Valuation of the Death Benefits Provided by the New York Compensation Law.

WINFIELD W. GREENE.

Death Benefits Provided by the New York Compensation Law.

The death benefits of the various compensation laws effective in the United States may be classified as follows:

I. Benefits limited as to the sum total of compensation payments.

II. Pensions limited in duration to a stipulated period.

III. Pensions ceasing at the death of the beneficiary,—and, in case of certain classes of pensioners, terminating also either when the beneficiary attains a certain age, or when he remarries.

The benefits provided by the New York law for dependents of workmen fatally injured in the course of their employment belong in the third class, as is indicated by the following summary of Sec. 16:

(*a*) To surviving wife (or dependent husband) 30 per cent. of deceased's wages until death; or until remarriage, when the pension is terminated by a payment equal to 60 per cent. of the annual earnings of the deceased. To each child, 15 per cent. of the deceased's wages until age 18, except that during the lifetime of the surviving wife (or dependent husband) the child's pension is 10 per cent. instead of 15.

(*b*) To any of the following relatives of the deceased,—grandchild, brother, sister, parent or grandparent,—15 per cent. of deceased's wages. The pensions to the grandchildren, brothers and sisters are payable until age 18, while the pensions to parents and grandparents are payable for life.

(*c*) The total amount of compensation payable per annum is not to exceed 66⅔ per cent. of the wages of the deceased. The wife and children of the deceased are a preferred class of claimants; their pensions are apportioned first, and any balance remaining is divided among the other dependents.

At least three other states in the Union have compensation acts with death benefits similar to those of the New York law. The

West Virginia law provides \$20 per month to the widow until her death or remarriage, with \$5 per month additional for each child under age 14, total not to exceed \$35 per month; the Washington law has the same death benefits as the West Virginia law, except that the children's pensions continue until age sixteen; while the Oregon law differs only in fixing the widow's pension at \$30 per month, with \$6 additional for each child under sixteen, total not to exceed \$50 per month.

Although there are only a few statutes of what we may term "the life pension type" in this country, the majority of compensation laws effective on the continent of Europe follow that pattern. In fact, a solution of the problems of the New York Compensation Law will go far to solve many questions of actuarial principle involved in the compensation acts of Austria, Belgium, Finland, France, Germany, Hungary, Holland, Norway, Portugal, Roumania, Russia, Sweden and Switzerland, and, as we have already suggested, of West Virginia, Oregon and Washington as well.

Formulae for Present Value of Death Benefits in Simple Cases.

In order to calculate the present value of compensation payable to beneficiaries of the New York law in fatal cases, we must first equip ourselves with appropriate and practicable formulae. In many instances this is not a difficult task.

Assuming, for convenience, that the average annual earnings of the deceased were 100, the present value of compensation where a widow (x) is the only dependent may be written as follows:

$$(1) \qquad 30\bar{a}_{x'} + 60\bar{E}_{x''},$$

where $\bar{a}_{x'}$ indicates an annuity of 1 per annum, payable momently, and terminating at death or remarriage; and where $\bar{E}_{x''}$ indicates the present value of 1 payable at the moment of remarriage.

Where the only dependent is a child, grandchild, brother or sister (y) the present value equals

$$(2) \qquad 15\bar{a}_{y\,\overline{18-y}|}.$$

Where a father, mother, grandfather or grandmother (w) is the only person entitled to compensation the appropriate expression is

$$(3) \qquad 15\bar{a}_{w}.$$

Where there are several dependents receiving in all less than $66\frac{2}{3}$ per cent. of the deceased's wages, the total present value of compensation will be the sum of several expressions like the foregoing, corresponding to the particular dependents involved.

Where a widow (x) and child (y) of the deceased are both entitled to compensation, the expression for the child's interest becomes

(4) $$15\bar{a}_{y\,\overline{18-y}|} - 5\bar{a}_{xy\,\overline{18-y}|}$$

instead of merely

$$15\bar{a}_{y\,\overline{18-y}|}, \text{ as in (2).}$$

The introduction of this negative term is due to the fact that during the lifetime of the widow the child receives 10 per cent. of the deceased's wages, instead of 15.

The following is a simple illustration of the foregoing principles:

Example.

$$\text{Dependents} \begin{cases} \text{Widow, age 35,} \\ \text{Child, age 10,} \\ \text{Child, age 2,} \\ \text{Mother, age 65.} \end{cases}$$

Formula:

(5)
$$\begin{aligned} &30\bar{a}_{35'} + 60\bar{E}''_{35} \\ &+ 15\bar{a}_{2\,\overline{16}|} - 5\bar{a}_{35:2\,\overline{16}|} \\ &+ 15\bar{a}_{10\,\overline{8}|} - 5\bar{a}_{35:10\,\overline{8}|} \\ &+ 15\bar{a}_{65}. \end{aligned}$$

In the above example it has not been necessary to take account of the provision in the law (Sec. 16) that "the total amount payable shall in no case exceed $66\frac{2}{3}$ per centum of such wages," i. e., of the "average wages of the deceased." *Had there been, in addition to the dependents assumed in the example, another dependent entitled to 10 or 15 per cent. of the deceased's wages, it is clear that the limitation imposed by the law would affect the aggregate present value of the benefits.* Assuming this additional dependent to be a father 70 years of age and disregarding the limitation, the present value in the latter case would be $15\bar{a}_{70}$ plus the expression already obtained, or

$$(6)\qquad \begin{aligned} &30\bar{a}_{35'} + 60\bar{E}_{35''} \\ &+ 15\bar{a}_{2\,\overline{16}|} - 5\bar{a}_{35:2\,\overline{16}|} \\ &+ 15\bar{a}_{10\,\overline{8}|} - 5\bar{a}_{35:10\,\overline{8}|} \\ &+ 15\bar{a}_{65} + 15\bar{a}_{70}. \end{aligned}$$

We will now derive a mathematical expression for the effect of the limitation in this particular case by determining to what extent the above formula conflicts with the maximum of 66 2/3 per cent. stipulated by the law.

In the first place, the above formula assumes that while all of the dependents remain entitled to compensation, an aggregate of 80 per cent. of the deceased's wages will be payable,—30 per cent. to the widow, 10 per cent. to each of two children, and 15 per cent. to each of two dependent parents. The limitation of $66\frac{2}{3}$ per cent. will accordingly effect a deduction which may be written as follows:

$$(7)\qquad (80 - 66\tfrac{2}{3})\bar{a}_{35':2\,\overline{16}|:10\,\overline{8}|:65:70} = 13\tfrac{1}{3}\bar{a}_{35':2:10:65:70\,\overline{8}|},$$

where the accent over the "35" indicates that with respect to this life (that of the widow) probabilities of surviving unmarried,—not of survival alone,—are taken into account.

The foregoing formula assumes further that after one of the children ceases to receive compensation, as long as all of the other dependents continue entitled to their pensions, compensation aggregating 70 per cent. will be payable. The limit of $66\frac{2}{3}$ per cent. will accordingly introduce a second deduction equal to

$$(8)\qquad \begin{aligned} &(70 - 66\tfrac{2}{3})[\bar{a}_{35':2\,\overline{16}|:65:70} - \bar{a}_{35':2\,\overline{16}|:10\,\overline{8}|:65:70}] \\ &\qquad + (70 - 66\tfrac{2}{3})[\bar{a}_{35':10\,\overline{8}|:65:70} - \bar{a}_{35':2\,\overline{16}|:10\,\overline{8}|:65:70}] \\ &\qquad = 3\tfrac{1}{3}[\bar{a}_{35':2:65:70\,\overline{16}|} + \bar{a}_{35':10:65:70\,\overline{8}|} - 2\bar{a}_{35':2:10:65:70\,\overline{8}|}]. \end{aligned}$$

As we have discussed all of the conditions in which the assumptions of our original formula conflict with the maximum stipulated in the law, *we may now write a new formula which makes due allowance for the maximum by taking the difference between (6) and the algebraic sum of (7) and (8), as follows:*

$$
(9) \qquad \begin{aligned}
& 30\bar{a}_{35'} + 60\overline{E}_{35''} \\
& + 15\bar{a}_{2\,\overline{16}|} - 5\bar{a}_{35:2\,\overline{16}|} \\
& + 15\bar{a}_{10\,\overline{8}|} - 5\bar{a}_{35:10\,\overline{8}|} \\
& + 15\bar{a}_{65} + 15\bar{a}_{70} \\
& - 3\tfrac{1}{3}[\bar{a}_{35':2:65:70\,\overline{16}|} + \bar{a}_{35':10:65:70\,\overline{8}|}] \\
& - 6\tfrac{2}{3}\bar{a}_{35':2:10:65:70\,\overline{8}|}.
\end{aligned}
$$

The particularized method just employed cannot easily be extended to cases involving many dependents. In the first place, without the aid of a general rule it is extremely difficult in complicated cases to enumerate all the situations where the $66\frac{2}{3}$ per cent. limitation will affect the compensation payable. Moreover, since the number of terms in the algebraic expression corresponding to the deduction on account of the limit increases so rapidly with the number of dependents involved, that our formula soon becomes unwieldy and impracticable, it seems advisable to investigate some more general method.

A Mathematical Expression for the Deduction on Account of the $66\frac{2}{3}$ Per Cent. Limit as to Compensation.

Where the Beneficiaries are a Widow and any Number of Children Whatsoever.

Let us assume that the victim of an industrial accident is survived by a widow (x), and children (y_1), (y_2), (y_3), $\cdots$, (y_n), where (y_1) is the youngest, (y_2) is next to the youngest, and so on. We shall first investigate the influence of the limit

While the wife survives, and has not remarried. The wife is entitled to 30 per cent., each child to 10 per cent., of the deceased's wages. While the widow and only one, two, or three children receive compensation, the amount payable will not be affected by the limitation, as this amount is always less than the maximum permitted by the law.

While the widow and at least four children receive compensation, the pensions payable, if there were no prescribed limit, would equal or exceed 70 per cent. of the deceased's wages (30 per cent. to the widow, and at least 40 per cent. to the children). Consequently, during this status the limitation of $66\frac{2}{3}$ per cent. will effect a deduction of at least $3\frac{1}{3}$ per cent. in the compensation payable. The deduction on account of the limit will therefore equal, or include, the following expression:

$$3\tfrac{1}{3}\bar{a}_{x'\,\overset{4}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}},$$

where the temporary annuity represented is to run while (x) survives unmarried, and while at least four of the n children survive and have not attained age 18.

While the widow and at least five children remain beneficiaries, the total compensation which would be payable if there were no limit is equal to or greater than 80 per cent. During this latter status the limit will therefore effect a deduction of at least $13\tfrac{1}{3}$ per cent. or 10 per cent. in addition to the $3\tfrac{1}{3}$ per cent. deduction which obtains while the wife and four children continue to receive compensation. This deduction may be expressed algebraically as follows:

$$10\bar{a}_{x'\,\overset{5}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}}.$$

By similar reasoning we may show that as long as a widow and six children are receiving compensation, there will obtain a further deduction of 10 per cent., which may be written

$$10\bar{a}_{x'\,\overset{6}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}}.$$

By use of the foregoing process we may obtain an expression in actuarial symbols for the total effect of the limit *while the widow remains entitled to compensation.* This expression may be extended indefinitely, but its significant terms will be limited by the actual number of children who are beneficiaries in any particular case. We may write this expression in the following perfectly general terms:

$$(10)\quad 3\tfrac{1}{3}\bar{a}_{x'\,\overset{4}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}} + 10\bar{a}_{x'\,\overset{5}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}}$$
$$+\,10\bar{a}_{x'\,\overset{6}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}}\cdots\cdots$$
$$\cdots\cdots+10\bar{a}_{x'\,\overset{n}{\overline{y_1\,\overline{18-y_1}|\,:\,y_2\,\overline{18-y_2}|\,\ldots\,y_n\,\overline{18-y_n}|}}}.$$

We shall next consider the influence of the limit

During the lifetime of the wife but after her remarriage. The children are now the only beneficiaries, receiving 10 per cent. each, as before. While at least seven children are receiving compensation, the limit will effect a deduction of at least $(7 \times 10$ per cent.

— 66⅔ per cent.) or 3⅓ per cent., which may be written

$$(11, a)\quad 3\tfrac{1}{3}\left[\bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,7}} - \bar{a}_{x'\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,7}}\right].$$

As before, the total deduction takes the form of a series. Each term involves the difference between two temporary annuities, one payable during the lifetime of the widow, and the other payable until her death or remarriage; in this way the probability that the widow survives after having remarried is taken into account.

The second term is

$$(11, b)\quad 10\left[\bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,8}} - \bar{a}_{x'\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,8}}\right].$$

And the final significant term is

$$(11, c)\quad 10\left[\bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,n}} - \bar{a}_{x'\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,n}}\right].$$

By a similar process we may write an expression for the effect of the limit

After the death of the wife. The children are now each entitled to 15 per cent. The compensation payable to five children will accordingly be affected by the limit to the extent of (5 × 15 per cent. — 66⅔ per cent.), or 8⅓ per cent. The formula for this deduction will be

$$\begin{aligned}(12)\quad & 8\tfrac{1}{3}\left[\bar{a}_{\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,5}} - \bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,5}}\right] \\ & + 15\left[\bar{a}_{\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,6}} - \bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,6}}\right] \\ & \quad \cdots\cdots\cdots\cdots\cdots\cdots \\ & + 15\left[\bar{a}_{\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,n}} - \bar{a}_{x\,\overline{y_1\,\overline{18-y_1|}:y_2\,\overline{18-y_2|}\dots y_n\,\overline{18-y_n|}}^{\,n}}\right].\end{aligned}$$

In the foregoing expression, the coefficient of the second and of each subsequent term is 15, instead of 10 as in the other two series, because after the widow's death the compensation which would be payable if there were no limit is increased 15 per cent. by each additional child. The probability that the widow no longer survives has to be accounted for by making each term of the above series involve the difference between two temporary annuities, the one conditional upon probabilities of the widow's survival and the other independent of such probabilities.

The Text Book of the Institute of Actuaries (Part II, page 133) illustrates a method which we may extend to temporary annuities and thereby express

THE FORMULA FOR THE TOTAL DEDUCTION IN A COMPACT FORM.

If Z^r signifies the sum of the values of the temporary annuities on r joint lives for all the combinations of r lives that can be made out of n lives,—where for our purposes each life ceases at age 18,—then

$$\bar{a}_{\overline{y_1\, \overline{18-y_1}| : y_2\, \overline{18-y_2}| \cdots y_n\, \overline{18-y_n}|}}^{\;r} = \frac{\underset{1}{Z^r}}{(1+\underset{1}{Z})^r},$$

where $\underset{1}{Z}$ relates to temporary annuities which depend upon the lives (y_1), (y_2) $\cdots$ (y_n) only. We may employ the symbol $\underset{2}{Z}$ where (x), and $\underset{3}{Z}$ where (x') is involved, so that (10) becomes

$$(13)\quad 3\tfrac{1}{3}\,\frac{\underset{3}{Z^4}}{(1+\underset{3}{Z})^4} + 10\,\frac{\underset{3}{Z^5}}{(1+\underset{3}{Z})^5} + 10\,\frac{\underset{3}{Z^6}}{(1+\underset{3}{Z})^6} \cdots + 10\,\frac{\underset{3}{Z^n}}{(1+\underset{3}{Z})^n},$$

(11, a), (11, b) and (11, c) are replaced by

$$(14)\quad 3\tfrac{1}{3}\left[\frac{\underset{2}{Z^7}}{(1+\underset{2}{Z})^7} - \frac{\underset{3}{Z^7}}{(1+\underset{3}{Z})^7}\right] + 10\left[\frac{\underset{2}{Z^8}}{(1+\underset{2}{Z})^8} - \frac{\underset{3}{Z^8}}{(1+\underset{3}{Z})^8}\right]$$

$$+\,10\left[\frac{\underset{2}{Z^9}}{(1+\underset{2}{Z})^9} - \frac{\underset{3}{Z^9}}{(1+\underset{3}{Z})^9}\right]\cdots\cdots$$

$$\cdots\cdots + 10\left[\frac{\underset{2}{Z^n}}{(1+\underset{2}{Z})^n} - \frac{\underset{3}{Z^n}}{(1+\underset{3}{Z})^n}\right],$$

and (13) becomes

$$(15)\quad 8\tfrac{1}{3}\left[\frac{\underset{1}{Z^5}}{(1+\underset{1}{Z})^5} - \frac{\underset{2}{Z^5}}{(1+\underset{2}{Z})^5}\right] + 15\left[\frac{\underset{1}{Z^6}}{(1+\underset{1}{Z})^6} - \frac{\underset{2}{Z^6}}{(1+\underset{2}{Z})^6}\right]$$

$$+\,15\left[\frac{\underset{1}{Z^7}}{(1+\underset{1}{Z})^7} - \frac{\underset{2}{Z^7}}{(1+\underset{2}{Z})^7}\right]\cdots\cdots$$

$$\cdots\cdots + 15\left[\frac{\underset{1}{Z^n}}{(1+\underset{1}{Z})^n} - \frac{\underset{2}{Z^n}}{(1+\underset{2}{Z})^n}\right].$$

By taking the sum of (13), (14) and (15), we may, after cancellation, write the following

Formula for the aggregate deduction on account of the 66 2/3 per cent. limit in compensation, where the only dependents are a wife and n children:

$$(16)\quad 3\tfrac{1}{3}\frac{Z_3^4}{(1+Z_3)^4}+10\frac{Z_3^5}{(1+Z_3)^5}+10\frac{Z_3^6}{(1+Z_3)^6}+6\tfrac{2}{3}\frac{Z_3^7}{(1+Z_3)^7}$$

$$-8\tfrac{1}{3}\frac{Z_2^5}{(1+Z_2)^5}-15\frac{Z_2^6}{(1+Z_2)^6}-11\tfrac{2}{3}\frac{Z_2^7}{(1+Z_2)^7}$$

$$-5\left[\frac{Z_2^8}{(1+Z_2)^8}+\frac{Z_2^9}{(1+Z_2)^9}\cdots+\frac{Z_2^n}{(1+Z_2)^n}\right]$$

$$+8\tfrac{1}{3}\frac{Z_1^5}{(1+Z_1)^5}+15\left[\frac{Z_1^6}{(1+Z_1)^6}+\frac{Z_1^7}{(1+Z_1)^7}\cdots\right.$$

$$\left.\cdots+\frac{Z_1^n}{(1+Z_1)^n}\right].$$

It is very easy to pass from the foregoing formula to an expression involving joint temporary annuities ceasing at first death, —or ceasing either at first death or at the remarriage of one of the annuitants. As temporary annuities of this sort can readily be calculated, it is theoretically possible to determine the numerical value of the deduction by the use of the above formula, after substituting for the powers of

$$\frac{Z_1}{(1+Z_1)},\quad \frac{Z_2}{(1+Z_2)}\quad \text{and}\quad \frac{Z_3}{(1+Z_3)}$$

their linear expansions.

To illustrate the practical obstacles in the way of the method just suggested, let us assume that the present value of the deduction on account of the $66\frac{2}{3}$ per cent. limit has to be computed where the dependents are a widow, and nine children less than 18 years of age. The following table shows the powers of Z_1, Z_2 and Z_3 which would have to be evaluated, and the number of joint temporary annuities which enter into the value of each of these powers, according to formula (16).

TABLE A.

EXHIBIT OF THE NUMBER OF TEMPORARY ANNUITIES TO BE CALCULATED BY FORMULA (16).

Dependents—A Widow and Nine Children.

Powers of Z Appearing in Linear Expansion of Formula (16).	Number of Temporary Annuities to be Calculated.		Summary.
	Symbol.	Actual Number.	
$\underset{1}{Z^5}$	${}^9c_5$	126	Joint temporary annuities upon lives of children only 256
$\underset{1}{Z^6}$	${}^9c_6$	84	
$\underset{1}{Z^7}$	${}^9c_7$	36	
$\underset{1}{Z^8}$	${}^9c_8$	9	
$\underset{1}{Z^9}$	${}^9c_9$	1	
$\underset{2}{Z^5}$	${}^9c_5$	126	Joint temporary annuities upon lives of children *during lifetime* of mother. 256
$\underset{2}{Z^6}$	${}^9c_6$	84	
$\underset{2}{Z^7}$	${}^9c_7$	36	
$\underset{2}{Z^8}$	${}^9c_8$	9	
$\underset{2}{Z^9}$	${}^9c_9$	1	
$\underset{3}{Z^4}$	${}^9c_4$	126	Joint temporary annuities upon lives of children *during widowhood* of mother. 382
$\underset{3}{Z^5}$	${}^9c_5$	126	
$\underset{3}{Z^6}$	${}^9c_6$	84	
$\underset{3}{Z^7}$	${}^9c_7$	36	
$\underset{3}{Z^8}$	${}^9c_8$	9	
$\underset{3}{Z^9}$	${}^9c_9$	1	
Total number of temporary annuities to be calculated. . .			894

The above example suggests that in many instances it will be wholly impracticable to employ formula (16) as long as $\underset{1}{Z^r}$, $\underset{2}{Z^r}$ and $\underset{3}{Z^r}$ retain their original meaning. This remains true although the the number of temporary unities to be calculated drops rapidly with the number of children entitled to compensation. For instance, where the beneficiaries are a widow and seven children, formula (16) requires the calculation of 122 temporary annuities and, incidentally, the computation of equivalent equal ages for 122 different groups of lives. Accordingly it is desirable to investigate some method of reducing the number of equal ages and annuities to be calculated.

The Effect of Disregarding Differences in Age Among the Children, as Regards Mortality.

With respect to temporary annuities, as well as whole life annuities, $\underset{1}{Z^r}$ is the sum of the annuities for all the combinations of a given n lives taken r at a time. The n lives (y_1), (y_2), $\cdots$, (y_n) are respectively associated with the following terms, $\overline{18-y_1}|$, $\overline{18-y_2}|$, $\cdots$, $\overline{18-y_n}|$. If $n=9$, and $r=5$, one of the annuities which form $\underset{1}{Z^r}$ will be

$$\bar{a}_{y_1\,\overline{18-y_1}|\,:\,y_3\,\overline{18-y_3}|\,:\,y_4\,\overline{18-y_4}|\,:\,y_6\,\overline{18-y_6}|\,:\,y_7\,\overline{18-y_7}|}$$

which may be abbreviated to

$$\bar{a}_{y_1:y_3:y_4:y_6:y_7\,\overline{18-y_7}|}$$

since the shortest term is the only one having any practical significance. As the oldest life in any group of children entitled to compensation fixes the term for which the annuity is to run, it is clear that whenever r is less than n, $\underset{1}{Z^r}$ will include annuities for more than one term.

The longest term will be $\overline{18-y_r}|$, and there will be one annuity for this term, as there can be but one group of r lives in which (y_r) is the oldest.

There will be ${}^rc_{r-1}$ groups of r lives where y_{r+1} is the oldest, as, with y_{r+1} a fixture in the group, there remain $(r-1)$ places to fill from the r lives younger than (y_{r+1}). Thus there will be rc_{r-1} annuities for the term of $(18-y_{r+1})$ years.

Similarly, there will be ${}^{r+1}c_{r-1}$ annuities for the term of $(18-y_{r+2})$ years.

If for the time being we disregard the particular ages involved in each annuity, and consider only the term for which the annuity runs, we may combine all annuities involving the same number of lives, and the same term. Then

$$\begin{aligned} \underset{1}{Z^r} &= \underset{r}{\bar{a}}_{\overline{18-y_r}|} + {}^rc_{r-1}\cdot\underset{r}{\bar{a}}_{\overline{18-y_{r+1}}|} \\ &\qquad + {}^{r+1}c_{r-1}\cdot\underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|}\cdots + {}^{n-1}c_{r-1}\cdot\underset{r}{\bar{a}}_{\overline{18-y_n}|} \\ (17)\qquad &= \underset{r}{\bar{a}}_{\overline{18-y_r}|} + r\cdot\underset{r}{\bar{a}}_{\overline{18-y_{r+1}}|} + \frac{r\cdot\overline{r+1}}{2}\,\underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|} \\ &\qquad \cdots + \frac{\underline{|n-1}}{\underline{|r-1}\;\underline{|n-r}}\,\underset{r}{\bar{a}}_{\overline{18-y_n}|}, \end{aligned}$$

where the "r" under the "$\bar{a}$" indicates the number of lives (of children) upon which the annuity is based, and where the original value of $\underset{1}{Z^r}$ will be preserved if we define $\underset{r}{\bar{a}}\ \overline{18-y_s}|$ as the average of the temporary annuities for the term $(18-y_s)$ years, for all the groups consisting of r lives of which y_s is the oldest, which may be selected from the n lives (y_1), (y_2), $\cdots$, (y_n). Employing notation analogous to the above, we may write similar expressions for $\underset{2}{Z^r}$ and $\underset{3}{Z^r}$.

The number of annuities to be calculated to obtain the value of Z^r has now been reduced from ${}^nc_r$ to $(n-r+1)$. The deduction on account of the limit may now be obtained by computing 51 annuities in case of a widow and nine children, and 22 annuities in case of a widow and seven children.

The Deduction Expressed as the Sum of Several Convergent Series.

The symbol $\dfrac{\underset{1}{Z^r}}{(1+\underset{1}{Z})^r}$ stands for its expansion

$$\underset{1}{Z^r} - r\underset{1}{Z^{r+1}} + \frac{r\cdot\overline{r+1}}{2}\underset{1}{Z^{r+2}}\cdots.$$

Substituting for $\underset{1}{Z^r}$ its value as shown in (17), we obtain

$$(18)\quad \frac{\underset{1}{Z^r}}{(1+\underset{1}{Z})}r = \underset{r}{\bar{a}}_{\overline{18-y_r}|} + r(\underset{r}{\bar{a}}_{\overline{18-y_{r+1}}|} - \underset{r+1}{\bar{a}}_{\overline{18-y_{r+1}}|})$$
$$+\frac{r\cdot\overline{r+1}}{2}(\underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|} - 2\,\underset{r+1}{\bar{a}}_{\overline{18-y_{r+2}}|} + \underset{r+2}{\bar{a}}_{\overline{18-y_{r+2}}|}).$$

For our purposes we may consider $\underset{r}{\bar{a}}_{\overline{18-y_s}|}$ as a function of r, so that $(\underset{r+1}{\bar{a}}_{\overline{18-y_s}|} - \underset{r}{\bar{a}}_{\overline{18-y_s}|})$ may be written $\Delta\underset{r}{\bar{a}}_{\overline{18-y_s}|}$, and $(\underset{r+2}{\bar{a}}_{\overline{18-y_{r+2}}|} - 2\,\underset{r+1}{\bar{a}}_{\overline{18-y_{r+2}}|} + \underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|})$ may be written $\Delta^2\underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|}$ whereupon (18) becomes

$$(19)\quad \frac{\underset{1}{Z^r}}{(1+\underset{1}{Z})^r} = \underset{r}{\bar{a}}_{\overline{18-y_r}|} - r\,\Delta\underset{r}{\bar{a}}_{\overline{18-y_{r+1}}|} + \frac{r\cdot\overline{r+1}}{\underline{|2}}\Delta^2\underset{r}{\bar{a}}_{\overline{18-y_{r+2}}|}\cdots.$$

By interpreting $\underset{2}{Z^r}$ and $\underset{3}{Z^r}$ as we have $\underset{1}{Z^r}$, we may expand formula (16) as is shown in "Table B." The several expressions for $\frac{Z^r}{(1+Z)^r}$ have not been extended to include higher powers of Δ than Δ^2. In cases involving seven or more children, some of these omitted terms become significant, but each such term may easily be deduced from the law of the series of which it forms a part. Moreover, it seems highly probable that no serious error will be introduced if we disregard Δ^3 and higher differences altogether.

If the higher differences are taken into account, the only error introduced by the use of the formula shown in "Table B" arises from combining annuities for the same term, and upon the *same number of lives* (of children), regardless of the *different groups of ages* upon which the annuities so combined depend. Where we employ a mortality table which follows Makeham's law from age "0" on, this error may be kept within narrow limits. By computing the present value of the deduction on account of the limit twice, *first* basing each temporary annuity $\bar{a}_{\underset{r}{}\overline{18-y_s}|}$ upon the r equal ages corresponding to the r lives $y_s, y_{s-1}, y_{s-2}, \cdots, y_{s-r+1}$, and the *second* time upon the r equal ages corresponding to the r lives $y_1, y_2, y_3, \cdots, y_r$, we obtain two values,—the true value of the deduction lying between the two, and nearer to the first than to the second. The first method yields a smaller present value for the deduction, and consequently a larger net present value of compensation.

TABLE B.

Reduction in Present Value Because of $66\frac{2}{3}$ Per Cent. Limit in Compensation.

Where the Dependents Are a Widow and Several Children.

Subtract value obtained by use of significant terms of this formula from present value of compensation which would be payable if there were no $66\frac{2}{3}$ per cent. limitation.

$$3\tfrac{1}{3}\left[{}_4\bar{a}_{x'\,\overline{18-y_4}|} - 4\Delta\,{}_4\bar{a}_{x'\,\overline{18-y_5}|} + \frac{4.5}{\underline{|2}}\Delta^2\,{}_4\bar{a}_{x'\,\overline{18-y_6}|}\cdots\right]$$

$$+\,10\left[{}_5\bar{a}_{x'\,\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{x'\,\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\Delta^2\,{}_5\bar{a}_{x'\,\overline{18-y_7}|}\cdots\right]$$

$$+\,10\left[{}_6\bar{a}_{x'\,\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{x'\,\overline{18-y_7}|} + \frac{6.7}{\underline{|2}}\Delta^2\,{}_6\bar{a}_{x'\,\overline{18-y_8}|}\cdots\right]$$

$$+\,6\tfrac{2}{3}\left[{}_7\bar{a}_{x'\,\overline{18-y_7}|} - 7\Delta\,{}_7\bar{a}_{x'\,\overline{18-y_8}|} + \frac{7.8}{\underline{|2}}\Delta^2\,{}_7\bar{a}_{x'\,\overline{18-y_9}|}\cdots\right]$$

$$-\,8\tfrac{1}{3}\left[{}_5\bar{a}_{x\,\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{x\,\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\Delta^2\,{}_5\bar{a}_{x\,\overline{18-y_7}|}\cdots\right]$$

$$-\,15\left[{}_6\bar{a}_{x\,\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{x\,\overline{18-y_7}|} + \frac{6.7}{\underline{|2}}\Delta^2\,{}_6\bar{a}_{x\,\overline{18-y_8}|}\cdots\right]$$

$$-\,11\tfrac{2}{3}\left[{}_7\bar{a}_{x\,\overline{18-y_7}|} - 7\Delta\,{}_7\bar{a}_{x\,\overline{18-y_8}|} + \frac{7.8}{\underline{|2}}\Delta^2\,{}_7\bar{a}_{x\,\overline{18-y_9}|}\cdots\right]$$

$$-\,5\left[{}_8\bar{a}_{x\,\overline{18-y_8}|} - 8\Delta\,{}_8\bar{a}_{x\,\overline{18-y_9}|} + \frac{8.9}{\underline{|2}}\Delta^2\,{}_8\bar{a}_{x\,\overline{18-y_{10}}|}\cdots\right.$$

. . . .

$$\left.{}_n\bar{a}_{x\,\overline{18-y_n}|} - n\Delta\,{}_n\bar{a}_{x\,\overline{18-y_{n+1}}|} + \frac{n\cdot\overline{n+1}}{\underline{|2}}\Delta^2\,{}_n\bar{a}_{x\,\overline{18-y_{n+2}}|}\cdots\right]$$

$$+\,8\tfrac{1}{3}\left[{}_5\bar{a}_{\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\Delta^2\,{}_5\bar{a}_{\overline{18-y_7}|}\cdots\right]$$

$$+\,15\left[{}_6\bar{a}_{\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{\overline{18-y_7}|} + \frac{6.7}{\underline{|2}}\Delta^2\,{}_6\bar{a}_{\overline{18-y_8}|}\cdots\right.$$

. . . .

$$\left.{}_n\bar{a}_{\overline{18-y_n}|} - n\Delta\,{}_n\bar{a}_{\overline{18-y_{n+1}}|} + \frac{n\cdot\overline{n+1}}{\underline{|2}}\Delta^2\,{}_n\bar{a}_{\overline{18-y_{n+2}}|}\cdots\right]$$

A Workable Formula for the Reduction in Present Value Due to the $66\frac{2}{3}$ Per Cent. Limit as to Compensation.

For calculations involving more than seven children it will be found desirable to modify the formula of Table B somewhat. A study of this formula indicates that where the present value of the deduction has been computed, assuming as dependents a widow and seven children, taking into account one more child, the oldest of the entire eight, will increase the deduction by the value of the following terms (disregarding Δ^3 and higher differences).

$$(20) \qquad -6\tfrac{2}{3}\cdot 7\Delta \underset{7}{\bar{a}_{x'\,\overline{18-y_8}|}} + 10\,\frac{6.7}{\underline{|2}}\,\Delta^2 \underset{6}{\bar{a}_{x'\,\overline{18-y_8}|}}$$

$$-\,5\underset{8}{\bar{a}_{x\,\overline{18-y_8}|}} + 11\tfrac{2}{3}\cdot 7\Delta \underset{7}{\bar{a}_{x\,\overline{18-y_8}|}} - 15\,\frac{6.7}{\underline{|2}}\,\Delta^2 \underset{6}{\bar{a}_{x\,\overline{18-y_8}|}}$$

$$+\,15\left[\underset{8}{\bar{a}_{\overline{18-y_8}|}} - 7\Delta \underset{7}{\bar{a}_{\overline{18-y_8}|}} + \frac{6.7}{\underline{|2}}\,\Delta^2 \underset{6}{\bar{a}_{\overline{18-y_8}|}}\right].$$

The last line of the above expression is symbolically equivalent to

$$15\underset{8}{\bar{a}_{\overline{18-y_8}|}}\left[1 - 7\Delta(1+\Delta)^{-1} + \frac{6.7}{\underline{|2}}\,\Delta^2(1+\Delta)^{-2}\right]$$

on basis of our previous assumption that $\underset{r}{\bar{a}_{\overline{18-y_8}|}}$ is a function of r. The expression within the foregoing brackets is the first three terms of the binomial expansion of $\left(1 - \dfrac{\Delta}{1+\Delta}\right)^7$, that is, of $(1+\Delta)^{-7}$. As a convenient approximation, we may therefore write the last line of (20)

$$15\underset{8}{\bar{a}_{\overline{18-y_8}|}}(1+\Delta)^{-7}$$

that is

$$15\underset{1}{\bar{a}_{\overline{18-y_8}|}}$$

which is equal to

$$15\bar{a}_{y_8\,\overline{18-y_8}|}$$

according to the definition of $\underset{r}{\bar{a}_{\overline{18-y_8}|}}$ already promulgated.

In general, we may, as an approximation, write the following:

$$K\left[\bar{a}_{\underset{n}{}\overline{18-y_n}|} - (n-1)\Delta\, \bar{a}_{\underset{n-1}{}\overline{18-y_n}|} + \frac{\overline{n-1}\,\overline{n-2}}{\lfloor 2}\Delta^2\, \bar{a}_{\underset{n-2}{}\overline{18-y_n}|}\right] = K\bar{a}_{y_n\overline{18-y_n}|} \quad (21)$$

The same principle applies with respect to temporary annuities conditioned upon the survival, or upon the survival and continued widowhood of the wife of the deceased.

"Table C" which follows shows the formula of "Table B" amended in accordance with the foregoing principle. It is of interest that the formula for the deduction where the dependents are a widow and ten children,—according to "Table C,"—contains the following terms which would not appear if only nine children were entitled to compensation:

$$15\bar{a}_{y_{10}\overline{18-y_{10}}|} - 5\bar{a}_{xy_{10}\overline{18-y_{10}}|}. \quad (22)$$

It will be recalled that the above terms constitute the formula for the present value of the compensation due the tenth child in ascending order of age, where the deceased's widow is entitled to compensation at the date of valuation. Consequently it appears that for practical purposes we may disregard the tenth child and older children, as taking one or more of them into account would result in increasing both the positive and negative elements in our present value by the value of identical expressions of the form of (22). Where the number of children is eight or nine it will be found that certain terms in the formula of "Table C" are identical with, and therefore cancel, certain terms in the formula for the present value of compensation which would be payable if no limitation of 66⅔ per cent. obtained, although this cancellation does not permit us to leave the eighth and ninth children out of our reckoning altogether.

It is quite clear that the true present value of compensation payable will be increased somewhat whenever we take an additional dependent into account. It is by no means surprising, however, that when this additional dependent is the oldest of ten children (whose mother is entitled to compensation) this increase will be small enough to neglect in practice. For in such a case the existence of the tenth child cannot increase the compensation actually payable except in the event of the occurrence of one of the following extremely remote contingencies.

TABLE C.

Reduction in Present Value Because of 66⅔ Per Cent. Limit in Compensation.

Where the Dependents Are a Widow and n Children.

Subtract value obtained by use of significant terms of this formula from present value of compensation which would be payable if there were no 66⅔ per cent. limitation.

$$3\tfrac{1}{3}\left[\,{}_4\bar{a}_{x'\,\overline{18-y_4}|} - 4\Delta\,{}_4\bar{a}_{x'\,\overline{18-y_6}|} + \frac{4.5}{\underline{|2}}\,\Delta^2\,{}_4\bar{a}_{x'\,\overline{18-y_6}|}\right]$$

$$+\,10\left[\,{}_5\bar{a}_{x'\,\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{x'\,\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\,\Delta^2\,{}_5\bar{a}_{x'\,\overline{18-y_7}|}\right]$$

$$+\,10\left[\,{}_6\bar{a}_{x'\,\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{x'\,\overline{18-y_7}|} + \frac{6.7}{\underline{|2}}\,\Delta^2\,{}_6\bar{a}_{x'\,\overline{18-y_8}|}\right]$$

$$+\,6\tfrac{2}{3}\left[\,{}_7\bar{a}_{x'\,\overline{18-y_7}|} - 7\Delta\,{}_7\bar{a}_{x'\,\overline{18-y_8}|} + \frac{7.8}{\underline{|2}}\,\Delta^2\,{}_7\bar{a}_{x'\,\overline{18-y_9}|}\right]$$

$$-\,8\tfrac{1}{3}\left[\,{}_5\bar{a}_{x\,\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{x\,\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\,\Delta^2\,{}_5\bar{a}_{x\,\overline{18-y_7}|}\right]$$

$$-\,15\left[\,{}_6\bar{a}_{x\,\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{x\,\overline{18-y_7}|} + \frac{6.7}{\underline{|2}}\,\Delta^2\,{}_6\bar{a}_{x\,\overline{18-y_8}|}\right]$$

$$-\,11\tfrac{2}{3}\left[\,{}_7\bar{a}_{x\,\overline{18-y_7}|} - 7\Delta\,{}_7\bar{a}_{x\,\overline{18-y_8}|} + \frac{7.8}{\underline{|2}}\,\Delta^2\,{}_7\bar{a}_{x\,\overline{18-y_9}|}\right]$$

$$-\,5[\,{}_8\bar{a}_{x\,\overline{18-y_8}|} - 8\Delta\,{}_8\bar{a}_{x\,\overline{18-y_9}|} + {}_9\bar{a}_{x\,\overline{18-y_9}|}$$

$$+\,\bar{a}_{xy_{10}\,\overline{18-y_{10}}|} + \bar{a}_{xy_{11}\,\overline{18-y_{11}}|} \cdots + \bar{a}_{xy_n\,\overline{18-y_n}|}\,]$$

$$+\,8\tfrac{1}{3}\left[\,{}_5\bar{a}_{\overline{18-y_5}|} - 5\Delta\,{}_5\bar{a}_{\overline{18-y_6}|} + \frac{5.6}{\underline{|2}}\,\Delta^2\,{}_5\bar{a}_{\overline{18-y_7}|}\right]$$

$$+\,15[\,{}_6\bar{a}_{\overline{18-y_6}|} - 6\Delta\,{}_6\bar{a}_{\overline{18-y_7}|} + {}_7\bar{a}_{\overline{18-y_7}|}$$

$$+\,\bar{a}_{y_8\,\overline{18-y_8}|} + \bar{a}_{y_9\,\overline{18-y_9}|} \cdots \bar{a}_{y_n\,\overline{18-y_n}|}\,]$$

1. The death of at least six of the nine youngest children, during the lifetime and continued widowhood of the wife of the deceased, before the tenth child attains age eighteen.

2. The death of at least three of the nine youngest children, and the remarriage of the widow, before the tenth child attains age eighteen.

3. The death of the widow and at least five of the nine youngest children, before the tenth child attains age eighteen.

Although it is not feasible to include in this paper an investigation as to the extent of the error which will result from disregarding Δ^3 and higher orders of differences in the formula shown in "Table B" and "Table C," such an investigation is of considerable practical importance. Taking these higher differences into account, the formula of "Table B" calls for the calculation of 51 temporary annuities where the beneficiaries are a widow and nine children; while if we may disregard these higher differences, only 39 annuities need be calculated. Only 31 temporary annuities need be computed if we are justified in employing "Table C." It is of interest to repeat that according to the usual method of computing temporary annuities to run during the continuance of at least r out of n stati, it would be necessary to calculate 894 values (see "Table A," and context).

Where combinations of beneficiaries other than a widow and several children are entitled to compensation under the New York Law, the present value of their benefits may be computed in the great majority of instances by an adaptation of the formula exhibited in "Table C," or by elementary methods similar to those suggested in the opening pages of this paper.

The author is indebted to Mr. Edward Olifiers, A. I. A., A. A. S., for several suggestions which were helpful in the preparation of this paper.