A New Criterion of Adequacy of Exposure.

BY

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Except in a few unusual lines of industry compensation insurance rates must for a long time in the future be based upon a minimum of data supplemented by expert judgment. This must be so for at the present time there are some 1,400 classifications in the workmen's compensation rate manual. We may look forward to changes from time to time in the phraseology of the classifications and the elimination of some existing classifications. I doubt if we can look forward to any material reduction in the total number. We must allow a certain time to elapse for the collection of our statistical data, yet so rapid is the flux and change of conditions we must measure and deal with, that we can not give much credence for future rate-making to data of even a very few years back.

This combination of conditions requires that we make the best possible use of every scrap of data we have, skilfully combining it with others as we find ourselves justified, and modifying its indications for rate-making where our judgment points out the need of so doing. Under such circumstances grouping of classifications for rate-making purposes and judgment of modification of experience indications before acceptance seem inevitable. Any extension of a group, however, to include data from an additional classification, even though pure premiums be made separately for the several parts of the hazard, breaks down to that extent the homogeneity of the group and, therefore, the applicability of its indication for all the classifications in the group. Hence the greatest skill and care is necessary in such work.

Generally speaking, the grouping of classifications is to give an increased spread of experience, though it may be in certain instances for the purpose of determining rates for some of the less important classifications by linking them up with a more important one in which there has been a large exposure.

Judgment is resorted to to correct apparent aberrations. These may be of two kinds,

1. Those due to a volume of statistical data insufficient to overcome the influence of pure chance.

2. Those due to persistent disturbing factors, such as the presence of a particular risk or risks of large size, which are distinctly better or worse than the true class type or average, a change in general industrial conditions, etc.

The proper correction of the latter type of cases is further analysis of the data so as to present homogeneous material and/or its modification to measure and allow for the differences in conditions of the past and future so far as known. This further analysis may so reduce the data as to introduce aberrations of the first type which may or may not be recognized as such.

It would seem there could be a substantial reduction in the need for judgment modification of experience indications if the hazards covered by the premium be segregated and separately measured. The big variations in indication arise from the hazards of low probability but high cost, such as death and permanent disability both total and partial. It may well be that we are fully justified in a much wider grouping basis for these elements than for those of high probability and low cost, such as medical expense. But even here we do not wish to extend our groupings too widely. There are many reasons why we may wish to confine them as closely as may be and give us a sufficient spread to give reasonably dependable indications. We need then a satisfactory criterion of exposure necessary to give such indications.

Back of this, of course, we must say what is a dependable indication. It would appear that this definition must be expressed in the form of a probability that the indication is within a certain fixed percentage of the true value. The definition in this form involves two free constants.

At the organization meeting of this Society, the writer presented a tentative solution which will be found in the *Proceedings,* Vol. I, p. 24. This solution was based upon two assumptions:

1. That the probability of hazard remained constant (or approximately constant) throughout the period observed so as to give a Bernoullian distribution of occurrences, and,
2. That over the critical region the normal frequency curve (Gauss's error curve) fitted the Bernoullian dispersion with a sufficiently close approximation to permit its use in place thereof.

Neither of these assumptions are fully realized in actual practise. But after due allowance and correction has been made for disturbing factors (such as for "increasing cost," "industrial activity," and other items considered at the last rate conference) the first may perhaps be taken as approximately true. Indeed, its approximate truth when so corrected seems to be a fundamental requirement of prospective rate-making. If this is so then we may properly approach the problem of the dependability of our data by first examining the conditions under which it was produced and making correction for general disturbing influences, and then determining from appropriate criteria its probable freedom for error due to lack of spread. Recently I have found a way to attack this latter problem without resorting to the second and more questionable assumption above cited. The method rests upon a theorem of Tchebycheff cited by Arne Fisher in his "Mathematical Theory of Probabilities," Vol. I, p. 108.

After pointing out the practical uselessness of the much discussed most probable value of a series of trials of an experiment with a given probability (p), Mr. Fisher shows that the expected value in a series of, say s, trials under a constant probability (i. e., in a Bernoullian series) is sp. Using the notation that $e(x)$ is the expected value in a series of trials of an event whose probability is $\phi(x)$ we may define the mean error of the series (x) by the equation

$$\epsilon^2(x) = \Sigma \ | x - e(x) |^2 \phi(x).$$

Tchebycheff's theorem then is:

"The probability that the absolute value of the difference $| x - e(x) |$ does not exceed the mean error by a certain multiplier λ $(\lambda > 1)$ is greater than $1 - (1/\lambda^2)$."

If we express this probability as P_T we have

$$P_T > 1 - \frac{1}{\lambda^2}. \qquad (1)$$

I will not reproduce here the proof of this theorem which will be found in Mr. Fisher's book as already referred to.

It can be readily shown that where the probability of x is p and the number of trials n then

$$\epsilon(x) = \sqrt{npq}$$

and if the relative frequency x/n is under consideration

$$\epsilon(x:n) = \sqrt{\frac{pq}{n}} \tag{2}$$

If we now choose a constant $k < 1$ such that $\lambda\epsilon = kp$; P_T becomes the probability that $|x - e(x)| \not> kp$, that is, the departure of the indicated probability from the true probability does not exceed a given percentage of the true probability. We are concerned, of course, with the relative rather than the absolute value of x.

Substituting in $\lambda\epsilon = kp$ the value of ϵ in (2) and squaring we get

$$\frac{\lambda^2 pq}{n} = k^2 p^2,$$

from which

$$n = \frac{\lambda^2}{k^2} \cdot \frac{q}{p} \tag{3}$$

In this equation and (1) above we have set up the conditions which enable us to determine the number of trials necessary to give a probability indication of which we may say the probability exceeds a certain value that it is not more than a given percentage from the true value. It will be noted that n is fixed by three parameters, λ determined from the value taken for P_T, k the permissible percentage of error, and p the true probability. The latter is the unknown in our work, but the experience indication may generally be used as an approximation thereto.

The application of this theorem may be rapidly made to a wide range of conditions by the use of two simply constructed tables. Entering the first with P_T and k, which are the judgment constants defining the dependability of an indication, we take out λ^2/k^2. With this and the probability of the event we enter the second table and read off directly n the number of observations required.

Appendix I gives the first table for a limited range of values of P_T and k and Appendix II gives the second table for a limited range of values of λ^2/k^2 and q. It may be here noted that following the analogy of life-insurance work where p is taken as the probability of survival I have taken p to be the escape from accidental injury during the term and q the incurring of injury. As q is what we are interested in these tables are in terms of q.

A comparison of (3) with equation (7), *Proceedings,* Vol. I, p.

26, shows that they are of the same form, the simpler λ taking the place of $x=khnq$. The general conclusions noted on page 27, therefore, still stand, but since the tabular solution of the problem is so simple and the construction of the tables also so easy, it is not necessary to give much consideration to such general statements of tendencies.

The tables may also be used in a different way, which may prove of greater practical value, taking the problem from a slightly different point of view. Given a certain exposure and hazard indication we may wish to know the probability that this is within a given percentage of the true value. Entering the second table with the indicated value of q and the known value of n we may read of the value of λ^2/k^2 interpolating if the table is not sufficiently extensive. Using this and the percentage of accuracy whose probability is desired, we may read off from Table I the value of P_T the probability sought.

The second use may prove of particular value in connection with the making of premiums to cover complex benefits for widely varying hazards. Obviously the direct application of the theory can deal only with a simple hazard, but we may use the tables in the second way to judge the accuracy of the experience indication for the several kinds of benefits and make such correction as appears necessary in the total result, bearing in mind that deviation in excess in one part may be offset by deviations in deficiency for the other part while our theory deals only with absolute values of departures.

Throughout the above we have used the theory of probability without discussing its basic definitions. In general it has been assumed necessary in compensation work to express probabilities in terms of annual full-time workers, and then convert that result into terms of payroll. I do not believe this is necessary.

The fundamental definition of probability of an event is the ratio of conditions favorable to the occurrence to the totality of equally likely conditions governing the occurrence.

In general we look upon this as limited to occurrence in the physical world, e. g., the drawing of a ball from an urn, the falling of a die with a particular face up, the death of a person, etc. If the fundamental conditions of the definition are adhered to there appears to be no reason why there should be strict limitation to physical events—why it is not rational to substitute some other measure associated with the event for the event itself. For example, there seems to be no reason for concluding that the life companies are not justified in investigating mortality experience and constructing a table therefrom on the basis of policies or amounts of insurance rather than lives. Having regard to practical business conditions there are marked advantages in so doing.

The same principle would apply to compensation insurance, taking, for example, death cases. Unless the terms of the compensation act and variation in marital conditions of injured persons make it such that sharper distinctions should be recognized, there appears to be no impropriety in taking the amount of death claims rather than the number of deaths as the numerator of the probability fraction. Again, the denominator which expresses the total possibility has usually been taken as the number of persons exposed during the year from which the deaths are presumed to arise. There is nothing sacred about the year as the unit for the probability and we might equally well express our probabilities in terms of weeks or months or some other unit, as for example, unit of payroll exposure, and we might use as our probability a probability of death or a fixed monetary loss within the term during which a given amount of payroll would be expended. It would thus seem that if we deal only with occurrences having approximately the same probability and cost we might be justified in treating the pure premiums for that element of benefit as its probability, or in erecting a probability which would compare the number of units of death loss, for example, with the corresponding number of similar units of payroll exposure. The probability might then be expressed as the probability that 100 per cent. of the payroll unit would be required for compensation for fatal accidents arising out of the expenditure of that unit.

It would seem that the fundamental requirement of the definition of equal likelihood of each condition entering into the denominator would be violated if we did not confine ourselves in this work to hazards of a like nature and of approximately like freqency of occurrence.

The advantage of this treatment, if it is logical, as I believe it is, is that it avoids the necessity of arbitrary assumption or statistical investigations as to average wages by which we may pass from a probability expressed in terms of the individual into a rate based upon wages or payroll. We only require to determine the average compensation cost per occurrence of the type under consideration.

The theories presented above will probably be made more clear by a few illustrative examples.

Let us consider the medical cost in a low-rated group where the pure premium for medical and hospital service is about 5 cents or 6 cents per \$100 of payroll. Let us take the average medical bill at say \$10. A pure premium of 5 cents or 6 cents per \$100 means a probability of .0005 or .0006 that a unit of payroll exposure is required for medical cost. We may wish to know how large a volume of data will be required in order that we may say of the indicated value of the medical pure premium there is less than one chance in ten that there is an error in it exceeding 10 per cent. of itself. Here P_T is .9 and k is .10. Using Table I we find λ^2/k^2 is 1,000. Then from Table II we find the number of units required if q is .0005 is 1,999,000 and if q is .0006 is 1,656,667. Since the unit is \$10 this means the payroll exposure required is between \$16,000,000 and \$20,000,000.

Of course, this standard of accuracy is very high. Were we content with a probability somewhat more than eight in 10 that the error did not exceed 10 per cent., we would find the value of λ^2/k^2 is 500, and from Table II that about half the payroll exposure indicated above would be required. For a probability of more than nine in ten that the error does not exceed 20 per cent. λ^2/k^2 equals 250, whence we require about 500,000 units or \$5,000,000 of payroll exposure. And if we are content with a probability of eight in ten that the error does not exceed 20 per cent., the value of λ^2/k^2 becomes 125 and we require but 250,000 units or \$2,500,000 of payroll exposure.

As another illustration we may take a death benefit which is assumed to cost on the average \$3,000. With the pure premium in the same neighborhood, and with the same standards of accuracy we would have the same number of units required, but the unit would here be \$3,000 or 300 times as much as in the other case, so that for a probability greater than 90 per cent. that the error doesn't exceed 10 per cent. we would require an exposure of \$6,000,000,000. For the case of a probability of more than 80 per cent. that the error doesn't exceed 20 per cent. we would require an exposure of \$150,000,000.

Let us take as a further example of the theory temporary disability where the pure premium is about 20 cents per \$100 of payroll and the average cost is say \$50. Here the probability of total loss of a unit of exposure is .20/100 or .002, and the unit is $50. Let us take as our standard that the probability shall exceed three in four (75 per cent.) that the error in the indicated pure premium does not exceed 15 per cent. Here P_T is .75 and k is .15 from which by Table I $\lambda^2/k^2 = 177$ and using this and $q = .002$ we get by Table II that n is about 8,500 units. Since the unit is $50 this means we must have $425,000 of payroll exposure to give us a pure premium indication of the desired dependability.

We may illustrate the second application of these theories by using the following data from the returns of all companies on Massachusetts Schedule Z, 1916, Part II, courteously furnished by Mr. E. S. Cogswell of the Massachusetts Insurance Department.

Classification.	Actual 2,660.	Pure Prem. per $100.	2,222.	Pure Prem. per $100.	2 286.	Pure Prem. per $100.
Total pay roll....	$78,943,253		$65,343,542		$39,593,977	
Total incurred losses for death.	12,861	.0163	26,412	.0404	7,091	.0179
Specific indemnity	14,596	.0185	13,721	.0210	9,642	.0244
Perm. total......	3,538	.0045	10,048	.0154	4,000	.0101
Perm. partial....	7,646	.0097	8,874	.0136	6,526	.0165
Temporary......	67,924	.0860	166,543	.2549	60,829	.1536
Medical hospital .	45,961	.0582	53,878	.0825	32,094	.0811

Taking the permanent partial element of classification 2,660, for example, we may fairly assume $1,000 as the average cost per case. We then have 7.6 occurrences out of 78,943 exposures giving a probability of .000097 or approximately .0001. We have to ex-

TABLE I.

VALUES OF "λ^2/k^2."

	k equals				
P_T.	.05.	.10.	.15.	.20.	.25.
.95.....	8,000	2,000	888	500	320
.90.....	4,000	1,000	444	250	160
.85.....	2,666	666	296	166	107
.80.....	2,000	500	222	125	80
.75.....	1,600	400	177	100	64
.70.....	1,333	333	148	83	53
.65.....	1,143	286	127	71	46
.60.....	1,000	250	111	62.5	40
.55.....	888	222	98.8	55.6	35

trapolate on line 10 of Table II to get the value of λ^2/k^2 which we can see will be very small, viz., about 8. Again this value of λ^2/k^2 is not found in Table I but by using the formula we can see that the probability does not greatly exceed $\frac{1}{2}$ that the indication is within 50 per cent. of the true indication. This seems to be in accord with what might be our subjective judgment on the problem.

If we assume the average temporary case to cost about $25 for temporary disability compensation, we have 2,717 occurrences in classification 2,660 out of 3,157,730 exposures. Our q is .0009 nearly. Table II shows that λ^2/k^2 is slightly more than 3,000. (This value is not shown in the table but is easily seen by noting the value of p/q and dividing 3,157,730 by it.) Table I showed a

TABLE II.

VALUES OF "n."

		λ^2/k^2 equals				
q.	p/q.	λ^2/k^2 1,000.	900.	800.	700.	600.
.00001	99,999	99,999,000	89,999,100	79,999,200	69,999,300	59,999,400
.00002	49,999	49,999,000	44,999,100	39,999,200	34,999,300	29,999,400
.00003	33,332	33,332,333	29,999,100	26,665,866	23,332,633	19,999,400
.00004	24,999	24,999,000	22,499,100	19,999,200	17,499,300	14,999,400
.00005	19,999	19,999,000	17,999,100	15,999,200	13,999,300	11,999,400
.00006	16,665.6$\dot{6}$	16,665,667	14,999,100	13,332,533	11,665,967	9,999,400
.00007	14,285	14,284,714	12,856,243	11,427,771	9,999,300	8,570,828
.00008	12,499	12,499,000	11,249,100	9,999,200	8,749,300	7,499,400
.00009	11,110.1	11,110,111	9,999,100	8,888,089	7,777,078	6,666,067
.0001	9,999	9,999,000	8,999,100	7,999,200	6,999,300	5,999,400
.0005	1,999	1,999,000	1,799,100	1,599,200	1,399,300	1,199,400
.0010	999	999,000	899,100	799,200	699,300	599,400
.0015	665.$\dot{6}$	665,667	599,100	532,534	465,967	399,400
.0020	499	499,000	449,100	399,200	349,300	299,400
.0025	399	399,000	359,100	319,200	279,300	239,400
.003	332.$\dot{3}$	332,333	299,100	265,867	232,633	199,400
.004	249	249,000	224,100	199,200	174,300	149,400
.005	199	199,000	179,100	159,200	139,300	119,400
.006	165.$\dot{6}$	165,667	149,100	132,533	115,967	99,400
.007	141.857	141,860–	127,671	113,486	99,300	85,114
.008	124	124,000	111,600	99,200	86,800	74,400
.009	110.11	110,111	99,100	88,089	77,078	66,067
.01	99	99,000	89,100	79,200	69,300	59,400
.02	49	49,000	44,100	39,200	34,300	29,400
.03	32.3	32,333	29,100	25,867	22,633	19,400
.04	24	24,000	21,600	19,200	16,800	14,400
.05	19	19,000	17,100	15,200	13,300	11,400
.06	15.6	15,667	14,100	12,533	10,967	9,400
.07	13.28	13,280	11,957	10,629	9,300	7,971
.08	11.5	11,500	10,350	9,200	8,050	6,900
.09	10.1	10,111	9,100	8,087	7,078	6,067
.10	9.0	9,000	8,100	7,200	6,300	5,400

q.	p/q.	λ^2/k^2 500.	400.	300.	200.	100.
.00001	99,999	49,999,500	39,999,600	29,999,700	19,999,800	9,999,900
.00002	49,999	24,999,500	19,999,600	14,999,700	9,999,800	4,999,900
.00003	33,332	16,666,167	13,332,933	9,999,700	6,666,467	3,333,233
.00004	24,999	12,499,500	9,999,600	7,499,700	4,999,800	2,499,900
.00005	19,999	9,999,500	7,999,600	5,999,700	3,999,800	1,999,900
.00006	$16.665.6\dot{6}$	8,332,833	6,666,267	4,999,700	3,333,133	1,666,567
.00007	14,285	7,142,357	5,713,886	4,285,414	2,856,943	1,428,471
.00008	12,499	6,249,500	4,999,600	3,749,700	2,499,800	1,249,900
.00009	11,110.1	5,555,056	4,444,044	3,333,033	2,222,022	1,111,011
.0001	9,999	4,999,500	3,999,600	2,999,700	1,999,800	999,900
.0005	1,999	999,500	799,600	599,700	399,800	199,900
.001	999	499,500	399,600	299,700	199,800	99,900
.0015	$665.\dot{6}$	332,833	266,267	199,700	133,133	66,567
.002	499	249,500	199,600	149,700	99,800	49,900
.0025	399	199,500	159,600	119,700	79,800	39,900
.003	$332.\dot{3}$	166,167	132,933	99,700	66,467	33,233
.004	249	124,500	99,600	74,700	49,800	24,900
.005	199	99,500	79,600	59,700	39,800	19,900
.006	$165.\dot{6}$	82,833	66,267	49,700	33,133	16,567
.007	141.857	70,929	56,743	42,557	28,371	14,186
.008	124	62,000	49,600	37,200	24,800	12,400
.009	$110.\dot{1}$	55,056	44,044	33,033	22,022	11,011
.01	99	49,500	39,600	29,700	19,800	9,900
.02	49	24,500	19,600	14,700	9,800	4,900
.03	$32.\dot{3}$	16,167	12,933	9,700	6,467	3,233
.04	24	12,000	9,600	7,200	4,800	2,400
.05	19	9,500	7,600	5,700	3,800	1,900
.06	$15.\dot{6}$	7,833	6,267	4,700	3,133	1,567
.07	13.286	6,643	5,314	3,986	2,657	1,329
.08	11.5	5,750	4,600	3,450	2,300	1,150
.09	10.1	5,056	4,044	3,033	2,022	1,011
.10	9	4,500	3,600	2,700	1,800	900

very high probability that the indication is not in error 5 per cent. on account of the influence of chance.

The medical will be found to have about as high accuracy.

Taking the specific indemnity element and remembering the Massachusetts provision we may perhaps take \$200 as a fair round figure per case. This gives us approximately 400,000 exposures with 73 occurrences. $q = .0002$ nearly. Table II shows λ^2/k^2 is approximately 100. And from Table I we may say the probability exceeds 75 per cent. there is not a 20 per cent. error in the result, that it exceeds 85 per cent. there is not a 25 per cent. error, but it only exceeds about 57 per cent. that there is not an error of 15 per cent.

Other examples may be worked out at will. I believe a careful use of this criterion will be of great value in analyzing data to be used in rate-making. A little practice with the Tables will develop great rapidity in their use. The construction of the Tables is very simple and they may, therefore, easily be extended to cover a wider range or proceed by narrower graduations.

Although the above theorem has been presented primarily with a view to its application to workmen's compensation rate-making, there seems to be no reason why it may not be equally applicable to similar problems in personal accident insurance, fire insurance and elsewhere, and indeed, I am not sure that it may not be possible to develop in this way methods by which the data at different ages might be appropriately weighted preliminary to the graduation of a mortality table.