n}|}^{(r)} = \frac{1-v^n}{j_{(r)}}.$$

In this example $n = 400/52.1775 = 7.666139620$
$r = 52.1775$ and $j_{(r)} = .036826963276$
(per Table II).

The value of v^n, per example (3), is .7541069649

$$\text{so} \quad a_{\overline{n}|}^{(r)} = \frac{.2458930351}{.036826963276} = 6.6769837$$

which must be multiplied by the *annual* annuity payment, namely 12.83×52.1775 or 669.437325: so the final answer is

$$669.437325 \times 6.6769838 = 4469.8221.$$

(5a) What would be the present value of the annuity given in example (5) if the year be assumed to consist of 52 weeks? Under this assumption $n = 400/52$

$$\text{and } \log v^n = -\frac{400}{52} \times \log 1.0375 = -.1229854260 = \bar{1}.8770145740.$$

$$\text{Antilog } .8770145740 = 7.44 \times 1.01260 \times 1.000008 \times 1.000000555 = .7533808451$$

so $v^n = 7.533808451$

also $j_{(52)} = .036827007628$ (per Table II).

$$\text{Thus} \quad a_{\overline{n}|}^{(52)} = \frac{.2466191549}{.036827007628} = 6.6966927$$

which has to be multiplied by 12.83×52 or 667.16 giving as the final answer 4467.7655.

Note the slight difference between this and the answer to example (5).

(6) For how many weeks will a payment of 1000 suspend an annuity of 12 per week, at 3% per annum interest and assuming 52.1775 weeks to the year?

We have the following equation from which to find n

$$12 \times 52.1775 \times \frac{1-v^n}{j_{(52.1775)}} = 1000$$

whence $v^n = .9527778953$.

We find $\log v^n$ to be $\bar{1}.9789916728 = -.0210083272$ which we divide by $\log v = -\log(1+i) = -.0128372247$ to get $n = 1.63651628$ years.

Therefore the required number of weeks is

$$1.63651628 \times 52.1775$$
$$= 85.3893 \text{ weeks.}$$

(7) In consideration of a payment now of 1000, by how many weeks should we shorten an annuity of 12 per week payable for 300 weeks, at 3% per annum, 52.1775 weeks to the year?

We find first, as in example (5), the present value of the annuity for 300 weeks. This is 3309.7679. Subtracting 1000 we have 2309.7679 and we must find as in example (6) how many weeks annuity this is equivalent to. The number is 203.8673.

Thus the payment now of 1000 shortens the annuity from 300 to 203.8673 weeks, that is by 96.1327 weeks.

(8) To construct a short table that will quickly give the present value of a weekly annuity for any integral number of weeks not exceeding 900, assuming compound interest at the rate of 3½% per annum with 52.1775 weeks to the year.

The value of a weekly annuity of 1 per week for n weeks is

$$\mathrm{r}\, a^{(\mathrm{r})}_{\overline{\mathrm{n}/\mathrm{r}}|} = \mathrm{r}\,\frac{1-v^{\frac{\mathrm{n}}{\mathrm{r}}}}{j_{(\mathrm{r})}}$$

where $\mathrm{r} = 52.1775$. Now if $\mathrm{n} = 30\,p + q$ we can write the value of the annuity as

$$\frac{\mathrm{r}}{j_{(\mathrm{r})}} - \frac{\mathrm{r}\, v^{\frac{\mathrm{n}}{\mathrm{r}}}}{j_{(\mathrm{r})}} \times v^{\frac{30p}{\mathrm{r}}}$$

If we put $\frac{\mathrm{r}\, v^{\frac{2}{\mathrm{r}}}}{j_{(\mathrm{r})}} = A_q$ and $v^{\frac{30p}{\mathrm{r}}} = B_p$ and construct tables of A_q for $q = 0, 1, 2, \ldots 29$ and of B_p for $p = 0, 1, \ldots$ the required annuity value can be easily determined from the formula

$$A_0 - A_q B_p.$$

First $A_0 = \dfrac{\mathrm{r}}{j_{(\mathrm{r})}} = \dfrac{52.1775}{.034412769904} = 1516.2249405.$

We now calculate $v^{\frac{1}{\mathrm{r}}}$, $v^{\frac{10}{\mathrm{r}}}$, $v^{\frac{30}{\mathrm{r}}}$, $v^{\frac{300}{\mathrm{r}}}$, $v^{\frac{900}{\mathrm{r}}}$ from Tables I, etc. The table of values of A_q we now get by starting with A_0 and continually multiplying by $v^{\frac{1}{\mathrm{r}}}$, that is $A_1 - v^{\frac{1}{\mathrm{r}}} A_0$, $A_2 = v^{\frac{1}{\mathrm{r}}} A_1$, etc. We calculate $A_{10} = A_0 v^{\frac{10}{\mathrm{r}}}$, $A_{20} = A_{10} v^{\frac{10}{\mathrm{r}}}$, $A_{30} = A_{20} v^{\frac{10}{\mathrm{r}}} = A_0 v^{\frac{30}{\mathrm{r}}}$, to be used as check values.
The tables of values of B_p we get similarly by continuous multiplication by $v^{\frac{30}{\mathrm{r}}}$, thus $B_0 = 1$, $B_1 = B_0 v^{\frac{30}{\mathrm{r}}}$, etc. Check values are obtained from $B_{10} = v^{\frac{300}{\mathrm{r}}}$, $B_{20} = B_{10} v^{\frac{300}{\mathrm{r}}}$, $B_{30} = B_{20} v^{\frac{300}{\mathrm{r}}} = v^{\frac{900}{\mathrm{r}}}$.
The tables are calculated to 9 or 10 significant figures and later cut down to 7 figures. The completed tables follow:

TABLE FOR ASCERTAINING THE PRESENT VALUE OF A WEEKLY ANNUITY OF 1 FOR n WEEKS — INTEREST 3½% PER ANNUM — 52.1775 WEEKS TO THE YEAR

Present Value = 1516.225 − $A_q B_p$ where n = $30p + q$

q	A_q	$30p$	B_p
0	1516.225	0	1.0000000
1	1515.226	30	.9804149
2	1514.227	60	.9612133
3	1513.229	90	.9423878
4	1512.232	120	.9239310
5	1511.235	150	.9058357
6	1510.239	180	.8880948
7	1509.243	210	.8707014
8	1508.249	240	.8536486
9	1507.255	270	.8369297
10	1506.261	300	.8205384
11	1505.268	330	.8044680
12	1504.276	360	.7887124
13	1503.285	390	.7732653
14	1502.294	420	.7581209
15	1501.304	450	.7432730
16	1500.314	480	.7287159
17	1499.325	510	.7144439
18	1498.337	540	.7004514
19	1497.350	570	.6867330
20	1496.363	600	.6732832
21	1495.377	630	.6600969
22	1494.391	660	.6471688
23	1493.406	690	.6344939
24	1492.422	720	.6220673
25	1491.438	750	.6098840
26	1490.455	780	.5979393
27	1489.473	810	.5862286
28	1488.491	840	.5747473
29	1487.510	870	.5634908
30	1486.529	900	.5524547

e.g., 467 weeks: value is 1516.225 − 1499.325 × .7432730 = 401.8172.

Appendix I

(1) It will sometimes happen that values of $\log(1+i)$ or $j_{(r)}$ are required for a rate of interest not given in Tables I and II. In this case we can

either (a) calculate the value from Tables III, IV, V and VI. For $\log(1+i)$ this calculation will be merely the determination of a logarithm e.g. for $3\frac{5}{16}\%$ we have to find $\log 1.033125$ which can be readily done, but only to 10 place accuracy. For $j_{(r)}$ we must calculate $r\{(1+i)^{\frac{1}{r}}-1\}$ which involves finding the $\log(1+i)$, and the antilogarithm of one rth of this. The final result will be accurate only to about 7 places, and the process is fairly long:

or (b) we can interpolate in Table I or II as the case may be assuming (as will usually be the case) that the rate of interest for which the function is required is within the range of the Table. Now ordinary (first difference) interpolation is not sufficiently accurate neither is second difference interpolation. However, third difference interpolation is. The maximum error is not greater than $\frac{3h}{128}\left(\frac{d}{di}\right)^4$ where h is the interval between the values of i in the Table. For the first part of Table I (i.e. from 0% to 6%) $h = .00125$ and the maximum error is .00000 00000 0015 while from 6% to 10% $h = .0025$ and the maximum error is .00000 00000 024: as for Table II $h = .0025$ and the maximum error is .00000 00000 017 for $r=2$ rising to .00000 00000 055 for $r = \infty$.

The interpolation to third difference can be done by the usual central difference methods, but perhaps the easiest way is as follows:—
Use four tabulated values, two on each side of the value required. Then

(i) if, as will often be the case, the value is required for i half way, quarter way or three quarters way between the tabulated rates, use the appropriate one of the following formulas:—

$$u_{1\frac{1}{2}} = \frac{-u_0 + 9\,u_1 + 9\,u_2 - u_3}{16}$$

$$u_{1\frac{1}{4}} = \frac{-7\,u_0 + 105\,u_1 + 35\,u_2 - 5\,u_3}{128}$$

$$\text{or } u_{1\frac{3}{4}} = \frac{-5\,u_0 + 35\,u_1 + 105\,u_2 - 7\,u_3}{128}$$

Thus if $j_{(52)}$ is required for $3\frac{1}{16}\%$, u_0 is $j_{(52)}$ for $2\frac{3}{4}\%$, u_1 that for 3%, u_2 that for $3\frac{1}{4}\%$ and u_3 that for $3\frac{1}{2}\%$: we require $u_{1\frac{1}{4}}$ which we get at once on the machine as $105\,u_1$ plus $35\,u_2$ minus $5\,u_3$ minus $7\,u_0$, the net divided by 128. The answer is .030174165568.

(ii) but if we require a value for a rate of interest, not half or quarter or three quarters way between tabulated rates, say $\log(1+i)$ for 3.1%, proceed as follows:—choose u_0, u_1, u_2, u_3 as before. Let the required value be u_{1+x}. Interpolate (by ordinary or first difference interpolation) between u_1 and u_2, that is calculate $x\,u_2 + (1-x)\,u_1$: call the result u'_{1+x}. Do the same between u_0 and u_3 that is calculate $\dfrac{(1+x)\,u_3 + (2-x)\,u_0}{3}$ and call the result u''_{1+x}.

Then the required u_{1+x} is equal to

$$u'_{1+x} + \frac{(1-x)\,x}{2}(u'_{1+x} - u''_{1+x}).$$

For instance in our example $u_0 = \log 1.02875$, $u_1 = \log 1.03$, $u_2 = \log 1.03125$ and $u_3 = \log 1.0325$. We require $u_{1.8}$. Interpolating for 3.1% between 3% and 3.125% we have

$$u'_{1.8} = .8\,u_2 + .2\,u_1 = .013258614187.$$

Similarly interpolating between u_0 and u_3,

$$u''_{1.8} = (1.8\,u_3 + 1.2\,u_0)/3 = .013257975485.$$

Now $x\,(1-x)/2 = .08$ and so

$$u_{1.8} = u'_{1.8} + .08\,(u'_{1.8} - u''_{1.8}) = .013258665283.$$

From Tables III, etc., we get the value of log 1.031

as .0132586652. The correct value is .013258665284. As another example let us calculate $j_{(52)}$ for 3.1%.

$$u_0 = j_{(52)} \text{ for } 2\tfrac{3}{4}\%$$
$$u_1 = j_{(52)} \text{ for } 3\%$$
$$u_2 = j_{(52)} \text{ for } 3\tfrac{1}{4}\%$$
$$u_3 = j_{(52)} \text{ for } 3\tfrac{1}{2}\%.$$

We require $u_{1.4}$

$$u'_{1.4} = .4\,u_2 + .6\,u_1 = .030537476446$$

$$u''_{1.4} = \frac{1.4\;u_3 + 1.6\,u_0}{3} = .030531708137$$

Difference $= .000005768309$

Multiply by $\dfrac{.4 \times .6}{2} = .12$ $\quad .000000692197$

Add to $u'_{1.4}$ $\quad .030538168643 = u_{1.4}$.

The correct value is $\quad .030538168639$

and the best we can get by calculating from Tables III, IV, etc., is .030538144.

Note: When interpolating in Table I for i between $5\tfrac{5}{8}\%$ and $6\tfrac{1}{4}\%$ we must remember that the interval for i changes at 6% and be careful to take u_0, etc., at equal intervals, e.g. for 5.9% we must take $u_0 = \log(1+i)$ for $5\tfrac{1}{2}\%$, u_1 for $5\tfrac{3}{4}\%$, u_2 for 6% and u_3 for $6\tfrac{1}{4}\%$.

(2) If $j_{(r)}$ is required for a value of r not given in Table II, e.g. $j_{(6)}$ we must either calculate from Tables III, etc., as indicated in (1) (a) above or else get the value by summation of a series as for instance

$$j_{(r)} = i - \frac{r-1}{2r}\,i^2 + \frac{(2r-1)(r-1)}{6r^2}\,i^3 - \ldots.$$

except that if $j_{(r)}$ is required for weekly annuities when the number of weeks to be assumed in a year is neither 52 or 52.1775 but some other near number, e.g. $52\tfrac{1}{7}$, we can interpolate (or exterpolate if necessary) between $j_{(52)}$ and $j_{(52.1775)}$. Thus for $j_{(52\,1/7)}$ put this equal to

$$\frac{(52.1775 - 52\tfrac{1}{7})\,j_{(52)} + (52\tfrac{1}{7} - 52)\,j_{(52.1775)}}{52.1775 - 52}$$

$$\text{or } j_{(52\,1/7)} = \frac{97\,j_{(52)} + 400\,j_{(52.1775)}}{497}$$

Again if a year is assumed to equal 365¼ days we require $j_{(52\ 5/28)}$: put this equal to (this is an exterpolation)

$$\frac{(52.1775 - 52\tfrac{5}{28})\, j_{(52)} + (52\tfrac{5}{28} - 52)\, j_{(52.1775)}}{52.1775 - 52}$$

$$\text{or } j_{(55\ 5/28)} = \frac{-3\, j_{(52)} + 500\, j_{(52.1775)}}{497}.$$

For example $j_{(52\ 1/7)}$ 3% will be found equal to .029567182004.

Appendix II

Examples (6) and (7) involve weekly annuities payable for so many weeks *and a fraction of a week*. This brings up the question of the interpretation of the results. What is meant for example by an annuity of 10 a week for 106½ weeks?

The formula for $a_{\overline{n}|}^{(r)}$ has been used above, and is usually used, as though it held for such non-integral periods. This evidently requires that if we have an annuity for an integral number of periods plus $\frac{1}{p}$th of a period the value of the annuity payment for the final $\frac{1}{p}$th of a period is $\frac{1-(1+j)^{-\frac{1}{p}}}{j}$, valued at the beginning of such $\frac{1}{p}$th period (that is just after the last full payment). In this formula j is the effective rate of interest for a complete period. We now have two methods of making the final payment:—

(a) we can make it at the end of the $\frac{1}{p}$th of the complete period, when the amount of the payment should be

$$\frac{(1+j)^{\frac{1}{p}}-1}{j} = \frac{1}{p}\left(1-\frac{p-1}{2p}j+\ldots\right)$$

which is slightly less than $\frac{1}{p}$.

(b) We can make it at the end of the next *complete* period, when the amount of the payment should be

$$\frac{(1+j)-(1+j)^{1-\frac{1}{p}}}{j} = \frac{1}{p}\left(1+\frac{p-1}{2p}j-\ldots\right)$$

which is slightly more than $\frac{1}{p}$.

In practice, the amount of the final payment is invariably taken as $\frac{1}{p}$ so that the total actual payments made correspond with the total period of the annuity. To conform to the above theory such a final payment of $\frac{1}{p}$ should be made neither at the end of the final complete period nor at the end of $\frac{1}{p}$th of it but at a point approximately halfway between these two points. However, in practice the final payment, of $\frac{1}{p}$, is usually made at the end of $\frac{1}{p}$th of the period, except in the case of weekly annuities when it is often made at the end of the week. The theoretical error introduced by these sensible practical procedure is of course negligible.

Appendix III

So far in this paper and the examples it has been implicitly assumed that all the annuities dealt with are payable at the end of the period of payment; that is at the end of each year for yearly annuities, at the end of each week for weekly annuities and so on. The amounts and present values of annuities payable at the beginning of the period can be immediately derived from those of annuities payable at the end of the period as follows:—

Present value of an annuity of 1 per annum for n years payable (in installments of $\frac{1}{r}$) at the beginning of each $\frac{1}{r}$th of a year equals

$$a^{(r)}_{\overline{n-\frac{1}{r}}|} + \frac{1}{r} \text{ or alternatively } a^{(r)}_{\overline{n}|}\left(1+\frac{j_{(r)}}{r}\right)$$

Amount of an annuity of 1 per annum for n years payable (in installments of $\frac{1}{r}$) at the beginning of each $\frac{1}{r}$th of a year equals

$$s^{(r)}_{\overline{n+\frac{1}{r}}|} - \frac{1}{r} \text{ or alternatively } s^{(r)}_{\overline{n}|}\left(1+\frac{j_{(r)}}{r}\right)$$

Appendix IV

Up to this point it has been implicitly assumed that, in the case of an annuity payable r times a year, valued at rate of interest i, the rate of interest given is an *effective* annual rate and not a *nominal* annual rate convertible r times a year. If in any instances the given rate is a nominal one the valuation of the annuity is effected very readily by working in time units of $\frac{1}{r}$th of a year.

For example the present value of an annuity of 1 a month for 60 months, at 3% per annum effective rate of interest is

$$12\,\frac{1 - v^5}{j_{(12)}} \text{ at } 3\%$$

but at 3% per annum nominal rate convertible monthly it is

$$\frac{1 - v^{60}}{i} \text{ at } \tfrac{1}{4}\%.$$

Such calculations at nominal rates convertible with the same frequency as the annuity payments are relatively simpler than those at effective annual rates: the function $j_{(r)}$ does not have to be used. The only difficulty that may arise is in the determination of $\log(1 + i)$ with sufficient accuracy. In the above example, 3% convertible monthly, the value of $\log(1 + i)$ for ¼% is given in Table I, but if we required the value of 5⁄24% corresponding to 2½% convertible monthly we must proceed as in Appendix I, that is either we would have to calculate log 1.002083333 from Table III, etc., or we must interpolate in Table I.

In the case of weekly annuities, we are dealing with very low rates of interest (per week): e.g. at 3¼% convertible 52 times a year the weekly rate of interest is $\frac{1}{1600} = .000625 = \frac{1}{16}$ % and unless we need extreme accuracy for a large number of weeks it will be sufficient to calculate $\log(1 + i)$ from Tables III, etc. If we do need greater accuracy than this gives, it is usually quicker to calculate $\log(1 + i)$ from the series

$$\log(1 + i) = .4342944819\left(i - \frac{i^2}{2} + \frac{i^3}{3} - \dots\right)$$

only the first three or four terms of which need to be used.