GRADUATION OF AN AMERICAN REMARRIAGE TABLE FOR JOINT LIFE ANNUITIES

BY

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Messrs. Roeber and Marshall, in a paper entitled "An American Remarriage Table," printed in the *Proceedings*, Vol. XIX, p. 279, stated that a number of methods of graduating the average rates of the table referred to were tried including one of the two mentioned in my paper T.A.S., Vol. XXXI, p. 223, entitled "Graduation of Marriage and Remarriage Table by Mathematical Formulas." No mention was made, however, to the other formula used in the graduation of the Dutch remarriage table which is but one particular case of exponential curves which might be used for the graduation of similar tables.

The object of this paper is to draw attention:

(a) to the following exponential curves:

$$\text{Colog}\,(1-r_x^r) = -\Delta log(1+\beta_1\, s_1^{\,x}\, w^{x^2}\, g_1^{c^x}) \qquad (1)$$

$$\text{and Colog}\,(1-r_x^r) = -\Delta log(1+\beta_1\, s_1^{\,x}\, w^{x^2}\, v^{x^3}) \qquad (2)$$

and an application of these formulas to the graduation of the American remarriage probabilities which will establish values of joint life annuities, allowance for remarriage being made on one life.

(b) to the advantage of disposing of ungraduated rates of remarriage in addition to the probabilities of remarriage when, as is generally the case, the rates of mortality by another experience are to be used (see Appendix I).

Graduation of an American Remarriage Table Using Formula (1)

The values of β_1, s_1, w and g_1 of this formula were found from the ungraduated values of colog $(1-r_x^r)$ (see column 1 of Appendix 2) which, summed from the bottom upwards, i.e., from the older to the younger ages, give:

$$\Sigma\,\text{colog}\,(1-r_x^r) = \log\,(1+\beta_1\, s_1^{\,x}\, w^{x^2} g_1^{c^x}) + C$$

(see column 2 of Appendix 2).

Equalling the logarithms of the antilog of these sums minus one (see column 3 of Appendix 2) to

$$\log \beta_1 + x \log s_1 + x^2 \log w + c^x \log g_1,$$

(where c^x has the values used in the graduation of the American Experience table), summing both members of these equalities for groups of ages 18-31, 32-45, 46-59, 60-73 and solving those equations, the following values were found for the constants $\log \beta_1 = 2.0254$, $\log s_1 = -.090730$, $\log w = .00072815$ and $\log g_1 = -.00078428$.

From these constants $\log (\beta_1 s_1{}^x w^{x^2} g_1{}^{c^x})$ were found by a continuous process, computing the third differences from

$$\log (-\Delta^3 f(x)) = x \log c + 3 \log (c - 1) + \log (-\log g_1)$$

(the latter being computed for all ages by a continuous addition of $\log c$), the second differences $2 \log w + c^x (c - 1)^2 \log g_1$ from the third differences and the first differences $\log s_1 + (2x + 1) \log w + c^x (c - 1) \log g_1$, from the second. The initial value of age 18 was computed, using the formula $\log (\beta_1 s_1{}^x w^{x^2} g_1{}^{c^x})$ as well as the checking values at ten years intervals of ages by a five decimal places logarithm table.

From formula (1) may be seen the other operations necessary to be performed to find r_x^I, given in Appendix 3, column (3), as well as the deviations (ungraduated minus graduated values, giving weight to number of observations) and the accumulated deviations. In the first column of Appendix 3 are given the ungraduated remarriage probabilities. In the second column, the graduated probabilities obtained by the parabolic formula used by Messrs. Roeber and Marshall as well as the deviations and accumulated deviations. It may be seen that although the deviations for both the parabolic and exponential curves have the same signs for most ages, the accumulated deviations for the exponential curve show a tendency to be negative, having that sign from ages 51 onwards.

Diminishing the value of $\log \beta_1$, it is to be expected that the graduated probabilities will be decreased in greater degree at the younger ages than at older. To determine this decrement the differences were calculated between $\log (\beta_1 s_1{}^x w^{x^2} g_1{}^{c^x})$ and their corresponding ungraduated values given in Appendix 2, column 3 up to age 60. Twenty-five of those 43 differences proved to have the positive sign and their mean was found to be .00638. The new graduated probabilities found by thus reducing $\log \beta_1$ are given in column 4 entitled "exponential graduation $\log \beta_1 = 2.01902$ together with the deviations and accumulated deviations. It may be seen that the effect of diminishing $\log \beta_1$ is to change the signs of the accumulated deviations showing now a tendency to be positive being althrough positive from ages 41 onwards. If now instead of deducting .00638 we deduct say two-thirds of this number (.00425), the correct values found by formula (1) for the probabilities as well as the deviations and their accumulation will be approximately equal to those found by a linear interpretation taking one-third of the values obtained using $\log \beta_1 = 2.0254$ and two-thirds of the values obtained using $\log \beta_1 = 2.01902$. It thus appears that the accumulated deviations will be negative for the groups of ages 21-27, 38-41 and 57-64, whereas by the parabolic curve the accumulated deviations are negative for the groups of ages 20-26, 38-47 and 51-59 as can be seen from Appendix 3.

An interesting feature of these exponential formulas is that a change in value of $\log \beta_1$ does not affect the signs of the deviations as much as the signs of the accumulated deviations.

Joint Life Annuities with Allowance for Remarriage

Using formula (1) above, the logarithm of the probability that a person of age x will be alive at the end of one year and will not have contracted remarriage during that year of age is,

$$\log p_x + \log (1 - r'_x) = \log p_x + \Delta \log (1 + \beta_1 s_1{}^x w^{x^2} g_1{}^{c^x}),$$

and thus $p_x \dfrac{1 + \beta_1 s_1{}^{x+1} w^{(x+1)^2} g_1{}^{c^{x+1}}}{1 + \beta_1 s_1{}^x w^{x^2} g_1{}^{c^x}}$ denotes the probability

that a person of age x will be alive at the end of one year and will not have contracted remarriage during that year of age. The probability that a person of age x will be alive after t years and will not have remarried during that time is thus:

$${}_tp_x \frac{1 + \beta_1 s_1{}^{x+t} w^{(x+t)^2} g_1{}^{c^{x+t}}}{1 + \beta_1 s_1{}^x w^{x^2} g_1{}^{c^x}}$$

The probability that two persons of age x and y will be alive after t years and that x will not have remarried during that time is thus:

$$ {}_tp_{xy}\, \frac{1+\beta_1\, s_1^{x+t}\, w^{(x+t)^2}\, g_1^{c^{x+t}}}{1+\beta_1\, s_1^{x}\, w^{x^2}\, g_1^{c^{x}}} $$

Taking into account the element of interest we have the following formula for the value of an annuity payable during the joint life of x and y and until remarriage of x, when the mortality table follows Makeham's law:

$$ \frac{1}{1+\beta_1\, s_1^{x}\, w^{x^2}\, g_1^{c^{x}}} \Sigma\, v^t\, {}_tp_{xy} + \frac{\beta_1\, s_1^{x}\, w^{x^2}\, g_1^{c^{x}}}{1+\beta_1\, s_1^{x}\, w^{x^2}\, g_1^{c^{x}}}\ \Sigma\, v^t\, {}_tp_{xy} s_1^t\, w^{2xt+t^2}\ g_1^{c^x(c^t-1)} $$

t varying from one to w. These limits have not been inserted hereafter, it being understood that they are implied.

Putting in the above expression

$$ \frac{1}{1+\beta_1\, s_1^{x}\, w^{x^2}\, g_1^{c^{x}}} = \phi\,(x),\ \ {}_tp_{xy} = s^{2t}\ g^{(c^x+c^y)(c^t-1)} $$

and $\log g_1\, g = c^n \log g$, we have

$$ \phi\,(x)\ \Sigma v^t\,{}_tp_{xy} + (1-\phi(x))\ \Sigma v^t\ (ss_1)^t\, s^t\, w^{2xt+t^2}\ g^{(c^{x+n}+c^y)(c^t-1)} \qquad \text{(1a)} $$

The value of the second factor of the first term of (1a) may be found from equal ages annuity tables. As far as the second factor of the second term of (1a) is concerned we may also find its value from equal ages annuity tables calculated at varying rates of interest. Indeed, expressing the second factor of the above expression in terms of equal ages determined by the formula $c^{x+n}+c^y = c^{z+n}+c^z$ and multiplying and dividing by w^{2zt}, we have,

$$ a_{zz}^{(r)} = \Sigma \left(\frac{w^{2(x-z)}}{1+\iota}\right)^t (ss_1)^t\, s^t\, w^{2zt+t^2}\ g^{c^z(c^n+1)(c^t-1)} = \Sigma\, v^t\,{}_tp_{zz}^{(r)} = \Sigma \frac{D_{z+t\ z+t}^{(r)}}{D_{zz}^{(r)}} $$

$$ = \Sigma \frac{D_{z+t}^{(r)}}{D_z^{(r)}} \frac{l_{z+t}}{l_z}, D_z^{(r)} = v^z\, l_z^{(r)},\ v = \frac{1}{1+\iota'} = \frac{w^{2(x-z)}}{1+\iota} \text{ or } \iota' = \frac{1+\iota}{w^{2(x-z)}} - 1 \text{ and} $$

$$ l_z^{(r)} = k\,\beta_1 (ss_1)^z\, w^{z^2}\, (g_1\, g)^{c^z} \qquad \text{(1b)} $$

Expressing formula (1a) in terms of equal ages annuity we have thus:

$$ \phi(x)\ a_{z_1 z_1} + (1-\phi\,(x))\ a_{zz}^{(r)} $$

To find the values of the joint life annuities using formula (1b) one will need to tabulate the values of $\phi\,(x)$ and its complement and $a_{z_1 z_1}$ at a fixed rate of interest ι and $a_{zz}^{(r)}$ at the rates of interest $\iota' = \frac{1+\iota}{w^{2(x-z)}} - 1$, values of which must be tabulated as a function of $x - z$. Two uniform seniority tables must be given so as to find z_1 and z.

As an alternative to avoid the work to calculate $a_{zz}^{(r)}$ at the different rates of interest $\iota' = \frac{1+\iota}{w^{2(x-z)}} - 1$ one may expand $w^{2(x-z)t}$ in the second term below

$$\phi\,(x)\,a_{z_1 z_1} + (1-\phi\,(x))\,\Sigma\frac{D^{(r)}_{z+t}}{D^{(r)}_z}\,\frac{l_{z+t}}{l_z}\,w^{2(x-z)t}$$

putting $v = \frac{1}{1+\iota}$ in $D^{(r)}_z = v^z\, l^r_z$ we have thus

$$\phi(x)a_{z_1 z_1} + (1-\phi(x))\left[a^{(r)}_{zz} + 2\,(x-z)\log w\, Ia^{(r)}_{zz} + \frac{4(x-z)^2}{2!}\log^2 w\; I^2 a^{(r)}_{zz}\text{ etc.}\right] \quad (1c)$$

where log w must be calculated on Napieran basis

$$Ia^{(r)}_{zz} = \Sigma t\,\frac{D^{(r)}_{z+t\;z+t}}{D^{(r)}_{zz}} = \frac{\int^{(r)}_{zz}}{D^{(r)}_{zz}}\;;\; I^2 a^{(r)}_{zz} = \Sigma t^2\,\frac{D^{(r)}_{z+t\;z+t}}{D^{(r)}_{zz}} = \frac{2\Sigma\int^{(x)}_{zz} - \int^{(r)}_{zz}}{D^{(r)}_{zz}}$$

and generally the expression for $I^{x+1}a^{(r)}_{zz}$ will be found in terms of

$$\Sigma^x\frac{\int^{(r)}_{zz}}{D^{(r)}_{zz}}\quad\text{since}\quad \Sigma^x\int^{(r)}_{zz} = \Sigma\,\frac{t(t+1)\quad\quad(t+x)}{\underline{|x+1}}\,D^{(r)}_{z+t\;z+t}$$

$$\text{for example } I^3 a^{(r)}_{zz} = \frac{6\Sigma^2\int^{(r)}_{zz} - 6\Sigma\int^{(r)}_{zz} + \int^{(r)}_{zz}}{D^{(r)}_{zz}}$$

$$I^4 a^{(r)}_{zz} = \frac{24\Sigma^3\int^{(r)}_{zz} - 36\Sigma^2\int^{(r)}_{zz} + 14\Sigma\int^{(r)}_{zz} - \int^{(r)}_{zz}}{D^{(r)}_{zz}}$$

To find the values of the joint life annuities using formula (1c) one will need to tabulate besides the values of $\phi\,(x)$ and its complement, $a_{z_1 z_1}$ and $a^{(r)}_{zz}$ at a fixed rate of interest ι and two uniform seniority tables to find z_1 and z also $Ia^{(r)}_{zz}$, $I^2a^{(r)}_{zz}$, etc. and its coefficients as mentioned in (1c) values of which must be tabulated in a single entry table as a function of $x - z$.

Graduation Using Formula (2)

In formula (1) the factor $g_1^{c^x}$ is involved. The graduation of the American remarriage probabilities as explained above was made by giving to log c the same value as the one used in the American experience mortality table. The question arises whether this formula will suitably graduate that experience when the value of log c is changed to correspond to the one used in another mortality table graduated by Makeham's formula.

It is an observed fact that for most mortality tables graduated by Makeham's formula log c varies between .04 and .05 and one could, of course, find the graduated remarriage probabilities for the two extreme cases and thus see its influence on the other constants. A different process was, however, followed to obtain an indication as to whether log c has a great influence on the graduated remarriage probabilities. This process consists in substituting the factor $g_1^{c^x}$ for v^{x^3}. As a matter of fact, my first attempt to graduate the table referred to was to graduate applying the formula colog $(1 - r_x^r) = -\Delta \log (1 + \beta_1 s_1^x w^{x^2})$. The values of the constants were found by proceeding as mentioned above for formula (1) by solving three equations for the groups of ages 18-32, 33-47, 48-62 with the following results

$\log \beta_1 = 1.71895$, $\log s_1 = -.069167$ and $\log w = .00034775$.

The graduated probabilities gave values at ages 18-22 which were too low and otherwise did not bring out some of the features of the trend followed by the first and second differences of the ungraduated values of log $[\log^{-1} \Sigma \operatorname{colog} (1 - r_x^x) - 1]$ given in Appendix 2. We may see indeed that the tendency of the second differences is to decrease changing its sign from positive for the younger ages to negative for the older ages and thus give the shape to the first differences which are negatives throughout but go on increasing to a maximum. Formula (2) was, therefore, tested. The values of the constants were determined from four equations for groups of ages 18-28, 29-39, 40-50, 51-61 with the following results.

$$\log \beta_1 = 2.260707 \qquad \log w = .0015529$$
$$\log s_1 = -.11511 \qquad \log v = -.0000099088.$$

The graduated rates thus found were too great for the ages 18-22 and otherwise it was noted that the mean of the graduated probabilities found by the two graduations were nearer to the ungraduated for most ages. Therefore the mean of the constants above found were used in formula (2), i.e., $\log \beta_1 = 1.989828$, $\log s_1 = -.0921385$, $\log w = .00095032$ and $\log v = -.0000049544$. These graduated probabilities were found from the values of $\log (\beta_1 s_1^x w^{x^2} v^{x^3})$ by computing the latter expressions by a continuous process, the third differences being equal to $6 \log v$, the second differences being equal to $2 \log w + 6 (x+1) \log v$ and the first differences to $\log s_1 + (2x+1) \log w + (3x^2 + 3x + 1) \log v$. The initial values for age 18 were computed with a five decimal place logarithm table, as were also the checking values, at intervals of ten years of age.

In the fifth column of Appendix 3 entitled $\log \beta_1 = 1.989828$ (formula 2) the graduated probabilities are given as well as the deviations and accumulated deviations. It may be seen that the accumulated deviations have a clear tendency to be positive. This tendency will be counteracted by increasing the value of $\log \beta_1$, this causing the graduated rates to become greater all through the table, more so, however, at the younger ages than at the older. To determine this increment the differences between $\log (\beta_1 s_1^x w^{x^2} v^{x^3})$ and their corresponding ungraduated values given in Appendix 2, column 3 up to age 60 were calculated and the mean of the negative differences (25 out of 43 proved to have that sign) was found to be .00697. The new graduated probabilities thus found are given in the sixth column of Appendix 3, in the column entitled exponential graduation $\log \beta_1 = 1.996798$ formula 2 as well as the deviations and accumulated deviations. It may be seen that the effect of increasing $\log \beta_1$ is to give to the accumulated deviations a tendency to be negative, as one would expect. By adding a fraction of .00697 we will obtain probabilities, deviations and their accumulations lying between those found, being approximately those found by a linear interpolation.

Joint Life Annuities with Allowance for Remarriage

Using formula (2) above, the logarithm of the probability that a person of age x will be alive at the end of one year and will not have contracted remarriage during that year of age is:

$$\log p_x + \log (1 - r_x^r) = \log p_x + \Delta \log (1 + \beta_1 s_1^x w^{x^2} v^{x^3})$$

and thus $p_x \frac{1+\beta_1 s_1^{x+1} w^{(x+1)^2} v^{(x+1)^3}}{1+\beta_1 s_1^x w^{x^2} v^{x^3}}$ denotes the probability that a person of age x will be alive at the end of one year and will not have contracted remarriage during that year of age. The probability that a person of age x will be alive after t years and will have remarried during that time is thus:

$$_tp_x \frac{1+\beta_1 s_1^{x+t} w^{(x+t)^2} v^{(x+t)^3}}{1+\beta_1 s_1^x w^{x^2} v^{x^3}}$$

The probability that two persons of age x and y will be alive after t years and that x will not have remarried during that time is thus:

$$_tp_{xy} \frac{1+\beta_1 s_1^{x+t} w^{(x+t)^2} v^{(x+t)^3}}{1+\beta_1 s_1^x w^{x^2} v^{x^3}}$$

Taking into account the element of interest we have the following formula for the value of an annuity payable during the joint life of x and y and until remarriage of x, when the mortality table follows Makeham's law, t varying from 1 to the limit of the table

$$\frac{1}{1+\beta_1 s_1^x w^{x^2} v^{x^3}} \Sigma\, v^t\, {}_tp_{xy} + \frac{\beta_1 s_1^x w^{x^2} v^{x^3}}{1+\beta_1 s_1^x w^{x^2} v^{x^3}} \Sigma\, v^t\, {}_tp_{xy}\, s_1^t\, w^{2xt+t^2}\, v^{3xt^2+3x^2t+t^3}$$

putting in the above expression

$$\frac{1}{1+\beta_1 s_1^x w^{x^2} v^{x^3}} = \phi(x) \text{ and } {}_tp_{xy} = s^{2t}\, g^{(c^x+c^y)(c^t-1)}$$

$$\text{we have } \phi(x)\, \Sigma\, v^t\, s^{2t}\, g^{(c^x+c^y)(c^t-1)}$$

$$+(1-\phi(x))\, \Sigma\, v^t\, (s\, s_1)^t\, s^t\, w^{2xt+t^2}\, v^{3xt^2+3x^2t+t^3}\, g^{(c^x+c^y)(c^t-1)} \qquad (2a)$$

The value of the second factor of the first term of (2a) may be found from equal ages annuity tables since it has been assumed that the mortality table follows Makeham's law. As to the second factor of the second term of (2a) is concerned we may also find its value from equal ages annuity tables calculated at varying rates of interest. Indeed, putting in (2a) $c^x + c^y = 2c^w$ and multiplying and dividing by $w^{2zt}\, v^{3zt^2+3z^2t}$, we have

$$\phi(x)\, a_{zz} + (1-\phi(x)) \sum \frac{D^{(r)}_{z+t}}{D^{(r)}_z} \frac{l_{z+t}}{l_z} v^{3(x-z)t^2} \dots\dots\dots\dots\dots \quad (2b)$$

$$D^{(r)}_z = v^z\, l^{(r)}_z,\ v = \frac{1}{1+\iota'} = \frac{w^{2(x-z)}\, v^{3(x^2-z^2)}}{1+\iota},\quad l^{(r)}_z = K\beta_1(ss_1)^z w^{z^2}\, v^{z^3} g^{c^z}$$

Expanding $v^{3(x-z)t^2}$ we have

$$\sum \frac{D^{(r)}_{z+t}}{D^{(r)}_z} \cdot \frac{l_{z+t}}{l_z} v^{3(x-z)t^2} = a^{(r)}_{zz} + 3(x-z)\, \log v\ I^2 a^{(r)}_{zz} + \frac{q(x-z)^2}{2!} (\log v)^2\, I^4 a^{(r)}_{zz} + \text{etc.}$$

log v being taken on Napierian basis.

To find the values of joint life annuities using formula (2b) one will need to tabulate the value of $\phi(x)$ and its complement, a_{zz} at a fixed rate i and $a^{(r)}_{zz}$, $I^2a^{(r)}_{zz}$, $I^4a^{(r)}_{zz}$ etc., at various rates of interest found from the formula $\iota' = \dfrac{1+\iota}{w^{2(x-z)}\, v^{3(x^2-z^2)}} - 1$ to be tabulated in a double entry in terms of x and z.

As an alternative, to avoid the work of having to tabulate the functions referred to at different rates of interest, instead of including the factor $w^{2(x-z)t}\, v^{3(x^2-z^2)t}$ in the interest factor v^t, we may expand it, so that (2b) becomes

$$\phi(x)\, a_{zz} + (1-\phi(x)) \sum \frac{D^{(r)}_{z+t}}{D^{(z)}_z} \frac{l_{z+t}}{l_z} w^{2(x-z)t}\, v^{3(x-z)t^2+3(x^2-z^2)t}$$

$$D^{(r)}_z = v^z\, l^{(r)}_z,\ v = \frac{1}{1+\iota}$$

$$= \phi(x)a_{zz} + (1-\phi(x))[a^{(r)}_{zz} + {}_1K_{zz}\, Ia^{(r)}_{zz} + \frac{{}_2K_{zz}}{2!} I^2a^{(r)}_{zz} + \frac{{}_3K_{zz}}{3!} I^3a^{(r)}_{zz} \quad (2c)$$

putting $f(x,z) = 2 \log w + 3(x+z) \log v$, the logarithms being taken on Napierian basis

$${}_1K_{zz} = (x-z)\, f(x,z)$$

$$\frac{{}_2K_{zz}}{2!} = \frac{(x-z)^2}{2!} \{(f(x,z))^2 + 6\,(x-z)\log v\}$$

$$\frac{{}_3K_{zz}}{3!} = \frac{(x-z)^3}{3!} \{(f(x,z))^3 + 18\,(x-z)^2 \log v\, f(x,z)\}$$

and so on, values of which must be tabulated in a double entry table in terms of x and z.

In conclusion I wish to point out that it is not contended that the values obtained for the constants in formulas (1) and (2) would not be improved upon by giving weight to the observations. The method of finding their values above explained is simple and gave a good enough graduation to satisfy one of the objects of this paper as above mentioned.

It is also noteworthy that by the exponential formulas above mentioned the remarriage factor may be neglected from a certain age onwards as $\phi(x)$ approaches to one.

Appendix 1

To find the values of annuities with allowance for remarriage one has often to use the rates of mortality by one experience and either the rate of remarriage by another experience denoted by r_x or the probability of remarriage by another experience denoted by r^r_x. Messrs. Roeber and Marshall in their paper, page 296 give the formula for the adjustments to be made in the rates of mortality. When joint life annuity values have to be found, the rates of mortality may be graduated by a mathematical formula whose property permits their values to be easily found from equal ages annuities as is the case for Makeham's formula or otherwise. It is, therefore, advisable, if possible, to avoid those adjustments. In my paper (T.A.S., Vol. XXXI, p. 223) is given the formula used for graduating the remarriage experience when dependent probabilities and when independent probabilities are dealt with. What was meant by dependent and independent probabilities may be expressed by the symbols used in Messrs. Roeber and Marshall's paper by colog $(p^r_x - r^r_x)$ — colog p^r_x for dependent probabilities of death and remarriage and colog $(1-r_x)$ for independent probabilities (r_x is a notation I now use to denote rate of remarriage), q_x denoting the rate of mortality.

It is shown, hereafter, that for practical purpose one may express the relation between l^r_{x+1} and l^r_x in terms of factors of q_x and r_x and also of factors of q_x and r^r_x. Be it first noted that the relation between r_x, the rate, and r^r_x, the probability of remarriage, is:

$$r_x = \frac{m^r_x}{l^r_x - \frac{d^r_x}{2}} = \frac{r^r_x}{1 - \frac{q^r_x}{2}}$$

In this relation the deaths unmarried are given half a year of exposure, as half a year of exposure was given to the number remarrying at age x in Messrs. Roeber and Marshall's paper in the relation:

$$q_x = \frac{d^r_x}{l^r_x - \frac{m^r_x}{2}} = \frac{q^r_x}{1 - \frac{r^r_x}{2}}$$

By expressing in $l^r_{x+1} = l^r_x - m^r_x - d^r_x$, m^r_x and d^r_x in terms of $a|q_x\, l^r_x$ and $r_x\, l^r_x$ and of $(b)|q_x\, l^r_x$ and $r^r_x\, l^r_x$ we have:

(a) using the relations $q_x = \dfrac{d^r_x}{l^r_x - \dfrac{m^r_x}{2}}$ and $r_x = \dfrac{m^r_x}{l^r_x - \dfrac{d^r_x}{2}}$ the following equalities hold:

$$l^r_{x+1} = l^r_x - m^r_x - d^r_x = l^r_x \left[1 - \frac{r_x\left(1-\frac{q_x}{2}\right)}{1 - \frac{r_x\, q_x}{4}} - \frac{q_x\left(1-\frac{r_x}{2}\right)}{1 - \frac{r_x\, q_x}{4}}\right]$$

$$= l^r_x \, \frac{(1-r_x)(1-q_x) - \frac{q_x\, r_x}{4}}{1 - \frac{q_x\, r_x}{4}}$$

Thus $l^r_{x+1} = l^r_x - m^r_x - d^r_x$ is smaller than $l^r_x\,(1-q_x)\,(1-r_x)$ by

$$l^r_x\, \frac{q_x\, r_x}{4}\, \frac{1-(1-r_x)(1-q_x)}{1-\frac{q_x\, r_x}{4}}$$

(b) using the relations $q_x = \dfrac{d^r_x}{l^r_x - \dfrac{m^r_x}{2}}$ and $r^r_x = \dfrac{m^r_x}{l^r_x}$ the following equalities hold:

$$l^r_{x+1} = l^r_x - m^r_x - d^r_x = l^r_x \left[1 - r^r_x - q_x\left(1 - \frac{r^r_x}{2}\right)\right]$$

$$= l^r_x \left[(1-q_x)(1-r^r_x) - \frac{1}{2} q_x\, r^r_x\right]$$

Thus $l^r_{x+1} = l^r_x - m^r_x - d^r_x$ is smaller than $l^r_x\,(1-q_x)(1-r^r_x)$ by $\frac{l^r_x}{2}\, q_x\, r^r_x$

It may thus be seen that to express the relation between l^r_{x+1} and l^r_x in terms of factors of q_x and r_x is so much nearer to the exact relation than by expressing that relations in terms of factors of q_x and r^r_x. For age 18 $\frac{q_x\, r^r_x}{2}$ is equal to .0004588 whilst $\frac{q_x\, r_x}{2}\, \dfrac{1-(1-r_x)(1-q_x)}{1-\frac{q_x\, r_x}{4}}$ is equal to .00002861 q_{18} being the rate by the American Experience table. For older age those values will be smaller.

APPENDIX 2

Age	(1) colog $(1-r_x^r)$	(2) Σ colog $(1-r_x^r)$	(3) log $(log^{-1}(x)-1)$	(4) Δ	(5) Δ^2	(6) Δ^3	Age	(1) colog $(1-r_x^r)$	(2) Σ colog $(1-r_x^r)$	(3) log $(log^{-1}(x)-1)$	(4) Δ	(5) Δ^2	(6) Δ^3
18	.05443	.71292	0.61943	-.06859	.00410	.00183	46	.00568	.07685	$\bar{1}$.28691	-.03625	-.00382	.00590
19	.04949	.65849	0.55084	-.06447	.00593	-.00337	47	.00581	.07117	$\bar{1}$.25066	-.04007	.00208	.00224
20	.04340	.60900	0.48635	-.05856	.00256	-.00058	48	.00511	.06536	$\bar{1}$.21059	-.03799	.00432	-.00543
21	.04005	.56560	0.42779	-.05600	.00198	-.00039	49	.00423	.06025	$\bar{1}$.17260	-.03367	-.00111	.00911
22	.03720	.52555	0.37179	-.05402	.00159	-.00242	50	.00406	.05602	$\bar{1}$.13893	-.03478	.00800	-.01015
23	.03470	.48835	0.31777	-.05243	-.00083	-.00256	51	.00292	.05196	$\bar{1}$.10415	-.02678	-.00215	.00259
24	.03376	.45365	0.26534	-.05326	-.00339	.00412	52	.00301	.04904	$\bar{1}$.07737	-.02893	.00044	-.00647
25	.03423	.41989	0.21208	-.05665	.00073	-.00127	53	.00279	.04603	$\bar{1}$.04844	-.02849	-.00603	.01033
26	.03203	.38566	0.15543	-.05592	-.00054	.00148	54	.00314	.04324	$\bar{1}$.01995	-.03452	.00430	-.01336
27	.03054	.35363	0.09951	-.05646	.00094	.00160	55	.00261	.04010	$\bar{2}$.98543	-.03022	-.00906	.01322
27	.02826	.32309	0.04305	-.05552	.00254	.00514	56	.00309	.03749	$\bar{2}$.95521	-.03928	.00416	-.00849
29	.02530	.29483	$\bar{1}$.98753	-.05298	.00768	-.00865	57	.00257	.03440	$\bar{2}$.91593	-.03512	-.00433	-.00240
30	.02036	.26953	$\bar{1}$.93455	-.04530	-.00097	.00111	58	.00270	.03183	$\bar{2}$.88081	-.03945	-.00673	.00436
31	.01959	.24917	$\bar{1}$.88925	-.04627	.00014	.00188	59	.00283	.02913	$\bar{2}$.84136	-.04618	-.00237	.00930
32	.01836	.22958	$\bar{1}$.84298	-.04613	.00202	-.00096	60	.00270	.02630	$\bar{2}$.79518	-.04855	.00693	-.02588
33	.01646	.21122	$\bar{1}$.79685	-.04411	.00106	-.00419	61	.00213	.02360	$\bar{2}$.74663	-.04162	-.01895	.02054
34	.01507	.19476	$\bar{1}$.75274	-.04305	-.00313	.00832	62	.00274	.02147	$\bar{2}$.70501	-.06057	.00159	-.01352
35	.01511	.17969	$\bar{1}$.70969	-.04618	.00519	-.00182	63	.00231	.01873	$\bar{2}$.64444	-.05898	-.01193	-.00196
36	.01251	.16458	$\bar{1}$.66351	-.04099	.00337	.00129	64	.00244	.01642	$\bar{2}$.58546	-.07091	-.01389	-.02143
37	.01077	.15207	$\bar{1}$.62252	-.03762	.00466	-.00737	65	.00244	.01398	$\bar{2}$.51455	-.08480	-.03532	.02479
38	.00891	.14130	$\bar{1}$.58490	-.03296	-.00271	-.00048	66	.00279	.01154	$\bar{2}$.42975	-.12012	-.01053	-.01249
39	.00908	.13239	$\bar{1}$.55194	-.03567	-.00319	.00302	67	.00226	.00875	$\bar{2}$.30963	-.13065	-.02302	.05447
40	.00931	.12331	$\bar{1}$.51627	-.03886	-.00017	-.00159	68	.00192	.00649	$\bar{2}$.17898	-.15367	.03145	.00722
41	.00869	.11400	$\bar{1}$.47741	-.03903	-.00176	.00328	69	.00109	.00457	$\bar{2}$.02531	-.12222	.03867	-.08446
42	.00846	.10531	$\bar{1}$.43838	-.04079	.00152	-.00126	70	.00061	.00348	$\bar{3}$.90309	-.08355	-.04579	-.05267
43	.00758	.09685	$\bar{1}$.39759	-.03927	.00026	.00635	71	.00074	.00287	$\bar{3}$.81954	-.12934	-.09846	
44	.00700	.08927	$\bar{1}$.35832	-.03901	.00661	-.01046	72	.00087	.00213	$\bar{3}$.69020	-.22780		
45	.00542	.08227	$\bar{1}$.31931	-.03240	-.00385	.00003	73	.00126	.00126	$\bar{3}$.46240			

APPENDIX 3

(dev = Actual Marriages Minus Expected Remarriages)

Age	(1) Ungrad. Remarriage Probab.	(2) Parabolic Graduation	Dev	Acc Dev	(3) Exp. Grad. (form 1) log β_1 = 2.0254	Dev	Acc Dev	(4) Exp. Grad. (form 1) log β_1 = 2.01902	Dev	Acc Dev	(5) Exp. Grad. (form 2) log β_1 = 1.989828	Dev	Acc Dev	(6) Exp. Grad. (form 2) log β_1 = 1.996798	Dev	Acc Dev
18	.1178	.1128	6.1	6.1	.1113	8.0	8.0	.1109	8.5	8.5	.1069	13.4	13.4	.1073	12.9	12.9
19	.1077	.1060	3.0	9.1	.1058	3.3	11.3	.1054	4.0	12.5	.1017	10.4	23.8	.1021	9.7	22.6
20	.0951	.0995	−9.8	−0.7	.1002	−11.4	−.1	.0999	−10.8	1.7	.0964	−2.9	20.9	.0969	−4.1	18.5
21	.0881	.0932	−13.7	−14.4	.0946	−17.5	−17.6	.0943	−16.7	−15.0	.0913	−8.6	12.3	.0917	−9.7	8.8
22	.0821	.0872	−16.3	−30.7	.0891	−22.4	−40.0	.0887	−21.1	−36.1	.0861	−12.8	−.5	.0865	−14.1	−5.3
23	.0768	.0816	−17.4	−48.1	.0835	−24.3	−64.3	.0831	−22.8	−58.9	.0810	−15.2	−15.7	.0814	−16.7	−22.0
24	.0748	.0762	−5.5	−53.6	.0781	−13.0	−77.3	.0777	−11.5	−70.4	.0759	−4.4	−20.1	.0764	−6.4	−28.4
25	.0758	.0710	19.6	−34.0	.0728	12.2	−65.1	.0724	13.8	−56.6	.0710	19.6	−.5	.0715	17.5	−10.9
26	.0711	.0661	21.9	−12.1	.0676	15.3	−49.8	.0672	17.1	−39.5	.0663	21.1	20.6	.0667	19.2	8.3
27	.0679	.0615	28.5	16.4	.0627	23.1	−26.7	.0622	25.2	−14.3	.0617	27.6	48.2	.0622	25.3	33.6
28	.0630	.0571	27.6	44.0	.0579	23.8	−2.9	.0575	25.6	11.3	.0573	26.6	74.8	.0577	24.7	58.3
29	.0566	.0530	16.8	60.8	.0534	14.9	12.0	.0530	16.8	28.1	.0531	16.3	91.1	.0535	14.4	72.7
30	.0458	.0490	−16.0	44.8	.0490	−16.0	−4.0	.0487	−14.5	13.6	.0491	−16.5	74.6	.0495	−18.5	54.2
31	.0441	.0453	−6.0	38.8	.0451	−5.1	−9.1	.0447	−3.1	10.5	.0454	−6.5	68.1	.0458	−8.1	46.1
32	.0414	.0419	−2.5	36.3	.0414	0	−9.1	.0410	2.0	12.5	.0418	−2.1	66.0	.0422	−4.1	42.0
33	.0372	.0386	−7.2	29.1	.0378	−3.1	−12.2	.0375	−1.6	10.9	.0385	−6.7	59.3	.0389	−8.8	33.2
34	.0341	.0355	−7.2	21.9	.0347	−3.1	−15.3	.0343	−1.1	9.8	.0355	−7.2	52.1	.0357	−8.2	25.0
35	.0342	.0327	7.5	29.4	.0316	12.9	−2.4	.0313	14.4	24.2	.0326	8.0	60.1	.0330	5.9	30.9
36	.0284	.0300	−8.3	21.1	.0289	−2.6	−5.0	.0286	−1.1	23.1	.0299	−7.8	52.3	.0302	−9.3	21.6
37	.0245	.0275	−15.1	6.0	.0264	−9.6	−14.6	.0262	−8.6	14.5	.0275	−15.1	37.2	.0277	−16.2	5.4
38	.0203	.0252	−25.3	−19.3	.0242	−20.2	−34.8	.0238	−18.1	−3.6	.0251	−24.9	12.3	.0255	−26.9	−21.5
39	.0207	.0231	−12.4	−31.7	.0221	−7.3	−42.1	.0219	−6.2	−9.8	.0231	−12.4	−.1	.0234	−14.0	−35.5
40	.0212	.0211	0.5	−31.2	.0203	4.6	−37.5	.0200	6.2	−3.6	.0212	0.0	−.1	.0215	−1.6	−37.1
41	.0198	.0193	2.4	−28.8	.0185	6.3	−31.2	.0184	6.8	3.2	.0195	1.4	1.3	.0197	.4	−36.7
42	.0193	.0176	8.2	−20.6	.0171	10.6	−20.6	.0168	12.0	15.2	.0178	7.2	8.5	.0181	5.7	−31.0
43	.0173	.0161	5.4	−15.2	.0157	7.2	−13.4	.0155	8.1	23.3	.0164	4.1	12.6	.0166	3.1	−27.9
44	.0160	.0147	5.6	−9.6	.0144	6.8	−6.6	.0143	7.2	30.5	.0150	4.3	16.9	.0152	3.4	−24.5
45	.0124	.0134	−4.0	−13.6	.0134	−4.0	−10.6	.0132	−3.2	27.3	.0138	−5.5	11.4	.0140	−6.4	−30.9

APPENDIX 3—(Continued)

(dev = Actual Remarriages Minus Expected Remarriages)

Age	(1) Ungrad. Remar-riage Probab.	(2) Para-bolic Gradua-tion	Dev	Acc Dev	(3) Exp. Grad. (form 1) log β_1 = 2.0254	Dev	Acc Dev	(4) Exp. Grad. (form 1) log β_1 = 2.01902	Dev	Acc Dev	(5) Exp. Grad. (form 2) log β_1 = 1.989828	Dev	Acc Dev	(6) Exp Grad. (form 2) log β_1 = 1.996798	Dev	Acc Dev
46	.0130	.0123	2.7	−10.9	.0124	2.3	− 8.3	.0122	3.1	30.4	.0127	1.2	12.6	.0129	.3	−30.6
47	.0133	.0113	7.8	− 3.1	.0115	7.0	− 1.3	.0114	7.4	37.8	.0117	6.2	18.8	.0119	5.4	−25.2
48	.0117	.0104	5.1	2.0	.0108	3.4	2.1	.0106	4.2	42.0	.0107	3.8	22.6	.0109	3.0	−22.2
49	.0097	.0095	0.8	2.8	.0100	− 1.2	.9	.0099	− .8	41.2	.0099	− .8	21.8	.0100	− 1.2	−23.4
50	.0093	.0088	2.0	4.8	.0095	− 0.9	0.	.0094	− .5	40.7	.0091	.8	22.6	.0093	− .1	−23.5
51	.0067	.0082	− 5.9	− 1.1	.0090	− 9.0	− 9.0	.0088	− 8.3	32.4	.0085	− 7.0	15.6	.0086	− 7.5	−31.0
52	.0069	.0077	− 3.0	− 4.1	.0085	− 6.1	−15.1	.0084	− 5.7	26.7	.0078	− 3.4	12.2	.0079	− 3.8	−34.8
53	.0064	.0072	− 2.8	− 6.9	.0081	− 5.9	−21.0	.0080	− 5.6	21.1	.0072	− 2.8	9.4	.0073	− 3.2	−38.0
54	.0072	.0068	1.3	− 5.6	.0077	− 1.8	−22.8	.0076	− 1.4	19.7	.0066	1.9	11.3	.0068	1.3	−36.7
55	.0060	.0065	− 1.6	− 7.2	.0075	− 4.8	−27.6	.0073	− 4.2	15.5	.0062	− .7	10.6	.0062	− .7	−37.4
56	.0071	.0062	2.5	− 4.7	.0071	0.	−27.6	.0071	.0	15.5	.0057	3.9	14.5	.0058	3.6	−33.8
57	.0059	.0059	0.	− 4.7	.0069	− 2.6	−30.2	.0068	− 2.4	13.1	.0053	1.5	16.0	.0054	1.2	−32.6
58	.0062	.0057	1.3	− 3.4	.0066	− 1.1	−31.3	.0065	− .9	12.2	.0049	3.3	19.3	.0050	3.0	−29.6
59	.0065	.0056	2.1	− 1.3	.0064	0.2	−31.1	.0062	.7	12.9	.0046	4.4	23.7	.0046	4.4	−25.2
60	.0062	.0055	1.5	.2	.0062	0.	−31.1	.0061	.2	13.1	.0042	4.1	27.8	.0043	3.9	−21.3
61	.0049	.0053	− 0.8	− .6	.0059	− 2.0	−33.1	.0058	− 1.8	11.3	.0040	1.8	29.6	.0040	1.8	−19.5
62	.0063	.0052	2.1	1.5	.0055	2.5	−30.6	.0055	2.5	13.8	.0037	5.0	34.6	.0038	4.7	−14.8
63	.0053	.0051	0.3	1.8	.0053	0.	−30.6	.0053	0.	13.8	.0035	2.7	37.3	.0035	2.7	−12.1
64	.0056	.0051	0.8	2.6	.0050	0.9	−29.7	.0049	1.0	14.8	.0032	3.5	40.8	.0033	3.3	− 8.8
65	.0056	.0050	0.8	3.4	.0046	1.3	−28.4	.0046	1.3	16.1	.0030	3.3	44.1	.0031	3.1	− 5.7
66	.0064	.0049	1.7	5.1	.0042	2.5	−25.9	.0042	2.5	18.6	.0028	4.0	48.1	.0029	3.9	− 1.8
67	.0052	.0047	0.5	5.6	.0038	1.3	−24.6	.0037	1.4	20.0	.0026	2.5	50.6	.0027	2.4	.6
68	.0044	.0046	− 0.1	5.5	.0033	1.0	−23.6	.0033	1.0	21.0	.0025	1.7	52.3	.0025	1.7	2.3
69	.0025	.0044	− 1.1	4.4	.0029	− 0.3	−23.9	.0028	− .2	20.8	.0023	.1	52.4	.0023	.1	2.4
70	.0014	.0041	− 1.9	2.5	.0024	− 0.7	−24.6	.0024	− .7	20.1	.0021	− .5	51.9	.0022	− .6	1.8
71	.0017	.0039	− 1.3	1.2	.0020	− 0.2	−24.8	.0020	− .2	19.9	.0020	− .2	51.7	.0021	− .3	1.5
72	.0020	.0035	0.7	1.9	.0016	.2	−24.6	.0015	.3	20.2	.0019	.1	51.8	.0019	.1	1.6
73	.0029	.0031														