A NEW METHOD OF ESTIMATING LOSS RESERVES

COLIN M. RAMSAY

Abstract

This paper introduces a new method for estimating loss reserves. The method is fundamentally different from other loss reserving methods because it explicitly assumes that the evolution of the incremental incurred loss for an accident year is the result of a random split of the ultimate loss for that accident year into separate pieces that are observed in each development year over the claim settlement period. The nature of the random split and the pattern of the evolution of incremental incurred loss must be specified by the reserving actuary, thus giving the method tremendous flexibility. A key feature of this method is that it provides loss development factors without any knowledge of the distribution of the ultimate loss and without the actual cumulative incurred loss. Thus this method is suitable for calculating reserves for new lines of business where there is little or no loss settlement data.

1. INTRODUCTION

The loss reserving problem can be briefly described as follows. Let S_i denote the unknown ultimate incurred loss[1] for accident year i (excluding expected income from salvage and subrogation) and C_{ij} denote the best estimate of cumulative incurred loss amounts for accident year i and development year[2] j. The data used to estimate loss reserves are usually presented in the

---

[1]The loss reserving problem can also be described in terms of cumulative paid losses or incurred but not reported (IBNR) losses.

[2]The development year refers to the number of calendar years as measured from the accident year so that $j = 0$ refers to the accident year.

TABLE 1

LOSS DEVELOPMENT TRIANGLE

Accident Year (i)	Development Year (j)					
	0	1	2	$\cdots$	$k-1$	k
1	C_{10}	C_{11}	C_{12}	$\cdots$	$C_{1,k-1}$	C_{1k}
2	C_{20}	C_{21}	C_{22}	$\cdots$	$C_{2,k-1}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$k-1$	$C_{k-1,0}$	$C_{k-1,1}$				
k	$C_{k,0}$					

form of a loss development triangle as shown in Table 1. A basic assumption in loss reserving is that the data in the rows of Table 1 are mutually independent, i.e., C_{ij} and C_{rm} are independent if $i \neq r$. In other words, losses from different accident years evolve independently. Another assumption is that all losses are settled within a certain number of calendar years, N years, say, from their date of occurrence, regardless of the year of occurrence. This means that $C_{ij} = C_{iN}$ for $j \geq N$ and $i = 1,2,\ldots$. Sometimes, however, the data in the loss development triangle consist of incremental incurred losses, c_{ij}, where

$$c_{ij} = \begin{cases} C_{ij} - C_{i,j-1} & j = 1,2,\ldots \\ C_{i0} & j = 0. \end{cases}$$

The decision to use either incremental or cumulative values depends on the loss reserving method used.

Given C_{ij}, the ultimate incurred loss for accident year i, S_i, is estimated as:

$$S_i = C_{ij} \times \mathrm{LDF}_j \tag{1}$$

where LDF_j is the incurred loss development factor for development year j to ultimate. When the total paid loss for occurrence year i at the end of development year j (TPL_{ij}) is known, the loss reserve at that point in time (LR_{ij}) is then given by

$$\mathrm{LR}_{ij} = C_{ij} \times \mathrm{LDF}_j - \mathrm{TPL}_{ij}. \qquad (2)$$

There are numerous methods for estimating loss reserves. These include the chain ladder method and its many modifications, separation methods, probabilistic methods such as Bühlmann et al. [5], Bayesian methods (see De Alba [7] and references therein), and many ad hoc methods such as the Bornhuetter-Ferguson method [3]. For a detailed discussion of the practical issues involved in developing loss reserves, see Berquist and Sherman [1], Salzmann [16], Wiser [19], or Booth et al. [2, Chapter 16]. For an overview of many older actuarial loss reserving methods, see Van Eeghen [18]. A more modern treatment of loss reserves is given in Taylor [17] and England and Verrall [8].

The important common characteristic of established loss reserving methods is their reliance on the existence of a sufficiently long loss run-off triangle. This makes many of them unsuitable for estimating loss reserves for new lines of business, especially in the early years where the loss development process is immature.[3]

For new lines of business, practical approach to loss reserving may be as follows:

1. The actuary tries to get an understanding of the business by talking to the underwriters and claims-handlers; then
2. The actuary makes his/her best a priori guess of the reserve based on this knowledge.

The actuary's guess may be based on a simple loss ratio reserving method together with a rough conservative guess as to the development pattern (possibly based on the experience from some other similar business).

[3]One method that is suited for the early years is the Bornhuetter-Ferguson method.

The objective of this paper is to provide reserving actuaries with a method or process to assist them with their "best guess" in the early years of development and with loss reserving in general. The method introduced fundamentally is different from other loss reserving methods because it explicitly assumes the evolution of the incremental incurred loss for an accident year is the result of a random split of the ultimate loss for that accident year into separate pieces of losses that are observed in each development year over the claim settlement period. The nature of the random split and the pattern of the evolution of incremental incurred loss must be specified by the reserving actuary, thus giving the method tremendous flexibility. As this method provides loss development factors without any knowledge of the distribution of the ultimate loss or of the actual cumulative incurred loss, it is suitable for calculating loss reserves for new lines of business, where there is little or no loss development data. This method is suitable for paid and incurred loss, and can also be used in conjunction with the Bornhuetter-Ferguson method by providing the necessary loss development factors.

2. THE BASIC MODEL

As is common in many models of the property/casualty loss reserving process, we assume:[4]

1. The maximum number of years it takes for incurred losses to be completely paid and settled is fixed and known to be N, i.e., a claim occurring in accident year i is settled by the end of accident year $i+N$;
2. The incremental loss development processes from different accident years are mutually independent, i.e., c_{ij} and c_{kl} are independent if $i \neq k$; and
3. The incremental incurred loss in each accident year forms a non-negative decreasing sequence, i.e.,

[4]This model can also be described in terms of paid losses.

$c_{ij} > c_{i,j+1}$ for $j = 0, 1, \ldots, N-1$. (The case where the incremental incurred losses form an arbitrary sequence is considered later in Section 5.2.)

Clearly, from the definitions of N, S_i, and the c_{ij}'s,

$$S_i = c_{i0} + c_{i1} + \cdots + c_{iN}. \tag{3}$$

where $N = 1, 2, \ldots.$ is known. Equation (3) shows that S_i can be viewed as being split at random into $N+1$ pieces of loss $c_{i0}, c_{i1}, \ldots, c_{iN}$ with the jth piece of loss being revealed (i.e., made known) at the end of the jth development year. On the other hand, Assumption 3 implies that the sequence $c_{i0}, c_{i1}, \ldots, c_{iN}$ is an ordered sequence. It is unlikely that a purely random split will lead to an ordered sequence. Thus the precise nature of the split must be specified.

Suppose the total unknown incurred S_i is split at random under a uniform distribution into $N+1$ pieces of loss labeled $X_{i1}, X_{i2}, \ldots, X_{i,N+1}$ such that

$$S_i = X_{i1} + X_{i2} + \cdots + X_{i,N+1}. \tag{4}$$

We further assume that these pieces of loss are ordered and relabeled so that

$$X_{i(1)} \leq X_{i(2)} \leq \cdots \leq X_{i(N+1)}.$$

By Assumption 3 the incremental incurred loss is a realization of the ordered pieces of loss, i.e.,

$$c_{ij} = X_{i(N+1-j)} \qquad \text{and} \qquad C_{ij} = \sum_{k=0}^{j} X_{i(N+1-k)} \tag{5}$$

for $j = 0, 1, \ldots, N$.

At this point, it is important to clarify what is meant by the statement "S_i is split at random under a uniform distribution." Suppose we have N independent and identically distributed random variables, $U_1, U_2, \ldots, U_N$, which are uniformly distributed on $(0,1)$. The $U_{(j)}$'s are ordered and relabeled as

$$0 < U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(N)} < 1 \qquad (6)$$

and the end points are then defined as $U_{(0)} = 0$ and $U_{(N+1)} = 1$. Next we define the spacings[5] between the consecutive ordered U_j's as

$$Y_j = U_{(j)} - U_{(j-1)} \qquad (7)$$

for $j = 1, 2, \ldots, N+1$. Then a random split of S_i into $N+1$ pieces of loss $X_{i1}, X_{i2}, \ldots, X_{i,N+1}$ means

$$X_{ij} = S_i \times Y_j \qquad \text{for} \quad j = 1, 2, \ldots, N+1. \qquad (8)$$

Ordering the Y_j's as $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(N+1)}$ and an application of Assumption 3 immediately yields

$$c_{ij} = S_i Y_{(N+1-j)} \qquad \text{and} \qquad (9)$$

$$C_{ij} = S_i \sum_{k=0}^{j} Y_{(N+1-k)} \qquad (10)$$

for $j = 0, 1, \ldots, N$. However, as the the cumulative incurred tends to be more stable than the incremental incurred, Equation (9) is not used to estimate S_i. Instead, we use

$$S_i = \frac{C_{ij}}{\sum_{k=0}^{j} Y_{(N+1-k)}}. \qquad (11)$$

3. ESTIMATING ULTIMATE LOSS

Recall that the $Y_{(j)}$'s are not known until S_i is known, hence they must be estimated. The obvious estimator of $Y_{(j)}$ is its mean,

---

[5] For comprehensive treatment of the distribution of the spacings between successive ordered random variables, see Pyke [14].

which leads to the first estimate of the total incurred for accident year i given the incurred losses through development year j:

$$\hat{S}_i^{(1)} = \frac{C_{ij}}{\sum_{r=0}^{j} \mathbf{E}[Y_{(N+1-r)}]} \tag{12}$$

i.e., the loss development factor from j to ultimate is

$$\mathrm{LDF}_{j,N}^{(1)} = \frac{1}{\sum_{r=0}^{j} \mathbf{E}[Y_{(N+1-r)}]} \tag{13}$$

for $j = 0,1,\ldots,N$. Alternatively, we may use

$$\hat{S}_i^{(2)} = C_{ij}\mathbf{E}\left[\frac{1}{\sum_{r=0}^{j} Y_{(N+1-r)}}\right] \tag{14}$$

which yields the alternative loss development factor from j to ultimate

$$\mathrm{LDF}_{j,N}^{(2)} = \mathbf{E}\left[\frac{1}{\sum_{r=0}^{j} Y_{(N+1-r)}}\right]. \tag{15}$$

From Jensen's inequality, $\mathrm{LDF}_{j,N}^{(2)} \geq \mathrm{LDF}_{j,N}^{(1)}$ for every j. Before calculating the values of $\mathrm{LDF}_{j,N}^{(1)}$ and $\mathrm{LDF}_{j,N}^{(2)}$ for various values of j and N, the distribution of the $Y_{(j)}$'s will be provided.

From the theory of the random division of an interval of unit length (for example, David [6, chapter 5.4] or Feller [9, chapter 1]), the variables $Y_1, Y_2, \ldots, Y_{N+1}$ form an exchangeable sequence of dependent random variables with joint pdf:

$$f(y_1, y_2, \ldots, y_{N+1}) = \begin{cases} (N+1)! & \text{if} \quad y_j \geq 0 \quad \text{and} \quad \sum_{k=1}^{N+1} y_k = 1. \\ 0 & \text{otherwise.} \end{cases}$$

The marginal distribution of $Y_{(j)}$ is given by Maurer and Margolin [12, equation (4.4)] as

$$\Pr[Y_{(j)} > y] = \sum_{m=N+2-j}^{N+1} (-1)^{m-(N+2-j)} \binom{m-1}{N+1-j} \times \binom{N+1}{m} (1-my)^N I\left\{y < \frac{1}{m}\right\}$$

where $I\{A\}$ is an indicator of the occurrence of the event A. In addition, the moments of $Y_{(j)}$ satisfy the recursion

$$(n-j)\mu_{j:n}^{(k)} + j\mu_{j+1:n}^{(k)} = n\mu_{j:n-1}^{(k)} \tag{16}$$

where $\mu_{j:n}^{(k)} = E[Y_{(j)}^k]$ for sample size n.

Due to the difficulties in deriving the inverse moments needed in $\mathrm{LDF}_{j,N}^{(2)}$, however, Monte Carlo simulations[6] are used to determine both sets of loss development factors. Table A1 in the Appendix shows the loss development factors $\mathrm{LDF}_{j,N}^{(1)}$ and $\mathrm{LDF}_{j,N}^{(2)}$, respectively.

4. NUMERICAL EXAMPLES

Example 1: A Simple Data Set

Suppose a new product was introduced on January 1, 2000 and the loss development data as of December 31, 2002 are given in Table 2.

The total paid loss to date is thus $1{,}719 + 2{,}573 + 1{,}761 = 6{,}053$. To estimate the total ultimate loss, we must first specify N. If we assume $N = 3$, then the total estimated ultimate loss for

[6] The uniform (0, 1) random number generator run in Press et al. [13, chapter B7, page 1142] is used to perform all simulations.

TABLE 2

HYPOTHETICAL CUMULATIVE INCURRED AND PAID LOSSES (IN 000S)

Accident Year (i)	Earned Premiums (in 000s)	Development Year (j)					
		Incurred Loss (C_{ij})			Paid Loss		
		0	1	2	0	1	2
2000	4,500	1,447	1,976	2,454	401	1,166	1,761
2001	8,500	3,578	3,911		906	2,573	
2002	16,000	4,754			1,719		

2000–2002, as of December 31, 2002, is:

$$\hat{S}^{(1)} = 4{,}754 \times 1.9195 + 3{,}911 \times 1.2627 + 2{,}454 \times 1.0662$$
$$= 16{,}680.18$$
$$\hat{S}^{(2)} = 4{,}754 \times 2.0379 + 3{,}911 \times 1.2826 + 2{,}454 \times 1.0691$$
$$= 17{,}328.00$$

using the $N = 3$ rows of Table A1. The corresponding reserve estimates are 10,627.18 = 16,680.18 – 6,053 and 11,275.00 = 17,328.00 – 6,053, respectively.

If, on the other hand, we assume $N = 5$, then the estimated ultimate loss for 2000–2002, as of December 31, 2002, is:

$$\hat{S}^{(1)} = 4{,}754 \times 2.4564 + 3{,}911 \times 1.5412 + 2{,}454 \times 1.2384$$
$$= 20{,}744.39$$
$$\hat{S}^{(2)} = 4{,}754 \times 2.6221 + 3{,}911 \times 1.5800 + 2{,}454 \times 1.2505$$
$$= 21{,}713.57$$

using the $N = 5$ row of Table A1. These ultimates lead to reserve estimates of 14,691.39 = 20,744.39 – 6,053, and 15,660.57 = 21,713.57 – 6,053, respectively.

TABLE 3

HYPOTHETICAL PREMIUM AND LOSS DEVELOPMENT DATA (IN 000S)

Accident Year	Earned Premium	Development Year (j) 0	1	2	3	4	5
1997	5,000	2,500	3,650	4,200	4,325	4,335	4,330
1998	5,500	2,150	3,225	3,775	3,965	3,960	
1999	6,000	3,250	4,500	5,050	5,150		
2000	7,000	3,700	5,200	5,775			
2001	7,500	3,300	4,800				
2002	8,000	4,250					

Source: Based on the data in Bornhuetter and Ferguson [3, page 193, Exhibit A] with "Year of Origin" changed from 1966–1971 to 1997–2002.

TABLE 4

ANNUAL LOSS DEVELOPMENT FACTORS FOR TABLE 3

Accident Year	Earned Premium	Development Year (j) 0	1	2	3	4
1997	5,000	1.460	1.151	1.030	1.002	0.999
1998	5,500	1.500	1.171	1.050	0.999	
1999	6,000	1.385	1.122	1.102		
2000	7,000	1.405	1.111			
2001	7,500	1.455				
2002	8,000					

Example 2: A Modified Bornhuetter-Ferguson Method

This method can be used to provide the ultimate loss development factors needed in applications of the Bornhuetter-Ferguson (B-F) method. For example, using the data in Table 3, the IBNR reserves are estimated using the traditional B-F method and a modified B-F method based on the $\mathrm{LDF}_{j,N}^{(1)}$ given in Table A1. In deriving their estimates, Bornhuetter and Ferguson [3] assume losses in the three most recent calendar years are settled in 3 years. Table 4 shows the annual loss development factors. Table 5 provides a summary of the results.

TABLE 5

HYPOTHETICAL IBNR RESERVE COMPUTATION AS OF DECEMBER 31, 2002

Accident Year	(1) Expected Losses	LDFs (2) $\mathrm{LDF}_j^{(BF)}$	LDFs (3) $\mathrm{LDF}_{j,3}^{(1)}$	IBNR Factor (4) $1-1/\mathrm{LDF}_j^{(BF)}$	IBNR Factor (5) $1-1/\mathrm{LDF}_{j,3}^{(1)}$	Indicated IBNR (6) B-F	Indicated IBNR (7) Mod. B-F
1999	5,700	1.000	1.0000	0.000	0.0000	0	0
2000	6,650	1.032	1.0662	0.031	0.0621	206	413
2001	7,125	1.166	1.2627	0.142	0.2080	1,012	1,482
2002	7,600	1.650	1.9195	0.394	0.4790	2,994	3,640
						4,212	5,535

Notes: Expected Losses are 95% of the earned premium. The information in Columns (2), (4) and (6) are provided by Bornhuetter and Ferguson [3, page 194, Exhibit B]. The information in Columns (3) and (5) are derived from Table A1 with $N = 3$. Column (7) = Column (1) $\times$ Column (5).

5. GENERALIZATIONS AND PRACTICAL CONSIDERATIONS

The loss reserving method introduced above is flexible and can be generalized in at least two ways. For example, one can consider a non-uniform random split and/or consider an arbitrary ordered sequence of random spacings to reflect the evolution of the incremental incurred loss.

5.1. A Non-Uniform Random Split

One can observe in Appendix Tables A1 and A2 that, under the uniform random split, $C_{i,0}$ is a relatively small percentage of S_i then there is a fairly rapid development of incurred loss. For example, in Table A1, $1/\mathrm{LDF}_{j,N}^{(1)}$ and $1/\mathrm{LDF}_{j,N}^{(2)}$ are small for $j = 0, 1$ or 2, while Table A2 shows that the loss development factors for years 1 and 2 are high suggesting the rapid development of incurred losses.

If the actuary has loss development factors that are not similar to the quantities in Tables A1 and A2, another distribution defined on $(0,1)$ must be used to form the basis of the split. Unfortunately, there is no obvious alternative distribution, especially one that is intuitively appealing. It is up to the reserving actuary to specify a continuous distribution with support on $(0,1)$. Some alternative distributions with support on $(0,1)$ include the beta, the truncated gamma, and the truncated Pareto distributions. One strategy that can be used is to have on hand tables similar to Tables A1 and A2 for each potential alternative random split distribution. The actuary can then use the table (i.e., distribution) that best matches the observed loss development factors.

Suppose the actuary chooses a specific cumulative distribution function $F_U(u)$ with continuous support on $(0,1)$. As the length of the claims settlement period is $N+1$ years, we sample N independent and identically distributed random variables, $U_1,\ldots,U_j,\ldots,U_N$ from $F_U(u)$.[7] The sampled U_j's are then ordered and relabeled as before. The resulting spacings, $Y_j = U_{(j)} - U_{(j-1)}$, with $U_{(0)} = 0$ and $U_{(N+1)} = 1$, are then used to define the random split. As before, simulations are then used to determine the expectations needed to determine the loss development factors. As an example, Tables A3 and A4 provide the loss development factors to ultimate and the annual loss development factors in the case of the truncated exponential pdf of U_j defined by

$$f_U(u) = \frac{\lambda e^{-\lambda u}}{1 - e^{-\lambda u}}, \tag{17}$$

for $0 \le u \le 1$ and $\lambda > 0$.

5.2. *An Arbitrary Ordered Sequence*

Recall equation (9) in which we defined the sequence of incremental incurred loss as $c_{i0} \ge c_{i1} \ge \cdots \ge c_{iN}$. If the actuary be-

[7]For more on techniques for generating random variables from continuous distributions see Bratley, Fox, and Shrage [4, chapters 5 and 6]; Kalos and Whitlock [11, chapter 3]; Fishman [10, chapter 3]; and Ross [15, chapters 3 to 5].

lieves, however, that the pattern of incremental incurred loss is different, then tables of loss development factors to ultimate and annual loss development factors that are consistent with the specified pattern must be derived.

To be precise, for $j = 0,1,\ldots,N$ let θ_j denote the order of the set of order statistics $Y_{(1)}, Y_{(2)},\ldots,Y_{(N+1)}$ that is used to define c_{ij}. Note that $\theta_0,\theta_1,\ldots,\theta_N$ is a permutation of the elements of the set $\{1,2,\ldots,N+1\}$. (For example, equation (9) implies $\theta_j = N+1-j$. As another example, the actuary may specifically believe that $\theta_0 = N-1, \theta_1 = N, \theta_2 = N+1$, $\theta_j = N+1-j$ for $j = 3,\ldots,N$, which implies $c_{i0} \leq c_{i1} \leq c_{i2} \geq c_{i3} \geq \cdots \geq c_{iN}$.) It follows that c_{ij} and C_{ij} are defined as

$$c_{ij} = S_i Y_{(\theta_j)}, \qquad \text{and} \tag{18}$$

$$S_i = \frac{C_{ij}}{\sum_{k=0}^{j} Y_{(\theta_k)}}. \tag{19}$$

for $j = 0,1,\ldots,N$.

In general, the loss development factors can be obtained via a simulation of sample size M as follows:

STEP 1. For given settlement period of $N+1$ years, set $\text{TEMP}^{(1)}_{j,N} = 0$ and $\text{TEMP}^{(2)}_{j,N} = 0$ for $j = 0,1,2,\ldots,N$.

STEP 2. Create an $(N+1) = 2$ dimensional permutation vector $\theta = (\theta_0,\theta_1,\ldots,\theta_N)$ containing the actuary's specified pattern of incremental incurred losses.

STEP 3. Generate N random variables $U_1,\ldots,U_j,\ldots,U_N$ from the actuary's specified random splitting distribution, $F_U(u)$.

STEP 4. Order the sampled U_j's as $U_{(1)} \leq U_{(2)} \ldots \leq U_{(N)}$.

STEP 5. For $j = 1,2,\ldots,N+1$, define $Y_j = U_{(j)} - U_{(j-1)}$, with $U_{(0)} = 0$ and $U_{(N+1)} = 1$.

STEP 6. Order the $N+1$ Y_j's as $Y_{(1)} \leq Y_{(2)} \ldots \leq Y_{(N+1)}$.

STEP 7. For $j = 0,1,2,\ldots,N$, set

$$\mathrm{TEMP}^{(1)}_{j,N} = \mathrm{TEMP}^{(1)}_{j,N} + \sum_{r=0}^{j} \mathbf{E}[Y_{(\theta_r)}]$$

$$\mathrm{TEMP}^{(2)}_{j,N} = \mathrm{TEMP}^{(2)}_{j,N} + \frac{1}{\sum_{r=0}^{j} Y_{(\theta_r)}}.$$

STEP 8. Repeat Steps 3 to 7 a total of M times.

STEP 9. For $j = 0,1,2,\ldots,N$, the loss development factors are estimated as:

$$\widehat{\mathrm{LDF}}^{(1)}_{j,N} = \frac{M}{\mathrm{TEMP}^{(1)}_{j,N}} \tag{20}$$

$$\widehat{\mathrm{LDF}}^{(2)}_{j,N} = \frac{\mathrm{TEMP}^{(2)}_{j,N}}{M}. \tag{21}$$

5.3. Other Practical Considerations

In practice, other potential problems may occur such as different accident years having different claim settlement periods. For example, N depends on i, or the existence of negative incremental incurred loss amounts. Tables 6 and 7 display two hypothetical data sets with several problems. In Table 6, one can assume that losses are settled in three years, i.e., $N = 3$. However, the losses do not exhibit the pattern assumed by Tables A1 and A2. In fact, even though the losses are settled in 3 years, the total incurred loss changes only slightly after development year 1, and for year 2001 the cumulative incurred loss is decreasing. Our method is not ideally suited to the data in Table 6 because of the negative incremental losses. Further research is needed in this area.

TABLE 6

SECOND HYPOTHETICAL CUMULATIVE INCURRED LOSS DATA (IN $000S)

Accident Year (i)	Development Year (j)						
	0	1	2	3	4	5	6
1997	2,237	2,369	2,376	2,376	2,376	2,376	
1998	2,899	2,942	2,936	2,934	2,934		
1999	2,225	2,330	2,322	2,325			
2000	2,145	2,205	2,207				
2001	1,513	1,499					
2002	1,168						

TABLE 7

THIRD HYPOTHETICAL CUMULATIVE INCURRED LOSS DATA (IN $000S)

Accident Year (i)	Development Year (j)						
	0	1	2	3	4	5	6
1996	1,076	927	927	951	960	1,087	1,087
1997	957	1,193	1,312	1,295	1,220	1,392	
1998	1,421	1,788	2,086	2,236	2,252		
1999	1,473	1,910	2,235	2,192			
2000	1,447	1,976	2,454				
2001	3,578	3,911					
2002	4,754						

Table 7 presents similar challenges as some incremental incurred loss amounts are zero and some are negative. The sequence of incurred losses generated in 1996 and 1997 appear to have a pattern distinct from those in subsequent years. In addition, it may be incorrect to assume the accident years all have the same settlement period, i.e., N depends on i. In such cases where there are non-positive incremental incurred losses and/or different claim settlement periods, the ultimate incurred loss for accident

TABLE 8

LOSS DEVELOPMENT FACTORS ($C_{i,j+1}/C_{i,j}$) FROM TABLE 7

Accident Year	Development Year ($j/j+1$)					
	0/1	1/2	2/3	3/4	4/5	5/6
1996	0.8615	1.0000	1.0259	1.0095	1.1323	1.0000
1997	1.2466	1.0997	0.9870	0.9421	1.1410	
1998	1.2583	1.1667	1.0719	1.0072		
1999	1.2967	1.1702	0.9808			
2000	1.3656	1.2419				
2001	1.0931					
2002						

year i, S_i, can still be estimated as

$$S_i = \frac{C_{ij}}{\sum_{k=0}^{j} Y_{(N_i+1-k)}}. \tag{22}$$

Note, the length of the claim settlement period can be approximated by observing cumulative loss development factors. If the assumptions of this model (as stated in Section 2) hold, then the jth annual cumulative loss development factor for accident year i, $C_{i,j+1}/C_{i,j}$, should satisfy

$$\frac{C_{i,j+1}}{C_{i,j}} \approx \frac{\mathrm{LDF}_{j,N}^{(k)}}{\mathrm{LDF}_{j+1,N}^{(k)}} \tag{23}$$

for $k = 1,2$ and $j = 0,1,\ldots N-1$. Table A2 shows the values $\mathrm{LDF}_{j,N}^{(k)}/\mathrm{LDF}_{j+1,N}^{(k)}$ for $k = 1,2$, $j = 0,1,\ldots,9$ and $N = 1,2,\ldots,9$. The annual cumulative loss development factors should then be compared with those in Table A2. Table 8 shows the actual cumulative loss development factors generated by Table 7. Comparing the first two columns of Table 8 with those expected in Table A2 show that patterns of actual cumulative loss development factors for years 1997 to 2001 are too low, making the data in Table 7 inconsistent with the assumption of a uniform random split. Notice that the results of Tables A3 and A4 for $\lambda = 5$ seem to better fit the data from the later years in Tables 7 and 8 than the uniform random split.

6. SUMMARY AND CLOSING COMMENTS

For new lines of business, the practical approach to loss reserving requires the actuary to make his/her best guess of the reserve level based on prior knowledge. The actuary's guess may be based on a simple loss ratio reserving method together with a rough conservative guess as to the development pattern (possibly based on the experience from some other similar business). This paper provides reserving actuaries with a tool to assist them with their "best guess" of the reserves, especially in the early years of development. The method essentially uses an a priori pattern in the table of expected loss development factors to determine the loss reserves. The pattern of expected loss development factors is independent of the distribution of the cumulative incurred loss in the accident year and can be varied depending on the actuary's estimate of the length of the claim settlement period, and the random split used.

When there is a sufficiently large amount of data in the loss development triangle, the actuary can use the method of this paper to generate tables of expected loss development factors to see which ones match the observed loss development factors. The best matched tables can be used to estimate the loss reserves.

In closing, there are several important attributes of this method:

1. It can be used for new and old business.
2. It can be used in conjunction with other methods such as the Bornhuetter-Ferguson method.
3. It makes no assumptions about the underlying distribution of the ultimate losses.

4. The ultimate losses are estimated using only the most recent cumulative loss data.

5. The method can be used if the length of the settlement period varies by year of origin.

6. The factors $\mathrm{LDF}^{(1)}_{j,N}$, $\mathrm{LDF}^{(2)}_{j,N}$ and their ratios $\mathrm{LDF}^{(1)}_{j,N}/\mathrm{LDF}^{(1)}_{j+1,N}$ and $\mathrm{LDF}^{(2)}_{j,N}/\mathrm{LDF}^{(2)}_{j+1,N}$ do not depend on the actual loss development pattern.

7. Tables of factors and ratios can be created and saved for each combination of development year j and settlement period N, and for various types of random splits such as the uniform and beta distributions. The appropriate table can be chosen to:

   a. Match the observed pattern of loss development factors or to

   b. Match the actuary's or underwriter's best guess of what the pattern of loss development factors should be.

REFERENCES

[1] Berquist, J. R., and R. E. Sherman, "Loss Reserve Adequacy and Testing: A Comprehensive, Systematic Approach," *PCAS* LXII, 1977, pp. 123–184.

[2] Booth, P., R. Chadburn, D. Cooper, S. Haberman, and D. James, *Modern Actuarial Theory and Practice*, Chapman & Hall/CRC, 1999.

[3] Bornhuetter, R. L., and R. E. Ferguson, "The Actuary and IBNR," *PCAS* LIX, 1972, pp. 181–195.

[4] Bratley, P., B. L. Fox, and L. E. Schrage, *A Guide to Simulation*, Springer-Verlag, 1983.

[5] Bühlmann, H., E. Straub, and R. Schnieper, "Claims Reserves in Casualty Insurance Based on a Probabilistic Model," *Mitteilungen der Vereinidung Schweizerischer Versicherungsmathematiker* 1, 1980, pp. 21–46.

[6] David, H. A., *Order Statistics*, (2nd edition), John Wiley & Sons, 1981.

[7] De Alba, E., "Bayesian Estimation of Outstanding Claim Reserves," *North American Actuarial Journal* 6:4, 2002, pp. 1–20.

[8] England, P. D., and R. J. Verrall, "Stochastic Claims Reserving in General Insurance," Presented to the Institute of Actuaries, January 28, 2002, http://www.actuaries.org.uk/files/pdf/sessional/sm0201.pdf.

[9] Feller, W., *An Introduction to Probability and Its Application*, (2nd edition), John Wiley & Sons, 1971.

[10] Fishman, G. S., *Monte Carlo: Concepts, Algorithms, and Applications*, Springer-Verlag, 1996.

[11] Kalos, M. H., and P. A. Whitlock, *Monte Carlo Methods Volume 1: Basics*, John Wiley & Sons, 1986.

[12] Maurer, W., and B. H. Margolin, "The Multivariate Inclusion-Exclusion Formula and Order Statistics from Dependent Variates," *Annals of Statistics* 4:6, 1976, pp. 1190–1199.

[13] Press, W. H., S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in Fortran 90 The Art of Parallel Scientific Computing*, Cambridge University Press, 1996.

[14] Pyke, R., "Spacings," *Journal of the Royal Statistical Society, Series B (Methodlogical)* 27:3, 1965, pp. 395–449.

[15] Ross, S., *Simulation*, (2nd edition), Academic Press, 2002.

[16] Salzmann, R. E., *Estimated Liabilities for Losses and Loss Adjustment Expenses*, Prentice Hall, 1984.

[17] Taylor, G. C., *Claim Reserving in Non-Life Insurance*, North-Holland, 1986.

[18] Van Eeghen, J., *Loss Reserving Methods*, (Vol. 1 in the series *Surveys of Actuarial Studies*), Nationale-Nederlanden N.V., 1981.

[19] Wiser, R. F., "Loss Reserves," Chapter 4 in *Foundations of Casualty Actuarial Science*, (2nd edition), Casualty Actuarial Society, 1996, pp. 146–230.

APPENDIX

TABLE A1

LOSS DEVELOPMENT FACTOR FROM j TO ULTIMATE FOR VARIOUS DEVELOPMENT YEARS AND SETTLEMENT PERIODS (N)

	Development Year j									
N	0	1	2	3	4	5	6	7	8	9
	Results for $\mathrm{LDF}^{(1)}_{j,N}$:									
0	1.0000									
1	1.3345	1.0000								
2	1.6353	1.1253	1.0000							
3	1.9195	1.2627	1.0662	1.0000						
4	2.1903	1.4015	1.1495	1.0416	1.0000					
5	2.4564	1.5412	1.2384	1.0980	1.0288	1.0000				
6	2.6959	1.6715	1.3259	1.1590	1.0689	1.0208	1.0000			
7	2.9402	1.8007	1.4137	1.2232	1.1152	1.0516	1.0159	1.0000		
8	3.1853	1.9333	1.5038	1.2894	1.1650	1.0881	1.0403	1.0126	1.0000	
9	3.4169	2.0584	1.5905	1.3542	1.2153	1.1270	1.0691	1.0321	1.0101	1.0000
	Results for $\mathrm{LDF}^{(2)}_{j,N}$:									
0	1.0000									
1	1.3871	1.0000								
2	1.7247	1.1347	1.0000							
3	2.0379	1.2826	1.0691	1.0000						
4	2.3333	1.4312	1.1569	1.0428	1.0000					
5	2.6221	1.5800	1.2505	1.1015	1.0294	1.0000				
6	2.8804	1.7182	1.3422	1.1649	1.0707	1.0211	1.0000			
7	3.1417	1.8547	1.4343	1.2316	1.1185	1.0526	1.0161	1.0000		
8	3.4033	1.9938	1.5281	1.3003	1.1698	1.0901	1.0409	1.0127	1.0000	
9	3.6511	2.1259	1.6188	1.3676	1.2219	1.1301	1.0704	1.0325	1.0102	1.0000

Notes: Development year 0 refers to the year in which the claim was incurred. N is the number of calendar years it takes to settle all claims occurring in the same calendar year. Thus $N = 0$ implies claims are settled in the calendar year of their occurrence.

TABLE A2

The Ratio Loss Development Factors For Various Development Years and Settlement Periods

	Development Year $j/j+1$									
N	0/1	1/2	2/3	3/4	4/5	5/6	6/7	7/8	8/9	9/10
	Results for $\mathrm{LDF}^{(1)}_{j,N}/\mathrm{LDF}^{(1)}_{j+1,N}$:									
1	1.3345									
2	1.4531	1.1253								
3	1.5201	1.1843	1.0662							
4	1.5628	1.2192	1.1036	1.0416						
5	1.5938	1.2445	1.1279	1.0673	1.0288					
6	1.6129	1.2606	1.1440	1.0843	1.0472	1.0208				
7	1.6328	1.2737	1.1558	1.0968	1.0605	1.0351	1.0159			
8	1.6476	1.2856	1.1663	1.1068	1.0707	1.0460	1.0273	1.0126		
9	1.6600	1.2941	1.1745	1.1143	1.0784	1.0541	1.0359	1.0218	1.0101	
	Results for $\mathrm{LDF}^{(2)}_{j,N}/\mathrm{LDF}^{(2)}_{j+1,N}$:									
1	1.3871									
2	1.5200	1.1347								
3	1.5889	1.1997	1.0691							
4	1.6303	1.2371	1.1094	1.0428						
5	1.6595	1.2635	1.1353	1.0700	1.0294					
6	1.6764	1.2801	1.1522	1.0880	1.0486	1.0211				
7	1.6939	1.2931	1.1646	1.1011	1.0626	1.0359	1.0161			
8	1.7070	1.3047	1.1753	1.1115	1.0732	1.0472	1.0278	1.0127		
9	1.7175	1.3133	1.1836	1.1193	1.0813	1.0557	1.0367	1.0221	1.0102	

Notes: Development year 0 refers to the year in which the claim was incurred. N is the number of calendar years it takes to settle all claims occurring in the same calendar year. Thus $N = 0$ implies claims are settled in the calendar year of their occurrence.

TABLE A3

$\mathrm{LDF}_{j,N}^{(1)}$ FOR THE TRUNCATED EXPONENTIAL DISTRIBUTION WITH PARAMETER λ

	Development Year j									
N	0	1	2	3	4	5	6	7	8	9
	$\lambda = 1$:									
0	1.0000									
1	1.3261	1.0000								
2	1.6130	1.1209	1.0000							
3	1.8811	1.2514	1.0634	1.0000						
4	2.1304	1.3823	1.1424	1.0396	1.0000					
5	2.3776	1.5139	1.2264	1.0931	1.0273	1.0000				
6	2.6049	1.6376	1.3101	1.1513	1.0655	1.0197	1.0000			
7	2.8284	1.7576	1.3920	1.2117	1.1093	1.0487	1.0150	1.0000		
8	3.0482	1.8786	1.4756	1.2737	1.1562	1.0834	1.0381	1.0119	1.0000	
9	3.2585	1.9959	1.5570	1.3349	1.2036	1.1200	1.0654	1.0303	1.0095	1.0000
	$\lambda = 5$:									
0	1.0000									
1	1.2041	1.0000								
2	1.3468	1.0672	1.0000							
3	1.4597	1.1323	1.0331	1.0000						
4	1.5554	1.1912	1.0716	1.0200	1.0000					
5	1.6469	1.2461	1.1103	1.0456	1.0133	1.0000				
6	1.7291	1.2954	1.1467	1.0725	1.0315	1.0094	1.0000			
7	1.7950	1.3388	1.1802	1.0989	1.0515	1.0231	1.0070	1.0000		
8	1.8582	1.3814	1.2124	1.1244	1.0717	1.0385	1.0176	1.0054	1.0000	
9	1.9291	1.4239	1.2447	1.1508	1.0932	1.0555	1.0304	1.0141	1.0044	1.0000
	$\lambda = 10$:									
0	1.0000									
1	1.1094	1.0000								
2	1.1747	1.0346	1.0000							
3	1.2211	1.0656	1.0169	1.0000						
4	1.2589	1.0923	1.0357	1.0100	1.0000					
5	1.2895	1.1155	1.0536	1.0226	1.0066	1.0000				
6	1.3190	1.1360	1.0702	1.0355	1.0157	1.0047	1.0000			
7	1.3454	1.1551	1.0861	1.0486	1.0258	1.0117	1.0036	1.0000		
8	1.3649	1.1711	1.0997	1.0603	1.0355	1.0193	1.0089	1.0028	1.0000	
9	1.3904	1.1879	1.1136	1.0722	1.0456	1.0275	1.0152	1.0071	1.0022	1.0000

TABLE A4

$\mathrm{LDF}^{(2)}_{j,N}$ FOR THE TRUNCATED EXPONENTIAL DISTRIBUTION WITH PARAMETER λ

	Development Year j									
N	0	1	2	3	4	5	6	7	8	9
	$\lambda = 1$:									
0	1.0000									
1	1.3791	1.0000								
2	1.7023	1.1300	1.0000							
3	2.0004	1.2710	1.0662	1.0000						
4	2.2764	1.4115	1.1494	1.0408	1.0000					
5	2.5492	1.5530	1.2382	1.0964	1.0279	1.0000				
6	2.7976	1.6850	1.3263	1.1571	1.0673	1.0200	1.0000			
7	3.0423	1.8129	1.4124	1.2199	1.1124	1.0498	1.0152	1.0000		
8	3.2778	1.9405	1.5000	1.2843	1.1609	1.0853	1.0387	1.0120	1.0000	
9	3.5078	2.0654	1.5854	1.3480	1.2099	1.1229	1.0666	1.0307	1.0096	1.0000
	$\lambda = 5$:									
0	1.0000									
1	1.2424	1.0000								
2	1.4106	1.0717	1.0000							
3	1.5424	1.1419	1.0343	1.0000						
4	1.6542	1.2056	1.0744	1.0205	1.0000					
5	1.7630	1.2651	1.1150	1.0468	1.0135	1.0000				
6	1.8570	1.3179	1.1531	1.0745	1.0320	1.0095	1.0000			
7	1.9343	1.3650	1.1882	1.1018	1.0525	1.0234	1.0071	1.0000		
8	2.0093	1.4113	1.2220	1.1282	1.0732	1.0390	1.0177	1.0054	1.0000	
9	2.0895	1.4569	1.2559	1.1555	1.0953	1.0564	1.0307	1.0142	1.0044	1.0000
	$\lambda = 10$:									
0	1.0000									
1	1.1246	1.0000								
2	1.1978	1.0360	1.0000							
3	1.2486	1.0682	1.0172	1.0000						
4	1.2896	1.0959	1.0364	1.0101	1.0000					
5	1.3235	1.1199	1.0547	1.0229	1.0067	1.0000				
6	1.3546	1.1409	1.0716	1.0360	1.0158	1.0048	1.0000			
7	1.3847	1.1608	1.0879	1.0493	1.0260	1.0118	1.0036	1.0000		
8	1.4054	1.1773	1.1018	1.0612	1.0359	1.0194	1.0090	1.0028	1.0000	
9	1.4342	1.1947	1.1160	1.0733	1.0460	1.0277	1.0153	1.0071	1.0022	1.0000