NOTES ON POISSON'S EXPONENTIAL AND CHARLIER'S CURVES.

BY

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In his paper on Graduation of Frequency Distributions (*Proc.*, VI, p. 52 et seq., at p. 72) Professor Carver gives the formula for Charlier's Type B curve, stating that the basis is

$$\psi(x) = \frac{e^{-m}m^x}{\lfloor x}.$$

This last function is known as Poisson's Exponential Expansion or (the term is due to Bortkewitsch) "The Law of Small Numbers" and through it we can see something of the relationship between the Pearson and Charlier Systems of Frequency Curves, on the one hand, as well as Professor Carver's method of graduation, and on the other the Theory of Probabilities as it is usually taught. I believe this relationship is sufficiently interesting to our members to justify calling it to their attention in these notes.

We are all, of course, familiar with the elementary proposition in the theory of probabilities that the probabilities of the respective possible results in a series of n trials of any subject whose individual probability is p are given by the successive terms in the expansion of the binomial $(p+q)^n$.

Performing the expansion and considering as the general term the $(x+1)$th term we get the term

$$\begin{aligned}(p+q)^n = p^n + np^{n-1}q + \frac{n(n-1)}{2}p^{n-2}q^2 + \cdots \\ + \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}p^{n-x}q^x, \cdots\end{aligned} \tag{1}$$

If in the above we let $n\,q = m$ we may write

$$(p+q)^n = p^n + mp^{n-1} + \frac{1\left(1-\frac{1}{n}\right)}{2} p^{n-2}m^2 + \cdots$$
$$+ \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x+1}{n}\right)}{x!} p^{n-x}m^x, \cdots \tag{2}$$

$$= p^n + mp^{n-1} + \frac{m}{2}\cdot 1\cdot\left(1-\frac{1}{n}\right)p^{n-2} + \cdots$$
$$+ \frac{m^x}{x!}\cdot 1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x+1}{n}\right)p^{n-x}, \cdots \tag{3}$$

We may now let $q=0$ while m remains finite, i.e., $n=\infty$. Under these conditions, the factor $m^x/x!$ in the general term remains finite and the several factors $[1-(1/n)]$, $[1-(2/n)]$ $\ldots [1-(x+1/n)]$ approach unity. There remains for consideration p^{n-x}, which we may transform as follows:

$$p^{n-x} = (1-q)^{n-x} = \left(1-\frac{m}{n}\right)^{n-x} = \left(1-\frac{m}{n}\right)^{n}\left(1-\frac{m}{n}\right)^{-x}.$$

Passing to the limit when $n=\infty$ the first factor becomes e^{-m} and the second unity.

Hence we find as the limit of the general term when $n=\infty$ but nq remains finite $e^{-m}(m^x/x!)$ and

$$\lim_{\substack{n\doteq\infty \\ q=0 \\ nq \text{ finite}}} (p+q)^n = e^{-m}\left(1+m+m^2/2+\frac{m^3}{3!}+\cdots+\frac{m^x}{x!}\right).$$

It will be noted that the expression in parentheses is the expansion of e^m which must be so since $(p+q)^n \equiv 1$.

This representation of $(p+q)^n$ was first discovered by Poisson. Hence its name.

Although the formula is rigid only in the limit, it gives a fair approximation when q is relatively large. In general, in the type of probabilities we deal with, q is small, and even $n\,q$ is small. It is because of its application in such cases that it was called the "Law of Small" numbers.

Tables of the value of this function according to the values of m and x are tabulated for values of m by intervals of 0.1 up to

$m = 15$ in "Tables for Statisticians and Biometricians" edited by Professor Karl Pearson. As a means of graduating actual frequencies the formula is, however, of little value, as the function is not continuous in this form and in practice perturbing influences prevent natural events accurately following the Bernoullian law represented by the binomial expansion.

It can be shown that the so-called normal error curve is the limiting form of the same binomial expansion when p and q are nearly equal and $n = \infty$. One demonstration of this will be found, for example, in Bowley's "Elements of statistics."

Since the Charlier Type A curve is, as Professor Carver has pointed out, reducible to the form

$$y = \phi^{(x)} f^{(x)} = \phi^{(x)} (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \ldots).$$

Where

$$\varphi(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-b)^2}{\sigma^2}}$$

(the normal curve) it would seem that it should be found best fitted to graduate frequencies developed from large probabilities where p is nearly equal to q and that the relative size of the coefficients b_1, b_2, etc., to b_0 are determined by the amount of perturbation of the basic probability. This must be so, since if they all vanish but b_0 the curve reduces to the normal curve multiplied by a constant, which is, of course, a function of the scale of the y coordinates.

On the other hand, Charlier's Type B curve is based on Poisson's Exponential, which approximates the probability binomial when q is small. It would therefore seem that this curve would be most useful in graduating frequencies arising from events with low probabilities, the kind we have most need to deal with.

Since this curve can be reduced to the form

$$y = \psi(x) f(x) = \psi(x)[b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots]$$

it would again seem that the size of the coefficients b_1, b_2, b_3, etc., relative to b_0 are dependent upon and may serve as a measure of the perturbation of the basic probability in this type of cases.

It would appear these relationships should be of value to us in researches as to stability of statistical series and the nature of disturbing forces which affect their value for rate making. I hope to find an opportunity in the near future to look into this further and present some tests along these lines to the Society.