DISCUSSION OF PAPER PUBLISHED IN VOLUME LXXXIV

APPLICATION OF THE OPTION MARKET PARADIGM TO THE SOLUTION OF INSURANCE PROBLEMS

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DISCUSSION BY STEPHEN J. MILDENHALL

RESPONSE BY THE AUTHOR

In his 2000 discussion [1] of my 1997 paper [2], Stephen Mildenhall chided me for overstating the similarity between options and insurance. He accepted the main point of the paper; namely, that the close resemblance between call option and excess of loss concepts can lead to insights about insurance and reinsurance risk management and product development. However, at a detailed level he dismissed my assertion that "the pricing mathematics is basically the same" for options and insurance, politely describing it as "inappropriate." He was correct in doing so. Unfortunately, in emphasizing the difference in the details of the pricing of call options and excess insurance, he missed the opportunity to show how these differences *can* be explained within a single pricing framework, though different from the one I originally presented. The purpose of this response is first to acknowledge my error at the formula level, but then to move beyond it to illustrate how Black-Scholes and excess insurance pricing *are* consistent, even if the pricing formula details are different.

Mea Culpa

I recognize that I overreached in claiming that my Formula (1.3) is a general formula for European call option pricing, which, I said, reduces to the Black-Scholes Formula (1.1) under the right conditions. It does reduce to Formula (1.1) when the underlying asset's price distribution at expiry is lognormal and the expected annualized continuous rate of return on the asset, μ, equals r, the annualized continuous risk-free rate. But that hardly represents the general case. Most of the time Formula (1.1) produces a different value from that produced by Formula (1.3).

To illustrate this point, consider a call option on a stock currently priced at $P_0 = \$100$ that gives the holder the right to buy the stock at a price of $S = \$100$ at option expiry in 20 days ($t = 20/365$). Assume the stock's expected annualized return and volatility are $\mu = 13\%$ and $\sigma = 25\%$, respectively. If the stock price movements follow geometric Brownian motion through time, the stock price distribution at option expiry is lognormal with a mean of $P_0 e^{\mu t} = (100)(e^{(.13)(20/365)}) = \$100.7149$ and a coefficient of variation (c.v.) of 5.857%. The expected expiry value of the option, given correctly by Formula (1.2), is \$2.7174. If all the Black-Scholes conditions are present, and the annualized risk-free rate $r = 5\%$, then the correct price of the option is given by Formula (1.1) as \$2.4705. In contrast, the Formula (1.3) *pure* premium is \$2.7100.[1] Clearly, my contention that Formula (1.3) is a "general formula for European call option pricing" is not only "inappropriate," it is wrong.

Bear in mind that while (1.3) is not a *general* formula for pricing a European call option, it *is* correct in some circumstances. For example, suppose the same lognormal distribution we just used to describe the stock price distribution at option expiry describes a distribution of aggregate insurance claims. Since the Black-Scholes conditions are not present, Formula (1.1) cannot be used to price a call option (more commonly called an "aggregate excess" or "stop-loss" cover in insurance circles) on the

[1]Generally, a risk charge needs to be added to convert the (1.3) value to a premium. The Black-Scholes value does not require an additional risk charge.

aggregate claims. Instead, actuarial ratemaking theory tells us to use Formula (1.3).

Same Paradigm, Different Details

That the same liability at expiry can give rise to different premiums, each of which is appropriate in its own context, is a paradox. It is clear that the premium is not a function solely of the liability. Mildenhall attributes the pricing difference to the different risk management paradigms operative in the financial and insurance markets: financial risks are hedged, whereas insurance risks are diversified. Yet it *is* possible to bring these two apparently distinct pricing paradigms together within a single framework. While it is possible to do so by reference to martingale measures and incomplete markets theory (see, for example, Moller [3], [4]), my aim is to make this subject as accessible as possible to practicing actuaries who may not be familiar with those concepts. Accordingly, I present the common framework as the more tangible and familiar one of asset-liability matching. *Within that framework the price for the transfer of a liability is a function of both the liability and its optimal matching assets.*

Before we search for the optimal asset strategy, let us explore the nature of the option liability. If the stock price at expiry is represented by a lognormally distributed[2] random variable, x, the expected value at expiry of the payoff obligation of a European call option is given by

$$\begin{aligned}
\mathrm{E}(\mathrm{call}_t) &= \int_S^{\infty} (x - S) f(x) dx \\
&= \mathrm{E}(x) \cdot N(d_1^{(\mu)}) - S \cdot N(d_2^{(\mu)}) \\
&= P_0 e^{\mu t} \cdot N(d_1^{(\mu)}) - S \cdot N(d_2^{(\mu)}), \qquad (1)
\end{aligned}$$

[2]If the stock price moves through time in accordance with geometric Brownian motion, the distribution of prices at expiry is lognormal. Note, however, that while Brownian motion is sufficient for lognormality, it is not necessary.

where $N(z)$ is the cumulative distribution function (c.d.f.) of the standard normal distribution, and

$$d_1^{(\mu)} = \frac{\ln(P_0/S) + (\mu + 0.5\sigma^2)t}{\sigma\sqrt{t}} \qquad \text{and}$$

$$d_2^{(\mu)} = \frac{\ln(P_0/S) + (\mu - 0.5\sigma^2)t}{\sigma\sqrt{t}} = d_1^{(\mu)} - \sigma\sqrt{t}.$$

The first term in Formula (1) is the expected market value of the assets to be sold by the call option grantor to the option holder at expiry. The second term is the expected value of the sale proceeds from that transaction.

The variance of the call payoff obligation at expiry is given by

$$\begin{aligned}\mathrm{Var}(\mathrm{call}_t) &= \int_S^\infty (x-S)^2 f(x)dx - \mathrm{E}(\mathrm{call}_t)^2\\ &= \mathrm{E}(x^2)\cdot N(d_0^{(\mu)}) - 2S\cdot \mathrm{E}(x)\cdot N(d_1^{(\mu)})\\ &\quad + S^2\cdot N(d_2^{(\mu)}) - \mathrm{E}(\mathrm{call}_t)^2, \end{aligned} \tag{2}$$

where $N(z)$ is the c.d.f. of the standard normal distribution, and $d_1^{(\mu)}$ and $d_2^{(\mu)}$ are defined as in Formula (1) and $d_0^{(\mu)} = d_1^{(\mu)} + \sigma\sqrt{t}$.

Returning to the example of the 20-day call option with $P_0 = S = \$100$, $\mu = 13\%$, $r = 5\%$ and $\sigma = 25\%$, the expected payoff liability at expiry associated with that option is \$2.7174. That amount is the difference between the expected market value of the stock the grantor of the option will sell to the option holder (\$56.4009), given by the first term of Formula (1), and the expected value of his sale proceeds (\$53.6835), which is given by the second term of Formula (1). The variance, given by Formula (2), is \$14.4456, implying a standard deviation of \$3.8007.

We will illustrate the pricing of this expected payoff liability of \$2.7174 in various available asset scenarios. The premium that the market can be expected to ask for assuming this liability depends on the optimal strategy available for investment of the premium to fund the liability. We will assume that enough investors or traders will find and execute the optimal strategy to force the asking price[3] in the market to be no greater than the level indicated by this strategy. (This is the standard "no arbitrage" requirement.)

This market premium is equal to the minimum expected present value cost of acquiring sufficient assets to fund the expected value liability at expiry and a risk charge related to the undiversifiable variability of the net result. If the variance of the net result can be forced to zero, as it can be when Black-Scholes conditions are present, then the risk charge is zero and the premium is simply equal to the minimum cost of acquiring the assets to fund the liability.

Case A—Underlying Asset is Tradable

The traditional actuarial approach to valuing the liability, embodied in Formula (1.3), is to assume the matching assets are invested in risk-free Treasuries.[4] However, where the liability arises from an option on a traded stock, it is easy to improve on this approach. Since the expected value of the stock to be transferred to the option holder at expiry is $P_0 e^{\mu t} \cdot N(d_1^{(\mu)})$, the option seller can match this expected liability by buying $N(d_1^{(\mu)})$ shares of stock at inception and holding them to expiry. He can fund most of the cost of the purchase, $P_0 \cdot N(d_1^{(\mu)})$, by borrowing against his expected sale proceeds at expiry of $S \cdot N(d_2^{(\mu)})$. Assuming he can borrow at the risk-free rate, he can raise $Se^{-rt} \cdot N(d_2^{(\mu)})$ in this way. That leaves him short of the $P_0 \cdot N(d_1^{(\mu)})$ he needs to buy the shares by $P_0 \cdot N(d_1^{(\mu)}) - Se^{-rt} \cdot N(d_2^{(\mu)})$, which is the amount he should ask for the option, before consideration of a risk charge. This indicates a formula for the premium before

[3]We will focus on the seller's asking price. The question of whether there are buyers at this asking price is beyond the scope of this discussion.

[4]Throughout this paper Treasuries are treated as risk-free assets and their yield as the risk free rate. If other assets meet that definition, they may be substituted for Treasuries.

risk charge (λ) of

$$\text{call}_0 - \lambda = P_0 \cdot N(d_1^{(\mu)}) - Se^{-rt} \cdot N(d_2^{(\mu)}). \qquad (3)$$

In the case of the 20-day option we have been following, he would buy 0.560005 shares at a total cost of \$56.0005, borrow \$53.5366, and charge an option premium before risk charge of \$2.4639. This is a much lower pure premium than the \$2.7100 given by the traditional actuarial Formula (1.3). Moreover, despite the investment of assets in the stock, an ostensibly riskier strategy, the option seller faces less risk (as measured by the standard deviation of the net result) than he would if he invested in risk-free Treasuries. The standard deviation of the option seller's net result is \$1.7527, which is much lower than the \$3.8007 that arises from the Treasuries investment strategy.[5] (For the details of the standard deviation calculation, see Appendix A.) Clearly, this strategy of investing the assets in the stock underlying the option is superior to investing them in Treasuries, since it produces a lower pure premium and a lower standard deviation, which together imply a lower risk-adjusted price.

However, as Black and Scholes proved, this strategy, while better than Treasuries, does not represent the optimal one. Assume the option is on the stock of a publicly traded company whose shares trade in accordance with the Black-Scholes assumptions; i.e., the price follows geometric Brownian motion through time, the shares are continuously tradable at zero transaction costs, etc. Black and Scholes showed that, under these conditions, the optimal investment strategy is one of dynamic asset-liability matching conducted in continuous time.

To execute this strategy, at inception the option seller buys n_0 shares of the underlying stock,[6] financed by a loan of L_0 and call premium proceeds of $P_0 \cdot n_0 - L_0$. Then, an instant later, he ad-

[5]Since the true values of μ and σ are unknown, there is parameter as well as process risk that needs to be taken into account in setting the risk charge for both asset strategies.

[6]Where n_0 is the first derivative of the call option price with respect to the stock price.

justs the number of shares he holds (to n_1) to reflect any change in the stock price and the infinitesimal passage of time. He adjusts the loan accordingly (to L_1). If n_0 and L_0 have been chosen correctly and the time interval is short enough, the gain or loss in his net position (i.e., the value of the net stock position less the value of the option) is effectively zero. The mean and variance of his net result is also zero. He repeats this adjustment procedure continuously until the option expires. In this way he ends up with exactly the right amount of stock at expiry to generate the funds to meet the option liability and repay the outstanding loan. Provided the sequences of n_i and L_i have been chosen correctly, the cumulative net result and its variance are both zero. Since the variance is zero, there is no justification for a risk charge. Black and Scholes proved that $n_0 = N(d_1)$ and $L_0 = Se^{-rt} \cdot N(d_2)$ and thus that

$$\text{call}_0 = P_0 \cdot N(d_1) - Se^{-rt} \cdot N(d_2), \tag{1.1}$$

where $N(z)$ is the c.d.f. of the standard normal distribution and

$$d_1 = \frac{\ln(P_0/S) + (r + 0.5\sigma^2)t}{\sigma\sqrt{t}} \qquad \text{and}$$

$$d_2 = \frac{\ln(P_0/S) + (r - 0.5\sigma^2)t}{\sigma\sqrt{t}} = d_1 - \sigma\sqrt{t}.$$

Since (1.1) does not depend on μ, the option seller engaging in the hedging strategy underlying the formula not only faces no process risk but also no μ-related parameter risk. (There is still parameter risk associated with σ.) In our example, Formula (1.1) indicates a call premium of \$2.4705.

In the highly liquid, efficient market in which execution of this dynamic hedging strategy is possible, arbitrageurs will force the market's "ask" price of the option to \$2.4705. If the option seller seeks a higher price, he will find no buyers, since another trader can and will undercut him without assuming any additional risk, simply by executing the hedging strategy. Note, however, that the option seller cannot afford to sell the option for \$2.4705 without assuming risk, unless he engages in the Black-Scholes hedging strategy that underpins this price.

Clearly, in order to engage in the kind of hedging activity described above, it is necessary that the stock be continuously tradable at zero transaction costs. The less liquid the market for the stock and the greater the trading costs, the less accurate Formula (1.1) will be in predicting the market asking price of the call. This is because the option seller will have to assume either residual volatility exposure requiring a risk charge (see Esipov and Guo [5]) or expenses not contemplated by Formula (1.1).

For example, if the mix of assets held by the option seller to hedge the 20-day option is adjusted on a daily basis, then the expected present value funding cost (excluding trading costs) is \$2.4708. The standard deviation is \$0.4405. Daily rebalancing is not sufficient to force the funding cost to the Black-Scholes predicted value of \$2.4705 and the standard deviation to zero. In the real world, where transaction costs are not zero, the trade-off between further reducing residual volatility and the cost of doing so will be valued by the market, often resulting in some deviation from the price predicted by Black-Scholes.

Case B—Underlying Asset is not Tradable

Suppose the call option is on the stock of a private company that will go public in 20 days time. Assume there is no "when issued" or forward market for this stock prior to the IPO. The stock is valued today at $P_0 = \$100$. The other parameters are the same as in Case A: $S = \$100$, $\mu = 13\%$, $r = 5\%$ and $\sigma = 25\%$. How should an option seller price this option? The key question is how to invest the call premium to fund the expected payoff obligation at option expiry. Since the option seller cannot invest in the underlying stock, it seems a good strategy would be to invest in Treasuries, which has the virtue of not increasing the variance of the net result.[7] If he does so, he needs to collect an option premium of \$2.7100 to fund the expected obligation of \$2.7174, plus a risk charge to compensate for the variability of the net result. The standard deviation of the net result, given $\mu = 13\%$, is \$3.8007. Note that the option seller does not know the true value of μ, 13% being merely an estimate. This means that there is parameter risk in addition to the process risk of \$3.8007.

Note that this scenario is identical to that faced by the excess insurer writing a stop-loss cover attaching at \$100 on an insurance portfolio in which aggregate claims notified and payable in 20 days time are lognormally distributed with mean \$100.7149 and coefficient of variation 5.857%. Conventional ratemaking theory prescribes investment in Treasuries, which indicates a premium of \$2.7100 to fund the expected claims of \$2.7174, plus a risk charge.

As plausible as this Treasury oriented investment strategy is, it is not necessarily the optimal one. If there are no assets available for investment that are correlated with the liability, then the conventional Treasury strategy is optimal. Otherwise, other strategies produce lower prices, lower risk, or both.

Case C—Underlying Asset is Not Tradable, but Tradable Proxy Exists[8]

Taking the stock option example first, suppose there is a publicly traded competitor of our soon to be public company that shares the same characteristics of $P_0 = \$100$, $\mu = 13\%$ and $\sigma = 25\%$. In addition, assume the two stocks' price movements in continuous time are believed to be correlated with $\rho = 60\%$. Under these circumstances it is possible to use the competitor's stock to partially hedge the call option on the non-public company's stock at a lower cost than that implied by investing in

[7]Investing in risky assets uncorrelated with the stock would increase variability.

[8]My thanks to Stephen Mildenhall for suggesting this scenario.

Treasuries. The option seller employs exactly the same procedure that he would use if he were hedging the target company's stock directly, except that he invests in the competitor's stock.

For example, at the moment he sells the call option, he buys \$53.0321 of the competitor's stock (0.530321 shares at \$100 a share), financing the purchase with a loan of \$50.5616 and proceeds from the sale of the call. By pursuing the same dynamic hedging procedure that he would use if he were able to buy and sell the target company's stock directly, the option seller will accumulate the assets that match the option payoff liability at an expected present value cost of \$2.4705. The difference from the scenario in which he can invest in the stock directly is that in that case the \$2.4705 is exact, whereas here it is an expected value.

This scenario involves risk. For example, if the hedge is adjusted on a daily basis, we found from a Monte Carlo simulation consisting of 10,000 trials that the standard deviation of the net result was \$3.6870. While this implies much more risk than that associated with hedging the option directly with the underlying stock (where we found the standard deviation associated with daily rebalancing to be \$0.4405), it is less than the \$3.8007 standard deviation of the net result arising from investing the call proceeds in Treasuries. Clearly, since the call option can be funded at an expected cost of $\$2.4705 < \$2.7100$ with an associated standard deviation of $\$3.6870 < \$3.8007$ by investing in a correlated asset rather than in Treasuries, investment in Treasuries in this scenario must be dismissed as a suboptimal asset strategy.

The same must be said of the analogous excess insurance example. Suppose the natural logarithms of the aggregate insurance claims covered by the stop-loss contract are known to be correlated ($\rho = 60\%$) with the natural logarithms of the values of the consumer price index (CPI-U). In this situation, the insurer can reduce the variance of its net result by investing in the index-linked Treasury notes known as TIPS (Treasury Inflation Protected Securities) rather than in conventional Treasuries.

TIPS pay a fixed rate of interest on a principal amount that is adjusted twice a year based on the change in the CPI-U index.

To illustrate this, assume the expected annualized return on the TIPS is 5%, comprising a fixed interest rate of 2% and expected inflation adjustment of 3%, the same expected total return as the fixed $r = 5\%$ that is available from standard Treasuries. While we usually think of an excess of loss claim as being the amount by which a claim exceeds the retention, we can also think of it as a total limits claim net of reimbursement for the retention. This characterization is useful here. The expected total limits claim is \$56.4009. To fund this payment, the insurer invests $\$56.4009 \cdot e^{-(.05)(20/365)} = \$56.2466$ in TIPS. To finance the purchase of the TIPS, the insurer borrows the present value of the retention reimbursement, $\$53.6835 \cdot e^{-(.05)(20/365)} = \$53.5366$. The remainder, \$2.7100, the insurer collects from the insured. This is the same amount the insurer would collect as a premium before risk charge if the insurer had simply invested in ordinary Treasuries. The benefit of investing in TIPS, which are correlated with the aggregate claim costs, is that the insurer can reduce the variability of the net underwriting result.

The standard deviation associated with this strategy was measured in a Monte Carlo simulation of 10,000 trials. Given a CPI-U index value at inception of 100, the value of the index 20 days later was assumed to be lognormally distributed with mean 100.1645 and c.v. 2.341%, which is consistent with the assumption that the inflation rate is 3% per annum, continuously compounded. The simulation indicated a standard deviation of \$3.2946, which is about 13% less than the standard deviation associated with the otherwise comparable investment in uncorrelated Treasuries.

We saw in the stock option example that hedging with the competitor's stock resulted in a much lower funding cost with less risk than investing in risk-free Treasuries, even with imperfect correlation. This raises the intriguing question of whether an insurer could similarly lower both its risk *and* its required pricing by identifying and investing in higher return securities that are partially correlated with its liabilities. This is food for thought.

Analysis

In all of these scenarios the expected value of the payoff obligation at expiry is the same: \$2.7174. The only differences are the type and tradability of assets available for investment. The characteristics of the *asset* side of the asset-liability equation determine the optimal asking price! Thus, pricing is a function of both the liability *and* the nature of the assets needed to fund it. In insurance applications, where there are usually no suitable assets other than Treasuries available, the liability alone appears to drive the price. This is only because historically, actuaries have assumed that investing in Treasuries is the only reasonable choice. However, as we have seen, when other assets are available, investing in Treasuries is not *always* the only reasonable choice and, in the case of tradable assets, it is not the optimal one.

If the pricing of a given option liability is driven by the optimal asset strategy, then it is critical that the seller of the option actually invests consistently with the pricing assumptions. For example, if an option trader believes that $\mu = 13\%$, and sells the call option described in Case A for the Black-Scholes price of \$2.4705, then it would be a mistake for him to simply invest the option proceeds in Treasuries. If he does that, he faces an expected loss of $\$2.7174 - \$2.4705e^{rt} = \$0.2402$. Beyond that, he is also assuming a sizeable amount of risk, since the standard deviation of his net result is \$3.8007 (plus parameter risk) instead of the zero promised by Black-Scholes. The lesson here is that while the Black-Scholes price is based on assumptions that remove all risk of loss and variability of outcomes, the option seller is not automatically protected. He must actively *manage* his risk.

Summary

My main aim in this response to Mildenhall's review of my paper has been to acknowledge that my Formula (1.3) does not have the generality I originally claimed for it, but then to press on with my contention that, even if the pricing formulas are not identical, call options and excess insurance are still governed by the same pricing paradigm; in particular, one that rests on optimal asset-liability matching.

There is another point I hope I have made clear. The dynamic asset-liability matching regimen that underlies the Black-Scholes Formula (1.1) imposes a different burden on the seller of a call option than the more passive asset-liability matching seen in Case B and in insurance applications. As we saw in our discussion of Case A, it is foolhardy to sell a hedgable call for the Black-Scholes price and then fail to dynamically hedge it. There are other situations where hedging is not possible, because the asset is either not traded or extremely illiquid. In such cases, it is also a mistake to sell the call option for the Black-Scholes price, since it *cannot* be dynamically hedged. In the case of liquid tradable assets, arbitrageurs will drive the option price to the Black-Scholes level. In illiquid or non-traded markets, there will be no such arbitrage activity and in these markets, the pricing formulas used in Case B are applicable.

In closing, I would like to thank Stephen Mildenhall for his excellent discussion, which not only corrected shortcomings in my paper but also added greatly to the understanding (including my own) of option concepts among actuaries.

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APPENDIX A

MEAN AND VARIANCE OF A SIMPLE BUY-AND-HOLD OPTION HEDGE

Let $h = bx - y$ define a random variable for the value at expiry of a hedged portfolio comprising b shares and one short call (i.e., sold short). Here x is a lognormal random variable representing the stock price distribution at option expiry, and y is the random variable representing the value at expiry of the call option on the stock. The option strike price is denoted S.

Mean of Hedged Portfolio at Expiry

$$\begin{aligned} \mathrm{E}(h) &= \mathrm{E}(bx - y) \\ &= b\mathrm{E}(x) - \mathrm{E}(y) \\ &= b\mathrm{E}(x) - (\mathrm{E}(x) \cdot N(d_1^{(\mu)}) - S \cdot N(d_2^{(\mu)})) \\ &= \mathrm{E}(x) \cdot (b - N(d_1^{(\mu)})) + S \cdot N(d_2^{(\mu)}). \end{aligned} \tag{A.1}$$

For the special case of $b = N(d_1^{(\mu)})$,

$$\mathrm{E}(h) = S \cdot N(d_2^{(\mu)}). \tag{A.1a}$$

Second Moment of Hedged Portfolio at Expiry

$$\begin{aligned} \mathrm{E}(h^2) &= \mathrm{E}((bx - y)^2) \\ &= \mathrm{E}(b^2x^2 - 2bxy + y^2) \\ &= b^2\mathrm{E}(x^2) - 2b\mathrm{E}(xy) + \mathrm{E}(y^2) \\ &= b^2\mathrm{E}(x^2) - 2b(\mathrm{E}(x^2) \cdot N(d_0^{(\mu)}) - S\mathrm{E}(x) \cdot N(d_1^{(\mu)})) \\ &\quad + (\mathrm{E}(x^2) \cdot N(d_0^{(\mu)}) - 2 \cdot S\mathrm{E}(x) \cdot N(d_1^{(\mu)}) + S^2 \cdot N(d_2^{(\mu)})). \\ &= (b^2 + N(d_0^{(\mu)}) \cdot (1 - 2b)) \cdot \mathrm{E}(x^2) \\ &\quad - 2S \cdot N(d_1^{(\mu)}) \cdot (1 - b) \cdot \mathrm{E}(x) + S^2 \cdot N(d_2^{(\mu)}). \end{aligned} \tag{A.2}$$

Variance of Hedged Portfolio at Expiry

$$\begin{aligned}
\sigma_h^2 &= \mathrm{E}(h^2) - \mathrm{E}(h)^2 \\
&= (b^2 + N(d_0^{(\mu)}) \cdot (1 - 2b)) \cdot \mathrm{E}(x^2) \\
&\quad - 2S \cdot N(d_1^{(\mu)}) \cdot (1 - b) \cdot \mathrm{E}(x) \\
&\quad + S^2 \cdot N(d_2^{(\mu)}) - (\mathrm{E}(x) \cdot (b - N(d_1^{(\mu)})) \\
&\quad + S \cdot N(d_2^{(\mu)}))^2 \\
&= (b^2 + N(d_0^{(\mu)}) \cdot (1 - 2b)) \cdot \mathrm{E}(x^2) \\
&\quad - 2S \cdot N(d_1^{(\mu)}) \cdot (1 - b) \cdot \mathrm{E}(x) \\
&\quad + S^2 \cdot N(d_2^{(\mu)}) - \mathrm{E}(x)^2 \cdot (b - N(d_1^{(\mu)}))^2 \\
&\quad - 2S\mathrm{E}(x) \cdot (b - N(d_1^{(\mu)})) \cdot N(d_2^{(\mu)}) \\
&\quad - S^2 \cdot N(d_2^{(\mu)})^2 \\
&= \mathrm{E}(x^2) \cdot (b^2 + N(d_0^{(\mu)}) \cdot (1 - 2b) - \mathrm{E}(x)^2 \cdot (b - N(d_1^{(\mu)}))^2 \\
&\quad - \mathrm{E}(x) \cdot 2S \cdot (N(d_1^{(\mu)}) \cdot (1 - b) + (b - N(d_1^{(\mu)})) \cdot N(d_2^{(\mu)})) \\
&\quad + S^2 \cdot N(d_2^{(\mu)}) \cdot (1 - N(d_2^{(\mu)})). \qquad \text{(A.3)}
\end{aligned}$$

For the special case of $b = N(d_1^{(\mu)})$,

$$\begin{aligned}
\sigma_h^2 &= \mathrm{E}(x^2) \cdot ((N(d_1^{(\mu)})^2 + N(d_0^{(\mu)}) \cdot (1 - 2N(d_1^{(\mu)}))) \\
&\quad - \mathrm{E}(x) \cdot 2S \cdot (N(d_1^{(\mu)}) \cdot (1 - N(d_1^{(\mu)})) \\
&\quad + S^2 \cdot N(d_2^{(\mu)}) \cdot (1 - N(d_2^{(\mu)})). \qquad \text{(A.3a)}
\end{aligned}$$

In the example used in the paper,

$$\mathrm{E}(x^2) = 10178.28293, \qquad N(d_0^{(\mu)}) = 0.58297$$

$$\mathrm{E}(x) = 100.71487, \qquad N(d_1^{(\mu)}) = 0.56001$$

$$
\begin{aligned}
S &= 100, \qquad N(d_2^{(\mu)}) = 0.53683 \\
\sigma_h^2 &= 10{,}178.28293 \cdot 0.24364 - 100.71487 \cdot 49.27987 \\
&\quad + 10{,}000 \cdot 0.24864 \\
&= 2{,}479.86796 - 4{,}963.2156 + 2{,}486.432 \\
&= 3.80433 \\
\sigma_h &= 1.75623.
\end{aligned}
$$