SIMPSON'S PARADOX, CONFOUNDING VARIABLES, AND INSURANCE RATEMAKING

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Abstract

The insurance process is complex, with numerous factors combining to produce both premiums and losses. When compiling rates, actuaries often aggregate data from more than one source, while at the same time stratifying the data to achieve homogeneity. Such exercises may lead to biased and sometimes even surprising results, called Simpson's paradox, because the variables involved in the aggregation process or the stratification process are confounded by the presence of other variables. In this paper, we will describe Simpson's paradox and confounding and the statistical underpinning associated with those phenomena. We will further discuss how such bias may exist in P&C actuarial rating applications and solutions that can resolve the bias.

1. INTRODUCTION

An actuary is asked by the CEO for a small insurance company to examine the good student discount that the company offers. The discount is currently fifteen percent, but several competitors offer a twenty percent discount for qualifying youthful operators. As usual, the CEO is in a hurry, so the actuary compiles the experience and develops a relativity based on the pure premiums for all youthful operators (Age 15 to 25). Imagine the actuary's shock when the experience indicates, not the twenty percent discount for which the CEO had been hoping, but a twenty percent surcharge. The loss experience appears in Table 1.

TABLE 1

WITHOUT GOOD STUDENT DISCOUNT			
Exposures	Distribution	Losses	Pure Premium
18,980	86.3%	$44,210,062	$2,329

WITH GOOD STUDENT DISCOUNT				
Exposures	Distribution	Losses	Pure Premium	Relativity
3,020	13.7%	$8,475,292	$2,806	20%

TABLE 2

WITHOUT GOOD STUDENT DISCOUNT				
Age	Exposures	Distribution Within Age	Losses	Pure Premium
15–18	5,500	68.8%	$21,661,344	$3,938
19–21	5,580	93.0%	$12,488,608	$2,238
22–25	7,900	98.8%	$10,060,110	$1,273
Total	18,980		$44,210,062	$2,329

WITH GOOD STUDENT DISCOUNT					
Age	Exposures	Distribution Within Age	Losses	Pure Premium	Relativity
15–18	2,500	31.3%	$7,653,680	$3,061	−22%
19–21	420	7.0%	$705,002	$1,679	−25%
22–25	100	1.3%	$116,610	$1,166	−8%
Total	3,020		$8,475,292	$2,806	20%

The actuary knows of the problems incumbent with pure premiums, but certainly they can't cause this magnitude of a disparity. The actuary decides to review the experience by driver age that is available from the company's class plan. Table 2 displays that experience.

The relativities by class appear more reasonable, but the actuary still has a concern. How can the "average" of these three

TABLE 3

WITHOUT GOOD STUDENT DISCOUNT				
Age	Exposures	Distribution Within Age	Losses	Pure Premium
15	1,300	65.0%	$6,500,000	$5,000
16	1,300	65.0%	$5,525,000	$4,250
17	1,350	67.5%	$4,876,875	$3,613
18	1,550	77.5%	$4,759,469	$3,071
19	1,860	93.0%	$4,854,658	$2,610
20	1,860	93.0%	$4,126,459	$2,219
21	1,860	93.0%	$3,507,490	$1,886
22	1,920	96.0%	$3,077,540	$1,603
23	1,980	99.0%	$2,697,656	$1,362
24	2,000	100.0%	$2,316,169	$1,158
25	2,000	100.0%	$1,968,744	$984
Total	18,980		$44,210,062	$2,329

WITH GOOD STUDENT DISCOUNT					
Age	Exposures	Distribution Within Age	Losses	Pure Premium	Relativity
15	700	35.0%	$2,625,000	$3,750	–25%
16	700	35.0%	$2,231,250	$3,187	–25%
17	650	32.5%	$1,761,094	$2,709	–25%
18	450	22.5%	$1,036,336	$2,303	–25%
19	140	7.0%	$274,053	$1,958	–25%
20	140	7.0%	$232,945	$1,664	–25%
21	140	7.0%	$198,003	$1,414	–25%
22	80	4.0%	$96,173	$1,202	–25%
23	20	1.0%	$20,437	$1,022	–25%
24	—	0.0%	—	—	0%
25	—	—	—	—	0%
Total	3,020		$8,475,292	$2,806	20%

discounts produce a surcharge? The actuary is also concerned about the variation in the indicated relativities. The actuary requests data by driver age from the company's IS department and reviews the experience, which is displayed in Table 3.

By further stratifying the data, even more precision appears to be achieved and it appears that an even higher discount is

TABLE 4

	Male			Female		
School	Applying	Accepted	Acceptance Ratio	Applying	Accepted	Acceptance Ratio
Engineering	1000	400	40%	200	100	50%
Arts	200	20	10%	1000	125	13%
Total	1200	420	35%	1200	225	19%

justified. In addition, the same discount seems to be supported for all driver ages. Nevertheless, the question remains: "How does the accumulation of all these discounts produce a surcharge?" The answer is Simpson's paradox.

2. SIMPSON'S PARADOX

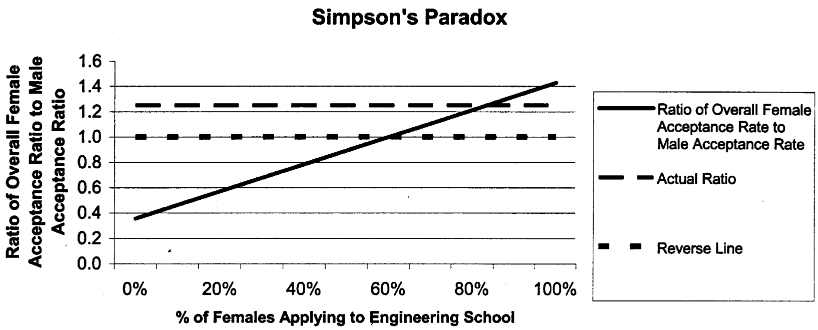
E. H. Simpson first described the paradox in 1951 in a paper titled "The Interpretation of Interaction in Contingency Tables" [14]. It is an interesting statistical phenomenon that causes a potential bias in certain data analyses. The paradox occurs when a relationship or association between two variables reverses when a third factor, called a confounding variable, is introduced. The paradox also occurs when a relationship/association reverses when the data is aggregated over a confounding variable.

2.1. The College Admissions Example

The classic illustration of the paradox involves college admissions by gender, which can be illustrated in the example in Table 4 [3].

In Table 4, the overall acceptance ratio for female applicants, 19%, is lower than the ratio for the male applicants, 35%. However, this relationship reverses when the factor of the school to which they apply is introduced. When this variable is considered, the acceptance ratio for female applicants is 25% higher

FIGURE 1



than male applicants for both the engineering school (50% to 40%) and the art school (13% to 10%).

The reason why Simpson's paradox occurs is that more female applicants apply to the art school, which has an overall lower acceptance rate than the engineering school. The engineering school has a 40% to 50% acceptance rate, while the art school has a 10% to 13% acceptance rate. In the above example, about 83% of female applicants apply to the art school, while 83% of male applicants apply to the engineering school.

Let's vary the percentage of the female applicants applying to the art school and assume all the other parameters in the example remain the same. Then, calculate the ratio of the overall female applicants to the male applicants.

In Figure 1, the solid line represents the ratio of the overall female acceptance rate to the overall male acceptance rate by varying the percentage of females applying to the engineering school. We know that the underlying ratio is 1.25 when we analyze the acceptance by school, and the dashed line represents the actual ratio of 1.25.

We can see that only when the percentage of female students applying to the engineering school is 83% is the overall ratio the same as the true ratio. This 83% is the same percentage as the male students applying to the engineering school. For all the other percentages, the overall ratio is different from the true ratio.

Another interesting point indicated in Figure 1 is that when the percentage of female students applying to the engineering school is less than 60%, the ratio of the overall acceptance rate of female to male is less than 1.00, represented by the dotted line, suggesting that the overall female acceptance rate is lower. This is a reversal of the fact that the female acceptance rate is higher than the male acceptance rate, which is Simpson's paradox [12].

From the example above, we can see that Simpson's paradox occurs when the distributions of the sample population are not uniform across the two predictive variables. When this takes place, the variable of "school" is confounding the acceptance rate and is confusing the relationship between the acceptance rate and applicants' gender. We will discuss the concept of confounding variables in detail later.

2.2. The Simple Math of Simpson's Paradox

Simpson's paradox arises from one simple mathematical truth. Given eight real numbers: a, b, c, d, A, B, C, D with the following properties: $a/A > b/B$ and $c/C > d/D$, then it is not necessarily true that $(a+c)/(A+C) > (b+d)/(B+D)$. In fact, it may be true that: $(a+c)/(A+C) < (b+d)/(B+D)$. This is Simpson's paradox. This is an obvious math reality, yet it has significant ramifications in Bayesian analysis, medical research, science and engineering studies, societal statistical analysis and yes, insurance ratemaking. It is of concern for any statistical activity involving the calculation and analysis of ratios of two measurements. This activity is prevalent in insurance; loss ratios, pure premium, frequency, severity and loss development factors are just some of the statistics involving the ratio of two measures.

3. CONFOUNDING VARIABLES

A variable can confound the results of a statistical analysis only if it is related (non-independent) to both the dependent variable and at least one of the other (independent) variables in the analysis. More specifically, a variable can confound the results of an insurance rate structure analysis only if it is related (non-independent) to both the experience measure (loss ratio, pure premium, etc.) and at least one of the other rating variables in the analysis.

3.1. Experimental Design

Confounding and Simpson's paradox are of great concern in the design of research studies. For example, in a typical design of medical research, researchers would like to know the impact of an intervention measure. Using the notation introduced in Section 2.2, assume that A and C are the number of observations where the intervention has taken place. B and D are the number of observations in the group where the intervention has not been executed (the control group). The distinction between the A and C (and also B and D) observations is the potential confounding variable. For example, in Cates [5], A and C would represent smokers attempting to quit with nurse intervention (the intervention) from two different studies (the potential confounding variable).[1] Also, in our previous college admission example, A and C might represent the number of females (the intervention) applying to the art and engineering schools (the potential confounding variable) respectively, as displayed in Table 5.

Further, the number of events is represented by a, b, c and d and the ratio a/A is the proportion of events per number of

[1]Cates [5] described the meta-analysis of smokers attempting to quit with and without high intensity nurse intervention. Cates illustrated several methods of combining studies from independent sources. Methods included Maentel-Hensel fixed effects method and a random effects methodology. Both of these methodologies produced weights that were used to combine the risk differences, rather than the underlying data. Cates showed that a reversal (Simpson's paradox) occurred when the raw data were combined.

TABLE 5

			Variables Under Study	
		Example	1 Females	2 Males
Confounding Variable Value 1	Number of Events	Art School	a	b
	Number of Observations		A	B
Confounding Variable Value 2	Number of Events	Engineering School	c	d
	Number of Observations		C	D

observations; e.g., the percentage of females being admitted to art school or the proportion of smokers in Study #1 (of the Cates paper) who quit with the aid of a nurse.

While both the college admission example and the smoking intervention example involve studies where existing data are observed and analyzed, assume for a moment that this is not the case—that we can design an experiment in such manner as to minimize the bias of any potential confounding variable. Ultimately, we find the bias is eliminated if the confounding variable and the variable under study are independent. The bias is also eliminated if either the groups are balanced (possess an equal number of observations) or are proportionally distributed (there is the same ratio of observations of the variable under study for each value of the confounding variable).[2] It is possible to illustrate this using the following argument.

Consider an experiment with groups A, B, C, D as described above. Also assume that the ratio differences are known and are equal to some K: $(a/A) - (b/B) = K = (c/C) - (d/D)$.

[2]Of course, the balance condition is a special case of the proportional condition. The balance condition is especially important in experiment design.

How can the experiment be designed so that $(a+c)/(A+C) - (b+d)/(B+D) = K$?

First assume that the potential confounding variable is independent of the variable under study, i.e., that $a/A = c/C$ and $b/B = d/D$. Therefore $A = aC/c$ and $B = bD/d$ and

$$\frac{a+c}{\dfrac{aC}{c}+C} - \frac{b+d}{\dfrac{bD}{d}+D} = \frac{a+c}{\dfrac{C}{c}(a+c)} - \frac{b+d}{\dfrac{D}{d}(b+d)} = \frac{c}{C} - \frac{d}{D} = K.$$

Therefore, if the potential confounding variable and the variable under study are independent then there is no confounding.

Now instead of assuming independence, assume that the experiment has a balanced distribution; i.e., there is the same number of observations in each group relative to the variable under study (the same number of females applying to the art school and the engineering school and the same number of males applying to both schools). Then $A = C$ and $B = D$. And

$$\begin{aligned}\frac{a+c}{A+C} - \frac{b+d}{B+D} &= \frac{a}{A+C} + \frac{c}{A+C} - \frac{b}{B+D} - \frac{d}{B+D}\\ &= \frac{a}{A+A} + \frac{c}{C+C} - \frac{b}{B+B} - \frac{d}{D+D}\\ &= \frac{1}{2}\left[\frac{a}{A} + \frac{c}{C} - \frac{b}{B} - \frac{d}{D}\right]\\ &= \frac{1}{2}\left[\frac{a}{A} - \frac{b}{B} + \frac{c}{C} - \frac{d}{D}\right]\\ &= \frac{1}{2}[K+K] = K.\end{aligned}$$

So there is no confounding if the observations possess a balanced distribution.

Now assume that the experiment is proportionally distributed; i.e., there is the same ratio of observations of the variable under study for each value of the confounding variable ($A/B = C/D$).

That is, the ratio of females applying to the art school to the number of males applying to the art school is the same as the ratio of females applying to the engineering school to the number of males applying to the engineering school. If $A/B = C/D$, then define $A/C = B/D = K'$. Then $A = CK'$, $B = DK'$. Therefore

$$\begin{aligned}
\frac{a+c}{A+C} - \frac{b+d}{B+D} &= \frac{a+c}{CK'+C} - \frac{b+d}{DK'+D} \\
&= \frac{1}{K'+1}\left(\frac{a+c}{C} - \frac{b+d}{D}\right) \\
&= \frac{1}{K'+1}\left(\frac{a}{C} + \frac{c}{C} - \frac{b}{D} - \frac{d}{D}\right) \\
&= \frac{1}{K'+1}\left(\frac{a}{C} - \frac{b}{D} + K\right) \\
&= \frac{1}{K'+1}\left(\frac{a}{\frac{A}{K'}} - \frac{b}{\frac{B}{K'}} + K\right) \\
&= \frac{1}{K'+1}\left[K'\left(\frac{a}{A} - \frac{b}{B}\right) + K\right] \\
&= \frac{1}{K'+1}(K'K + K) = \frac{K'+1}{K'+1}K = K.
\end{aligned}$$

Therefore, if the observations are proportionally distributed, there is no confounding.

In the example detailed in the introduction of the paper, the good student pure premiums and ultimately the indicated good student discount were confounded by driver age. It is not surprising that there is the observed relationship between the distribution of drivers by age and those with the good student discount. As driver age approaches 25, fewer are students, much less good students. The reversal occurs since there is a higher distribution of young drivers with good student discount and young drivers have higher pure premiums.

Important Principle: If there is independence between the potential confounding variable and the variable under study, or if the study is balanced or proportionally distributed, then there is no confounding.

Insurance ratemaking differs from most statistical studies in a number of ways:

1. It is generally not possible to design the makeup of groups of insureds so that classifications are balanced.
2. Generally there are far more values for each variable and probably more variables in insurance than in research analysis.
3. In most statistical studies, the objective is to accept or reject a hypothesis. The primary concern in insurance ratemaking is to properly calculate a rate, which requires a continuous rather than binary output.

In the next four sections, we will further examine and extend the Important Principle of confounding to more than two variables using general statistical models and experimental design theories. The two statistical models that we will use are the simple additive and the multiplicative models, both without an interaction term. Such additive and multiplicative multivariable models are the ideal models, and are similar to many insurance rating and class plan structures [1]. For illustrative purposes, we will use a 2-by-2 rating example with age of driver (youthful drivers vs. adult drivers) and territory (urban territories vs. suburban territories) throughout the sections. For more details of the additive and multiplicative statistical models and experimental design theories, please see Montgomery [9] and Neter, et al. [10].

3.2. The Confounding Effect on an Additive Model with No Interaction Term

Let's start with a 2-by-2 additive model. Assume that the observation or exposure distribution of each cell is $w_{i_1 i_2}$. Later we will extend the models to more dimensions and values.

Define: $w_{(i_1),1} = w_{i_1,1}/\sum_{i_2} w_{i_1,i_2}$; e.g.,

$$w_{(1),1} = \frac{w_{1,1}}{w_{1,1}+w_{1,2}}, \qquad w_{(1),2} = \frac{w_{1,2}}{w_{1,1}+w_{1,2}},$$

$$w_{(2),1} = \frac{w_{2,1}}{w_{2,1}+w_{2,2}}, \qquad w_{(2),2} = \frac{w_{2,2}}{w_{2,1}+w_{2,2}}.$$

Note: While this notation may be unfamiliar, please accept this verbal interpretation. If w_{i_1,i_2} represents the exposures in cell i_1, i_2, then $w_{(i_1),1}$ represents the marginal exposure distribution of cell i_1, i_2 for cells with $i_2 = 1$.

For a linearly independent additive model, the mean value (underlying rate) for each of the 2-by-2 cells can be represented as follows: $\mu_{i_1 i_2} = \mu + \mu_{i_1,\bullet} + \mu_{\bullet,i_2}$: $i_1 = 1,2$, $i_2 = 1,2$, where a dot ($\bullet$) index indicates the mean across that index.

By linearly independent we mean that there are no interaction terms. If the model were not linearly independent, the mean value (underlying rate) for each of the 2-by-2 cells would be represented as: $\mu_{i_1 i_2} = \mu + \mu_{i_1,\bullet} + \mu_{\bullet,i_2} + \varepsilon_{i_1,i_2}$: $i_1 = 1,2$, $i_2 = 1,2$, where ε_{i_1,i_2} is the interaction term.

More specifically, we define the following for the 2-by-2 age of driver and territory example: $\mu_{i_1 i_2} = \mu + \mu_{i_1,\bullet}$(Age of Driver) + $\mu_{\bullet,i_2}$(Vehicle Territory).

Now we want to compare the difference in the aggregate rate between adult and youthful drivers:

Then the aggregate rate for each i_1 is $m_{i_1,\bullet} = \mu_{i_1,1} w_{(i_1),1} + \mu_{i_1,2} w_{(i_1),2}$.

And

$$m_{1,\bullet} - m_{2,\bullet} = \mu_{1,1} w_{(1),1} + \mu_{1,2} w_{(1),2} - \mu_{2,1} w_{(2),1} - \mu_{2,2} w_{(2),2}.$$

Then

$$\begin{aligned} m_{1,\bullet} - m_{2,\bullet} = {} & w_{(1),1}(\mu + \mu_{1,\bullet} + \mu_{\bullet,1}) + w_{(1),2}(\mu + \mu_{1,\bullet} + \mu_{\bullet,2}) \\ & - w_{(2),1}(\mu + \mu_{2,\bullet} + \mu_{\bullet,1}) - w_{(2),2}(\mu + \mu_{2,\bullet} + \mu_{\bullet,2}). \end{aligned}$$

If $w_{(1),1} = w_{(2),1}$ and $w_{(1),2} = w_{(2),2}$, then

$$\begin{aligned} m_{1,\bullet} - m_{2,\bullet} &= w_{(1),1}(\mu + \mu_{1,\bullet} + \mu_{\bullet,1}) + w_{(1),2}(\mu + \mu_{1,\bullet} + \mu_{\bullet,2}) \\ &\quad - w_{(1),1}(\mu + \mu_{2,\bullet} + \mu_{\bullet,1}) - w_{(1),2}(\mu + \mu_{2,\bullet} + \mu_{\bullet,2}) \\ &= w_{(1),1}(\mu_{1,\bullet} - \mu_{2,\bullet}) + w_{(1),2}(\mu_{1,\bullet} - \mu_{2,\bullet}) = \mu_{1,\bullet} - \mu_{2,\bullet}. \end{aligned}$$

Since for the 2-by-2 case: $w_{(1),1} + w_{(1),2} = 1$, we can derive the same results for the other factor, the vehicle territory.

If $w_{1,(1)} = w_{2,(1)}$ and $w_{2,(1)} = w_{2,(2)}$, then $m_{\bullet,2} - m_{\bullet,1} = \mu_{\bullet,2} - \mu_{\bullet,1}$.

Therefore, territory does not confound the age of driver relativities for this 2-by-2 linearly independent additive model if territorial distribution of exposures is independent of the age of driver distribution of exposures. That is, if $w_{(1),1} = w_{(2),1}$, $w_{(1),2} = w_{(2),2}$, $w_{1,(1)} = w_{2,(1)}$ and $w_{1,(2)} = w_{2,(2)}$. This is a *proportional* distribution. Of course, a special case for such a distribution occurs when each cell has the same number of data points, $w_{1,1} = w_{1,2} = w_{2,1} = w_{2,2}$. This is a *balanced* distribution.

The following is a numerical example that illustrates such an additive model. The statistics for the example are as follows:

$$\mu_{i_1 i_2} = \mu + \mu_{i_1,\bullet}(\text{Age of Driver}) + \mu_{\bullet,i_2}(\text{Vehicle Territory})$$

Let $\mu = \$400$,

$$\begin{aligned} \mu_{1,\bullet} &= +\$100 \text{ for youthful drivers} \\ \mu_{2,\bullet} &= -\$100 \text{ for adult drivers,} \\ \mu_{\bullet,1} &= +\$100 \text{ for urban drivers,} \\ \mu_{\bullet,2} &= -\$100 \text{ for suburban drivers.} \end{aligned}$$

Therefore, the pure premiums for each of the four combinations are:

$$\begin{aligned} \mu_{1,1} &= \$600, \qquad \mu_{1,2} = \$400, \\ \mu_{2,1} &= \$400, \qquad \mu_{2,2} = \$200. \end{aligned}$$

TABLE 6

		Urban	Suburban	Total
Youthful	Total Loss	\$3,000	\$6,000	\$9,000
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$600	\$400	\$450
Adult	Total Loss	\$2,000	\$3,000	\$5,000
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$400	\$200	\$250
Total	Total Loss	\$5,000	\$9,000	\$14,000
	Exposures	10	30	40
	Distribution	25.0%	75.0%	100.0%
	Pure Premium	\$500	\$300	\$350

Also assume that

$$w_{1,1} = 12.5\%, \qquad w_{1,2} = 37.5\%,$$
$$w_{2,1} = 12.5\%, \qquad w_{2,2} = 37.5\%.$$

If we study Table 6, we can see that the difference between youthful driver underlying rate and the adult driver underlying rate is: $\mu_{1,\bullet} - \mu_{2,\bullet} = \$200$, which is the same as the difference between the aggregate rates, \$450 – \$250 = \$200. Therefore, in this case, confounding does not occur.

The data for the other factor, vehicle territory, yield the same result. The difference between underlying rates for the urban territory and the suburban territory is: $\mu_{\bullet,1} - \mu_{\bullet,2} = \$200$, which is the same as if we use the aggregate rates, \$500 – \$300 = \$200. Therefore, in this case as well, confounding does not occur.

Now consider a different distribution:

$$w_{1,1} = 12.5\%, \qquad w_{1,2} = 37.5\%,$$
$$w_{2,1} = 37.5\%, \qquad w_{2,2} = 12.5\%.$$

TABLE 7

		Urban	Suburban	Total
Youthful	Total Loss	\$3,000	\$6,000	\$9,000
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$600	\$400	\$450
Adult	Total Loss	\$6,000	\$1,000	\$7,000
	Exposures	15	5	20
	Distribution	37.5%	12.5%	50.0%
	Pure Premium	\$400	\$200	\$350
Total	Total Loss	\$9,000	\$7,000	\$16,000
	Exposures	20	20	40
	Distribution	50.0%	50.0%	100.0%
	Pure Premium	\$450	\$350	\$400

This distribution is neither balanced nor proportional. The confounding effect of territory on class (and vice versa) becomes apparent. Table 7 displays that in this case for the age of the driver factor, we can see that the difference between the underlying rate for youthful drivers and adult drivers is: $\mu_{1,\bullet} - \mu_{2,\bullet} = \$200.00$, as before. However the aggregate rate difference is \$450 – \$350 = \$100.

3.3. The Confounding Effect for an n-Dimensional Additive Model with No Interaction Term

Now we want to extend the linearly additive model from two dimensions to n dimensions. Also we will extend the number of values for each variable to more than two, that is, m values. This is because a typical insurance rating structure has many variables with multiple values. It is understood that the lower bound of the summation is equal to unity. Again, assume that the sample distribution of each cell is $w_{i_1 i_2 i_3 i_4 \ldots}$.

Define:

$$w_{(i_1),i_2,i_3,\ldots,i_n} = \frac{w_{i_1,i_2,i_3,\ldots,i_n}}{\sum_{i_2}^{m_2} \sum_{i_3}^{m_3} \sum_{i_4}^{m_4} \cdots \sum_{i_n}^{m_n} w_{i_1,i_2,i_3,\ldots,i_n}}.$$

For a linearly additive model, the mean value for each of the $\prod_i^n m_i$ cells can be represented as follows:

$$\mu_{i_1 i_2 i_3 \ldots i_n} = \mu + \mu_{i_1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n} :$$
$$i_1 = 1,2,3,\ldots,m_1, \quad i_2 = 1,2,3,\ldots,m_2,$$
$$i_3 = 1,2,3,\ldots,m_3, \quad i_n = 1,2,3,\ldots,m_n,$$

where a dot ($\bullet$) index indicates the mean across that index.

Again, we want to compare the difference in the aggregate rate and the underlying rate between any two values for the first factor, i_1.

Then the expected rate for each i_1 is

$$m_{i_1,\bullet,\bullet,\ldots,\bullet} = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{i_1,i_2,i_3,\ldots,i_n} w_{(i_1),i_2,i_3,\ldots,i_n}$$

and

$$m_{i_1,\bullet,\bullet,\ldots,\bullet} - m_{1,\bullet,\bullet,\ldots,\bullet} = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{i_1,i_2,i_3,\ldots,i_n} w_{(i_1),i_2,i_3,\ldots,i_n}$$
$$- \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{1,i_2,i_3,\ldots,i_n} w_{(1),i_2,i_3,\ldots,i_n}.$$

Then

$$\begin{aligned} m_{i_1,\bullet,\bullet,\ldots,\bullet} &- m_{1,\bullet,\bullet,\ldots,\bullet} \\ &= \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(i_1),i_2,i_3,\ldots,i_n} \\ &\quad \times (\mu + \mu_{i_1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n}) \\ &\quad - \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n} \\ &\quad \times (\mu + \mu_{1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n}). \end{aligned}$$

If $w_{(i_1),i_2,i_3,\ldots,i_n} = w_{(1),i_2,i_3,\ldots,i_n}$ then

$$
\begin{aligned}
&m_{i_1,\bullet,\bullet,\ldots,\bullet} - m_{1,\bullet,\bullet,\ldots,\bullet} \\
&\quad = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n} \\
&\qquad \times (\mu + \mu_{i_1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n}) \\
&\qquad - \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n} \\
&\qquad \times (\mu + \mu_{1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n}) \\
&\quad = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n} \\
&\qquad \times [(\mu + \mu_{i_1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n}) \\
&\qquad\quad - (\mu + \mu_{1,\bullet,\bullet,\ldots,\bullet} + \mu_{\bullet,i_2,\bullet,\ldots,\bullet} + \cdots + \mu_{\bullet,\bullet,\bullet,\ldots,i_n})] \\
&\quad = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}(\mu_{i_1,\bullet,\bullet,\ldots,\bullet} - \mu_{1,\bullet,\bullet,\ldots,\bullet}) \\
&\quad = (\mu_{i_1,\bullet,\bullet,\ldots,\bullet} - \mu_{1,\bullet,\bullet,\ldots,\bullet})\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}.
\end{aligned}
$$

But

$$\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n} = 1,$$

so $m_{i_1,\bullet,\bullet,\ldots,\bullet} - m_{1,\bullet,\bullet,\ldots,\bullet} = \mu_{i_1,\bullet,\bullet,\ldots,\bullet} - \mu_{1,\bullet,\bullet,\ldots,\bullet}$.

The *distribution* of the sample population is defined as proportional when:

$$
\begin{aligned}
w_{(i_1),i_2,i_3,\ldots,i_n} &= w_{(1),i_2,i_3,\ldots,i_n} && \text{for all} \quad i_1,\\
w_{i_1,(i_2),i_3,\ldots,i_n} &= w_{i_1,(1),i_3,\ldots,i_n} && \text{for all} \quad i_2,\\
w_{i_1,i_2,(i_3),\ldots,i_n} &= w_{i_1,i_2,(1),\ldots,i_n} && \text{for all} \quad i_3,\\
&\ldots\\
w_{i_1,i_2,i_3,\ldots,(i_n)} &= w_{i_1,i_2,i_3,\ldots,(1)} && \text{for all} \quad i_n.
\end{aligned}
$$

Confounding will not occur for the n-dimensional linearly additive model if the sample distribution is proportional.

3.4. The Confounding Effect on a Multiplicative Model with No Interaction Term

Let's start with a 2-by-2 multiplicative model without an interaction term. Assume that the sample distribution of each cell is $w_{i_1 i_2}$ as before.

Again define: $w_{(i_1),1} = w_{i_1,1}/\sum_{i_2} w_{i_1,i_2}$; e.g.,

$$
w_{(1),1} = \frac{w_{1,1}}{w_{1,1} + w_{1,2}}, \qquad w_{(1),2} = \frac{w_{1,2}}{w_{1,1} + w_{1,2}},
$$
$$
w_{(2),1} = \frac{w_{2,1}}{w_{2,1} + w_{2,2}}, \qquad w_{(2),2} = \frac{w_{2,2}}{w_{2,1} + w_{2,2}}.
$$

For a multiplicative model with no interaction term, the mean value for each of the 2-by-2 cells can be represented as follows:

$$
\mu_{i_1 i_2} = \mu \times \mu_{i_1,\bullet} \times \mu_{\bullet,i_2} : \qquad i_1 = 1,2, \quad i_2 = 1,2.
$$

More specifically, we define the following for the 2-by-2 age of driver and territory example:

$$
\mu_{i_1 i_2} = \mu \times \mu_{i_1,\bullet}(\text{Age of Driver}) \times \mu_{\bullet,i_2}(\text{Vehicle Territory}),
$$

where a dot ($\bullet$) index indicates the mean across that index.

Now we want to compare the difference in the aggregate rate and the underlying rate between adult and youthful drivers.

Then the expected rate for each i_1 is $m_{i_1,\bullet} = \mu_{i_1,1}w_{(i_1),1} + \mu_{i_1,2}w_{(i_1),2}$ and

$$\frac{m_{1,\bullet}}{m_{2,\bullet}} = \frac{\mu_{1,1}w_{(1),1} + \mu_{1,2}w_{(1),2}}{\mu_{2,1}w_{(2),1} + \mu_{2,2}w_{(2),2}}.$$

Then

$$\frac{m_{1,\bullet}}{m_{2,\bullet}} = \frac{w_{(1),1}(\mu\mu_{1,\bullet}\mu_{\bullet,1}) + w_{(1),2}(\mu\mu_{1,\bullet}\mu_{\bullet,2})}{w_{(2),1}(\mu\mu_{2,\bullet}\mu_{\bullet,1}) + w_{(2),2}(\mu\mu_{2,\bullet}\mu_{\bullet,2})}.$$

If $w_{(1),1} = w_{(2),1}$ and $w_{(1),2} = w_{(2),2}$, then

$$\frac{m_{1,\bullet}}{m_{2,\bullet}} = \frac{w_{(1),1}(\mu\mu_{1,\bullet}\mu_{\bullet,1}) + w_{(1),2}(\mu\mu_{1,\bullet}\mu_{\bullet,2})}{w_{(1),1}(\mu\mu_{2,\bullet}\mu_{\bullet,1}) + w_{(1),2}(\mu\mu_{2,\bullet}\mu_{\bullet,2})}$$

and

$$\frac{m_{1,\bullet}}{m_{2,\bullet}} = \frac{\mu_{1,\bullet}(w_{(1),1}\mu_{\bullet,1} + w_{(1),2}\mu_{\bullet,2})}{\mu_{2,\bullet}(w_{(1),1}\mu_{\bullet,1} + w_{(1),2}\mu_{\bullet,2})} = \frac{\mu_{1,\bullet}}{\mu_{2,\bullet}}.$$

Therefore, territory does not confound the age of driver relativities for this 2-by-2 multiplicative model if the territorial distribution of exposures is independent of the age of driver distribution of exposures. That is, if $w_{(1),1} = w_{(2),1}$, $w_{(1),2} = w_{(2),2}$, $w_{1,(1)} = w_{2,(1)}$ and $w_{1,(2)} = w_{2,(2)}$.

This occurs when the distributions among the predictive variables are independent and proportional to each other. Of course, a special case for such independent distributions is when each cell has the same number of data points; i.e., $w_{1,1} = w_{1,2} = w_{2,1} = w_{2,2}$. Again, this is a balanced distribution.

The following is a numerical example that illustrates such a multiplicative model. The statistics for the example are as follows:

$$\mu_{i_1 i_2} = \mu \times \mu_{i_1,\bullet}(\text{Age of Driver}) \times \mu_{\bullet,i_2}(\text{Vehicle Territory})$$

TABLE 8

		Urban	Suburban	Total
Youthful	Total Loss	\$3,750	\$3,750	\$7,500
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$750	\$250	\$375
Adult	Total Loss	\$2,250	\$2,250	\$4,500
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$450	\$150	\$225
Total	Total Loss	\$6,000	\$6,000	\$12,000
	Exposures	10	30	40
	Distribution	25.0%	75.0%	100.0%
	Pure Premium	\$600	\$200	\$300

Let $\mu = \$400$,

$$\mu_{1,\bullet} = 1.25 \text{ for youthful drivers}$$

$$\mu_{2,\bullet} = 0.75 \text{ for adult drivers,}$$

$$\mu_{\bullet,1} = 1.50 \text{ for urban drivers}$$

$$\mu_{\bullet,2} = 0.50 \text{ for suburban drivers.}$$

Therefore, the pure premiums for the four combinations are:

$$\mu_{1,1} = \$750, \qquad \mu_{1,2} = \$250,$$

$$\mu_{2,1} = \$450, \qquad \mu_{2,2} = \$150.$$

Also assume that $w_{1,1} = 12.5\%$, $w_{1,2} = 37.5\%$, $w_{2,1} = 12.5\%$, $w_{2,2} = 37.5\%$.

If we study Table 8 for the age of the driver factor, we can see that the underlying rate for the difference between youthful drivers and adult drivers is $\mu_{1,\bullet}/\mu_{2,\bullet} = 1.25/0.75 = 1.67$, which is the same as if we use the aggregate rate, \$375/\$225 = 1.67. Therefore, in this case, confounding does not occur.

TABLE 9

		Urban	Suburban	Total
Youthful	Total Loss	\$3,750	\$3,750	\$7,500
	Exposures	5	15	20
	Distribution	12.5%	37.5%	50.0%
	Pure Premium	\$750	\$250	\$375
Adult	Total Loss	\$6,750	\$750	\$5,000
	Exposures	15	5	20
	Distribution	37.5%	12.5%	50.0%
	Pure Premium	\$450	\$150	\$250
Total	Total Loss	\$10,500	\$4,500	\$15,000
	Exposures	20	20	40
	Distribution	50.0%	50.0%	100.0%
	Pure Premium	\$525	\$225	\$375

Now assume a different distribution:

$$w_{1,1} = 12.5\%, \qquad w_{1,2} = 37.5\%,$$
$$w_{2,1} = 37.5\%, \qquad w_{2,2} = 12.5\%.$$

This distribution is neither balanced nor proportional, and the confounding effect of territory on class (and vice versa) is again obvious. Table 9 shows that in this case for the age of the driver factor, the relationship between the underlying rates for youthful drivers and the adult drivers is $\mu_{1,\bullet}/\mu_{2,\bullet} = 1.25/0.75 = 1.67$, as before. However the aggregate rate is biased; \$375/\$250 = 1.50.

3.5. The Confounding Effect on an n-Dimensional Multiplicative Model with No Interaction Term

Now, we want to extend the multiplicative model from two dimensions to *n* dimensions. In addition, for each variable, we will extend the number of values to more than two, that is, *m* values. Again, assume that the sample distribution of each cell is $w_{i_1,i_2,i_3,i_4,\ldots}$.

Define:

$$w_{(i_1),i_2,i_3,\ldots,i_n} = \frac{w_{i_1,i_2,i_3,\ldots,i_n}}{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{i_1,i_2,i_3,\ldots,i_n}}.$$

For a multiplicative model, the mean value for each of the $\prod_1^n m_i$ cells can be represented as follows:

$$\mu_{i_1,i_2,i_3\ldots i_n} = \mu\mu_{i_1,\bullet,\bullet,\ldots,\bullet}\mu_{\bullet,i_2,\bullet,\ldots,\bullet}\cdots\mu_{\bullet,\bullet,\bullet,\ldots,i_n}:$$

$$i_1 = 1,2,3,\ldots,m_1, \qquad i_2 = 1,2,3,\ldots,m_2,$$

$$i_3 = 1,2,3,\ldots,m_3,\ \ldots\ ,i_n = 1,2,3,\ldots,m_n,$$

where a dot index indicates the mean across that index. Again we want to compare the difference in the aggregate rate and the underlying rate between any two values for the first factor, i_1.

Then the expected rate for each i_1 is

$$m_{i_1,\bullet,\bullet,\ldots,\bullet} = \sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{i_1,i_2,i_3,\ldots,i_n}w_{(i_1),i_2,i_3,\ldots,i_n}$$

and

$$\frac{m_{i_1,\bullet,\bullet,\ldots,\bullet}}{m_{1,\bullet,\bullet,\ldots,\bullet}} = \frac{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{i_1,i_2,i_3,\ldots,i_n}w_{(i_1),i_2,i_3,\ldots,i_n}}{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n}\mu_{1,i_2,i_3,\ldots,i_n}w_{(1),i_2,i_3,\ldots,i_n}}.$$

Then

$$\frac{m_{i_1,\bullet,\bullet,\ldots,\bullet}}{m_{1,\bullet,\bullet,\ldots,\bullet}}$$

$$= \frac{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(i_1),i_2,i_3,\ldots,i_n}(\mu\mu_{i_1,\bullet,\bullet,\ldots,\bullet}\mu_{\bullet,i_2,\bullet,\ldots,\bullet}\cdots\mu_{\bullet,\bullet,\bullet,\ldots,i_n})}{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}(\mu\mu_{1,\bullet,\bullet,\ldots,\bullet}+\mu_{\bullet,i_2,\bullet,\ldots,\bullet}\cdots\mu_{\bullet,\bullet,\bullet,\ldots,i_n})}.$$

If $w_{(i_1),i_2,i_3,\ldots,i_n} = w_{(1),i_2,i_3,\ldots,i_n}$ for all i_1 then

$$\frac{m_{i_1,\bullet,\bullet,\ldots,\bullet}}{m_{1,\bullet,\bullet,\ldots,\bullet}}$$

$$= \frac{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}(\mu\mu_{i_1,\bullet,\bullet,\ldots,\bullet}\mu_{\bullet,i_2,\bullet,\ldots,\bullet}\cdots\mu_{\bullet,\bullet,\bullet,\ldots,i_n})}{\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}(\mu\mu_{1,\bullet,\bullet,\ldots,\bullet}\mu_{\bullet,i_2,\bullet,\ldots,\bullet}\cdots\mu_{\bullet,\bullet,\bullet,\ldots,i_n})}$$

$$= \frac{(\mu+\mu_{\bullet,i_2,\bullet,\ldots,\bullet}+\cdots+\mu_{\bullet,\bullet,\bullet,\ldots,i_n})\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}\mu_{i_1,\bullet,\bullet,\ldots,\bullet}}{(\mu+\mu_{\bullet,i_2,\bullet,\ldots,\bullet}+\cdots+\mu_{\bullet,\bullet,\bullet,\ldots,i_n})\sum_{i_2}^{m_2}\sum_{i_3}^{m_3}\sum_{i_4}^{m_4}\cdots\sum_{i_n}^{m_n} w_{(1),i_2,i_3,\ldots,i_n}\mu_{1,\bullet,\bullet,\ldots,\bullet}}$$

$$= \frac{\mu_{i_1,\bullet,\bullet,\ldots,\bullet}}{\mu_{1,\bullet,\bullet,\ldots,\bullet}}.$$

Confounding will not occur for the n-dimensional multiplicative model if the sample distribution meets the above independence or proportionality conditions.

4. TYPES OF CONFOUNDING VARIABLES

A variable that confounds the results of a study does so in essentially the same way regardless of the nature of the variable under study or the confounding variable itself. However, the nature of the variable may affect its identification and treatment. For the purpose of this paper, confounding variables will be categorized as one of three types: stratification confounding variable, aggregation confounding variable or lurking confounding variable.

4.1. Stratification Confounding Variable

In order to properly price a pool of risks, it may be necessary to stratify those risks into smaller, more homogeneous groups. Often a structure is stratified using more than one criterion. An example that has already been discussed is personal automobile, which is usually rated by territory and by classification. Each of these rating variables is customarily analyzed separately and rating factors developed reflecting past loss experience. If territory is independent of classification, then the rates developed will be appropriate. If the distribution by classification varies between territories (that is classification is not independent of territory), then such a simple analysis will yield biased rates. For example, if there is a disproportionately high number of youthful operators in a particular territory and youthful operators are underpriced, a univariate analysis of each rating variable will yield rates that are too high for the youthful drivers in that territory. If the rating variable under analysis is territory, then classification is a potential stratification confounding variable.

4.2. Aggregation Confounding Variable

> "It's a well accepted rule of thumb that the larger the data set, the more reliable the conclusions drawn. Simpson' [sic] paradox, however, slams a hammer down on the rule and the result is a good deal worse than a sore thumb. Unfortunately, Simpson's paradox demonstrates that a great deal of care has to be taken when combining small data sets into a large one. Sometimes conclusions from the large data set are exactly the opposite of conclusion from the smaller sets. Unfortunately, the conclusions from the large set are also usually wrong." [6]

In order to stratify data into smaller and more homogeneous classes, actuaries gather data from as many sources as possible. Adding states, companies and years of experience are three ways that an actuary may maintain class homogeneity while increasing class size (and thus credibility). If the variable along which data is aggregated is correlated with one or more rating variables, then that variable may confound the results of any analysis of those rating variables. For example, assume that state B's loss experience is to be combined with state A's loss experience to increase the volume of data available for a class relativity analysis. Also assume that state B has a higher proportion of youthful operators as well as worse loss experience overall. While an analysis of each state's youthful operator experience alone might yield the same appropriate relativity, when combined the analysis will produce an indicated youthful operator relativity that is inappropriately high.

Exhibit 1 [15] illustrates the effect of two aggregation confounding variables. In this scenario both loss ratio and exposure distribution by class are related to both year and state. The loss experience displayed in Exhibit 1 (second page) arises from the required factors of 1.00 for class 01 and 2.10 for class 02. The derivation of the indicated class 02 relativity is displayed on the first page of Exhibit 1. The indicated relativities are 2.30 using the loss ratio method, 2.68 using the pure premium method and 2.30 using the modified loss ratio method. Although each of the indicated relativities are biased, the pure premium method is more susceptible to bias than either of the other two methods. Aggregation confounding variables (though not identified as such) were discussed at length by Stenmark [15]. The example of aggregation confounding variables given in Exhibit 1 will be discussed further in Section 5.5.

4.3. *Lurking Confounding Variable*

As displayed in the introduction to this paper, it is possible that a confounding variable may not be under examination. While many references use the terms lurking variable and confounding variable interchangeably, a more narrow definition of lurking confounding variable is being used here. A lurking confounding variable, then, is a variable that has not yet been uncovered as a stratification confounding variable or an aggregation confounding variable. A lurking confounding variable may exist outside of an actuary's ratemaking data, possibly to be detected using one of the many data mining techniques available. A lurking confounding variable may be a data element that is available only through demographic data, not captured through a company's processing system. Most discouraging of all, the piece of information may not exist anywhere as data.

Insurance companies have been collecting more and more information, and underwriters and actuaries have become sensitive to criteria that might affect the loss process. Hopefully, then, there are not too many undiscovered confounding variables lurking in our data that will significantly distort our rates. Regardless, one only needs to point at the use of credit scores to recognize an important lurking confounding variable that has only recently been utilized to its full potential.

There are two issues relative to the discussion of confounding in previously unused rating variables. First, prior to its use as a rating variable, the failure to segment insureds according to any credit measure may have caused confounding of those rating variables actually in use. For example, assume that a certain class of insureds often displays a poor credit rating and, as a result, that class manifests poor loss experience. The rates for insureds in that class with a better credit score would be inappropriately high.

Second, once credit score has been established as a rating variable, proper methods must be undertaken to prevent the continued confounding of the class rates through the use of one of the treatments described in the next section. For example, a company that provides a discount in automobile for insureds with a homeowner's policy might find that, after introducing a discount for good credit, the rates for automobile risks with an accompanying homeowner's policy are too low. This challenge is discussed at length by Guszcza and Wu [16].

5. TREATMENT OF POTENTIAL CONFOUNDING VARIABLES

We have presented the empirical and theoretical evidence for the existence of Simpson's paradox and confounding variables. In this section, we present several alternatives for the treatment of this phenomenon.

5.1. *No Treatment*

Pearl [12] concludes that there is no test for confounding. Much of Pearl's writing concerns the principle of causality [11], presumably because confounding is of such great concern in medical research and, in that research, causality is of prime importance. Since, in insurance, we are more concerned with statistical correlation than causality we allow a more liberal test for confounding. Therefore, we say that if a variable is unrelated to the variable under study or to the loss measure, then confounding will not result and no treatment is necessary. However, it is ill-advised for an actuary to assume that there is no confounding without extensive examination of the relationship of all the variables affecting the loss process.

5.2. *Controlling Confounding Through Experiment Design*

As discussed in Section 3.1, if we can carefully design an analysis and then collect the data accordingly, then we can control confounding. Whether we have prior knowledge of the relation between the potential confounding variable and the target information or not, we can control its effect if the confounding variable is unrelated to the variable(s) under study. This can be achieved through completely balanced design or proportional design of the experiment. That is, for each combination of the confounding variable and the variable(s) of interest, the same or proportional amount of data is collected. This concept is commonly used in many research areas such as medical, engineering, and scientific research projects.

When an actuary analyzes insurance data, the actuary typically cannot "design" the analysis. The actuary has to analyze whatever he or she is given. The data are mostly determined by the company's book of business, which is largely determined by the market segments that the company serves. Moreover, since there are multiple rating variables, and for each rating variable there exist many different values, it is possible that many combinations of the variables will have missing or very little data. In other words, insurance data is highly non-ideal for the confounding issue, and it is difficult, if not impossible, for us to control the bias through the experimental design approach.

5.3. Controlling Confounding Through Multivariate Analysis

If the insurance data is highly non-ideal and we cannot control confounding through standard experimental design, we can control it by using multivariate analysis. That is, we can perform the Bailey's [1] minimum bias analysis or GLM analysis [4, 8] by including the confounding variable along with the variable(s) of interest. By doing so, the confounding variable's relation with the target variable and the variable(s) in interest will be properly determined and be allocated through the multivariate approach. Therefore, the true relationship of the variable(s) under study can be revealed.

While multivariate analysis may be an ideal solution to deal with the confounding issue, there may exist practical issues against using the approach within insurance applications. One issue is that insurance applications constantly aggregate or stratify data with regards to states, years, and companies. In theory, we can include these potential confounding variables in the analysis, but the inclusion of these non-rating variables in the multivariate analysis may lead to other issues such as credibility of the data for analysis and reasonability and interpretation of the analysis result for the variables. Therefore, later we propose an alternative approach, using scaling factors, for actuaries to consider for addressing confounding. The alternative approach will be discussed in detail in Section 5.5.

5.4. Controlling Confounding Through the Use of Meta-Analysis

Researchers are often faced with situations that compel the use of data from several studies. In insurance we strive to increase the volume of our data to increase credibility, and medical researchers attempt to do the same through compilations of more than one study called meta-analyses [5].

A research study typically includes observations from two groups: an intervention group (N_i) and a control group (N_c). From these observations four pieces of data are derived: an intervention with an event (n_i), intervention without an event ($N_i - n_i$), control with an event (n_c) and control without an event ($N_c - n_c$). From these a statistic, generally called a "size effect," is calculated. The two size effects in general use are termed the "risk difference" and the "odds ratio." The risk difference is the difference between the ratio of the number of interventions with an event to the total observations of all interventions and the ratio of the number of control subjects with an event to the total observations of the control group. Risk difference = $(n_i/N_i) - (n_c/N_c)$. The reciprocal of the risk difference is termed the "number needed to treat" (or "harm") and represents the number of interventions required to achieve one event. The "odds ratio" is the ratio of the ratio of the number of interventions with an event to the number of interventions without an event to the ratio of the number of control subjects with an event to the number of control subjects without an event. Odds ratio = $(n_i/(N_i - n_i)) \div (n_c/(N_c - n_c))$.

If an analyst naively combines all of the observations, confounding can result and lead to biased findings because there is a different distribution of observations between studies. For example, in Cates [5] seven out of ten studies resulted in a positive number needed to treat, and the three that did not represented only 839 of the 6,121 observations in the meta-analysis. Regardless, combining the raw data produced a number needed to harm in contrast to the number needed to treat in the majority of studies.

A discipline has risen centered around the optimum method to be used to combine such studies. In general, methodologies focus on calculating a variance for each study. The reciprocal of this variance is used to weight the size effects themselves rather than the raw observations.

This treatment is analogous to calculating class relativities for each year and state and weighting those relativities to arrive at a composite relativity for each class. As such, it has some similarities to credibility weighting. However, one major difference between typical medical research and insurance ratemaking is that medical research results are binary outputs and insurance ratemaking results are relativities or rates on a continuous scale. Therefore, although meta-analysis provides an interesting example of the effect and treatment of confounding in medical research, it does not appear to have any direct application to insurance pricing.

5.5. Controlling Confounding Through the Use of Scaling Factors

In this section, we introduce a practical approach, called "scaling factors," to treat the confounding effect that may commonly exist in insurance rating applications. We believe that this approach was first proposed by Stenmark in his 1990 paper [15], and we are revisiting the approach from the perspective of confounding variables and Simpson's paradox. This approach is important because there are some confounding variables that are not optimally addressed using any of the treatments mentioned above.

It is not usually desirable or practical to include a multivariate analysis of most aggregation confounding variables as described in Section 4.2, since if data from several states are included, a multiplier by state is probably not a necessary rating model output. This is because each state's overall rate change requirements will be calculated through a statewide indication, possibly at some indeterminate time in the future. In addition, a multiplier for each experience year has no direct application or interpretation. Regardless, recognition of such variables in multivariate analysis, through the use of dummy variables, is an accepted and effective practice as will be discussed in Section 5.6. An alternative to that approach will be discussed in this section.

Is there a way that data from several experience years and several states can be aggregated to increase data volume without possibly confounding the results of the study and without the necessity of inclusion of the confounding variable in the analysis?

As stated previously, there are two conditions necessary for a variable to confound the results of an analysis:

1. There must be a relationship between that variable and the experience variable.
2. There must be a relationship between that variable and at least one of the rating variables under analysis.

If either of those two conditions is not met then there is no confounding of the results.

This leads to the question: if both conditions are met, can the data be modified so that one of the conditions is no longer met, eliminating the confounding? This must be done in such a manner that the important underlying relationships in the data are not disturbed. In the following sections, we will show the scaling factors approach using a class plan analysis example with two potential confounding variables—states and years.

5.5.1. The Loss Experience Model

To eliminate the confounding effect, it is first necessary to quantify that effect on a classification loss model. The model need not be complex and is composed, at the atomic level, of exposures, base rate, current rating factors, required rating factors and base class loss ratio. Appendix A outlines this model and the quantification of the impact of confounding. For example, the total earned premiums for class i on present rates$= P_i = \sum_y \sum_s e_{iys} B_{ys} c_i$ and the total incurred losses for class $i = L_i = \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i$. The notation introduced in Appendix A will be used throughout the remainder of Section 5.

The impact on indicated class relativities due to the confounding effect of aggregation of experience by year and by state is displayed for three classification ratemaking methods: the loss ratio method, the pure premium method and the modified loss ratio method. The modified loss ratio method bears some description. The premiums are calculated at base class rates so that the output of the method is the class relativity, not the indicated change to that relativity.

In addition to the three methods presented there is another subtle variation in methodology. It is possible to develop each class relativity as a ratio of the selected statistic (e.g., loss ratio) to that of a base class (special case) or to the statistic of the all-class experience (general case). The words "special" or "general" are used to identify each method. For example, in Personal Automobile Insurance it is common to divide the class loss ratio by the loss ratio for adult driver (pleasure use). This is the special case. It is not always the case that the base class has a large portion of the business, so the all-class loss ratio may provide a more stable base. This is the general case. The class relativities can be normalized back to the base class after the indicated relativities have been credibility weighted and selections have been made from those credibility-weighted relativities. The model introduced by Stenmark [15] was for the special case only. Including the general case adds further flexibility.

The bias produced by confounding is derived in Appendix A and is reproduced below:

Bias arising from confounding using pure premium method (special case):

$$\begin{aligned}\frac{g_i}{f_i} - 1 &= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys}} \bullet \frac{\sum_y \sum_s e_{bys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys} f_i} - 1 \\ &= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{E_i} \bullet \frac{E_b}{L_b} - 1. \qquad (5.1)\end{aligned}$$

Bias arising from confounding using loss ratio method (special case):

$$\frac{g_i}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_i} \bullet \frac{\sum_y \sum_s e_{bys} B_{ys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1$$
$$= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{P_i} \bullet \frac{P_b}{L_b} - 1. \qquad (5.2)$$

Bias arising from confounding using modified loss ratio method (special case)

$$\frac{g_i}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{\sum_y \sum_s e_{bys} B_{ys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1$$
$$= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{P_b}{L_b} - 1. \qquad (5.3)$$

Each of the above utilizes the base class experience as the base. If the relativity is calculated utilizing, instead, the all-class experience (general case), then the bias for the modified loss ratio method is shown in Equation (5.4).

$$\frac{g_i}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{\sum_y \sum_s e_{bys} B_{ys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1. \qquad (5.4)$$

5.5.2. Derivation of the Scaling Factor

Is it possible to scale the premiums or losses (or both) in such a manner that the bias is removed when the data from one state and/or year are combined with that of another state or year? What characteristics should such a scaling factor possess? Two criteria must be met by any scaling factor candidate:

Criterion 1: The scaling factors should maintain the relationship between class loss ratios by year and state (the scaling factors should not change the underlying relativities).

Criterion 2: The scaling factors should reduce the bias due to confounding to zero.

Any scaling factor that is applied uniformly to each class within a specific state for a particular year *or* is applied to both premiums and losses for a specific class will fulfill the requirements of Criterion 1. When either the exposure distribution or the base class loss ratio remains constant, the distortion is not present and any scaling factor that stabilizes either the base or total class loss ratio (in the special case or general case, respectively), or the exposure distribution should fulfill the requirements of Criterion 2.

A clue as to how to approach the derivation of a scaling factor was discussed in the section on experiment design. If the experience is balanced or there is no relationship between the experience and the confounding variable, then confounding does not occur. If a scaling factor candidate can promote either characteristic, then confounding should be mitigated.

Appendix B displays the evaluation of four types of scaling factors that meet the needs of both criteria. These scaling factors can be broken into two categories. One category applies to the special case and the other applies to the general case. Each category has one scaling factor that is used to address the non-independence of the confounding variable and the loss statistic (loss ratio, pure premium, etc.). The other scaling factor addresses the non-independence of the confounding variable and the rating element(s) under study (balance). *Only one type of scaling factor need be used in a rate analysis.* The type of factor to use is the choice of the actuary.

Please note that these factors were arrived at by inspection. This was not a trivial process, but the authors believe that a mathematical derivation of the factor is not possible. The factors are tested within Appendix B to display that the bias from confounding is eliminated.

The first scaling factor that is considered is the reciprocal of the base class loss ratio for each state and year. By applying this factor uniformly to the losses for each class, the relationship between each of the class loss ratios is maintained (Criterion 1) while the method error is reduced to zero (Criterion 2). This is shown in Appendix B.

The example given in Exhibits 1–5 is used to examine the scaling factors. Exhibit 2 displays the effect of scaling losses with the reciprocal of the base class loss ratio. Both Exhibits 2 and 3 utilize input parameters that were set forth in Exhibit 1. The modified loss ratio method is utilized in Exhibit 2. The premium is modified to the base class rate level by dividing by the class factor prior to calculating the loss ratio. For each class, the losses are scaled by the base class adjusted loss ratio for that year and state. For example, the incurred losses for state 01, year 1 ($500,000) are multiplied by the reciprocal of class 01 loss ratio (1.00/0.50 = 2.00) to yield the scaled losses of $1,000,000. The class 02 incurred losses ($525,000) are also multiplied by this factor to yield the scaled losses for that class of $1,050,000. These scaled losses maintain the relationship between the class loss ratios, but lose any information regarding the actual base class loss ratio. It is possible to apply a scaling factor (the base class loss ratio in this case rather than its reciprocal) to the premium rather than the losses. This method should be used only for larger, more stable lines of business. In cases where even the base class loss ratio can fluctuate wildly, it is more appropriate to scale the losses. The reason is that scaled losses are equal, in total, to premium. If the scaling factor were applied to premium, the result would be equal to (the more volatile) losses.

The second scaling factor derived in Appendix B addresses the different exposure distribution by year and state. The ratio of the total exposures for each class to the total exposures for the base class is multiplied by the ratio of the base class exposures in each state and year to the class exposures in each state and year to provide the scaling factor (algebraically, $S_{iys} = (\sum_y \sum_s e_{iys} / \sum_y \sum_s e_{bys}) \bullet (e_{bys}/e_{iys})$). As opposed to the first scaling factor, the second scaling factor is unique for each class, year and state. However, since the factor is applied to both premiums and losses, this scaling factor also satisfies the requirements of Criterion 1. When e'_{iys} replaces e_{iys} in the equation for the error developed in Appendix A, error is reduced to zero, thus satisfying the requirements of Criterion 2. Exhibit 3 displays the effect of utilizing the second scaling factor.

The third scaling factor is for the general case and it addresses the non-independence of the confounding variable and the loss experience, as did Scaling Factor 1. Appendix B displays the derivation of this factor. The reciprocal of the loss ratio for the state and year is shown to eliminate the bias in the loss experience.

The fourth scaling factor is similarly tested in Appendix B. This factor is applied in the general case and addresses balance. As displayed in the appendix this scaling factor is $S_{iys} = (\sum_y \sum_s e_{iys} / \sum_i \sum_y \sum_s e_{iys}) \bullet (\sum_i e_{iys} / e_{iys})$.

The advantages of Scaling Factors 1 and 3 are:

1. Ease of use: The base class and statewide loss ratios are directly obtainable from the data already necessary for the modified loss ratio method.

2. Since the scaling factor is applied uniformly for each class, the premium distribution by class for each year and state is left unaltered.

3. Many of the traditional adjustments to premium and loss data are no longer necessary. Any adjustment that applies uniformly to the premiums or losses of all classes is nullified by the application of that scaling factor. These adjustments would include present level adjustments for overall rate changes, development factors and trend factors. If, however, an adjustment is not applied uniformly by class, it will still be necessary. For example, if trend factors are applied by cause of loss, these factors will need to be applied prior to the scaling process.

TABLE 10

RESULT OF MINIMUM BIAS
USING DUMMY VARIABLES FOR STATE AND YEAR

Raw Output			
	1	2	Number of Iterations
State	1.1544	1.3853	11
Year	0.7432	1.1148	
Class	0.5828	1.2238	
Normalized			
	1	2	
State	0.5000	1.8900	
Year	1.0000	1.5000	
Class	1.0000	2.1000	

The advantage of Scaling Factors 2 and 4 is that if the exposure distribution is more stable than the loss ratios from year to year, then Scaling Factors 2 and 4 will result in less abrupt adjustments for most classes than will Scaling Factors 1 and 3.

5.6. *Comparison of Multivariate Analysis vs. Scaling*

It is common practice to include dummy variables for potential aggregation confounding variables in a multivariate analysis. Inclusion of a dummy variable for both year and state, for example, would allow the non-independence of those variables with the dependent variable (e.g., loss ratio) to be reflected in the dummy variables. Does this methodology compensate for the confounding observed previously? If it does, is this method more or less effective than the use of one of the scaling factors discussed in the previous section? Table 10 displays the results of such a computation. The resulting factors for States 1 and 2, Years 1 and 2 and Classes 01 and 02 are shown. In eleven iterations the minimum bias equations converged to the raw output displayed in Table 10.

The raw output was then normalized to base class (year and class) and the state factors were adjusted to correct for the normalization. The normalized class 02 factor is equal to the correct value, 2.10. It appears that both the scaling factors discussed in Section 5.5 and the multivariate analysis discussed above yield the correct factor in this deterministic scenario.

The world in which we live is hardly deterministic. It is necessary to test each method in a stochastic model. The deterministic model was used to parameterize such a model. Separate frequency and severity averages were derived assuming a frequency of 0.01 adjusted by the class and year loss ratio. The state loss ratio was reflected in the severity. The frequency distribution was assumed to be Poisson and the severity distribution was assumed to be Lognormal. Exhibit 4 displays the model output. One thousand iterations were simulated. Within each iteration for each exposure for each year, state, and class a number of claims was derived from the Poisson. For each of these claims, a claim size was determined from the Lognormal distribution. The loss ratio for each year, state, and class was determined and from these the Class 02 relativity was derived using the univariate (traditional) method, each of the four scaling factor methods, as well as Bailey's minimum bias. The authors acknowledge that the use of a linear model based on the Lognormal might have been more appropriate.

The values that emerged from the deterministic model are displayed as the expected values. Below these are the average values from all one thousand iterations. Finally, the next row displays the mean square error (MSE) for each column. The value used to calculate this error for the univariate method was the correct class relativity rather than the relativity emerging from the deterministic model (i.e., 2.10 rather than 2.2958).

The presence or absence of a loss limit might affect the sensitivity of each method to variability in losses. Therefore, the model was repeated, but this time losses were limited to $25,000. Of course, the Lognormal parameters had to be adjusted upward to compensate for the excluded losses at the top end of the distribution.

The mean square error for the univariate method was somewhat higher than that for the other methods with or without the loss limitation. This was expected since the method possessed a relatively large bias in the first place. On the other hand, there was no significant difference between the errors for Bailey's minimum bias and the four scaling methods. It appears that while use of an iterative bias reducing methodology does, in fact, reduce bias, so do each of the scaling factors described earlier in this paper.

6. CONFOUNDING VARIABLES AND CURRENT ACTUARIAL PRACTICE

6.1. Areas Where Confounding Variables Have Been Recognized

Bailey and Simon [2] first recognized the potential for bias from confounding in 1960, though they did not identify it as such. Are there other areas where actuaries have recognized this bias and compensated for it?

One answer is in the trending process that actuaries frequently employ in their rating and reserving applications to adjust premium and loss data. It is customary when preparing a rate indication to trend losses to recognize the increase in severity and changing frequency. It is also necessary to trend premium to recognize that some loss trend is from factors that will increase the premium over time. These inflation and coverage-sensitive rating factors confound the loss trends necessitating an adjustment. Since deductible, for example, is both the trend measure, pure premium, or frequency and severity, as well as time (deductibles tend to increase over time), deductible is a confounding variable for trend data. Other confounding variables for trend might be symbol, model year, limit of liability, and amount of insurance, to name just a few.

6.2. Areas Where Confounding May Be an Unrecognized Problem

Confounding is a frequent and serious problem in ratemaking. Obviously, almost all the rating variables can confound each other because their distributions are hardly independent. As discussed above, the premium and loss on-leveling and trending is a process that actuaries employ to deal with such confounding to the best we can. However, the process may not be able to remove all the potential confounding relationships between the variables.

Moreover, there are other potential confounding variables that exist outside the rating variables that may not be fully recognized and explored, i.e., lurking confounding variables. The following are a few examples, some of which have been discussed previously:

• *Geographic Information*: While a rating plan may include geographic rating variables, such as state and territory, these variables may not be enough to fully explain the confounding relationship in the rating data. The real underlying drivers for such geographic factors include the underlying demographic, consumer, economic, traffic, and weather information. This information includes, but is not limited to, information such as education, employment, credit, lifestyle, consumer spending, traffic volume, crime, cold, heat, hail, storms, etc. Especially for commercial lines of business, such geographic information is usually under-represented in the rating process.

• *Market Segment*: The distribution of rating variables is significantly influenced by market segments. For example, a non-standard book of business might be expected to have a much higher distribution of younger drivers, more risks with prior claims and violations, and insurance with lower coverages. Therefore, it might be prudent to aggregate or stratify data along different market segments. In many instances, companies or tiering will be used to separate different market segments. It is highly likely that classification experience will be confounded by rating tier or company. Variables used in company placement or tiers typically include both rating variables and non-rating variables. Company or rating tier can be used as a variable with classification in a linear model, or the experience should be treated with one of scaling factors introduced in Section 5.5.

• *Distribution Channels*: Our experience indicates that distribution channels will also affect the composition of and the information gathered for a book of business. This issue has become even more significant as many companies are marketing online in addition to the two traditional channels of direct writers and independent agents. We have found that business flowing through different channels may be of very different quality and contain differing amounts of information.

• *External Environment*: The insurance industry is not operating within an isolated world, and its performance is a part of the increasingly more integrated national or even worldwide economies. Therefore, in this fast-changing world, issues such as technological development, economic cycles, and recent terrorist activity will affect the insurance industry. The current hard market condition is clear evidence of how the insurance underwriting cycle is influenced by the external world. Therefore, combining multiple years with possible year-to-year changes and insurance cycles requires special care. Additional care must be rendered when projecting historical information into the future.

7. CONCLUSIONS

E. H. Simpson introduced the concept now known as Simpson's paradox. It is the extreme case of a phenomenon known as confounding. While such extreme cases may not occur frequently in actuarial calculations, the change in relationship due to confounding does.

A variable can confound the results of an insurance rate structure analysis only if it is related (non-independent) to both the experience measure (loss ratio, pure premium, etc.) and at least one of the other rating variables in the analysis. Confounding variables can be categorized as either a stratification confounding variable, an aggregation confounding variable, or a lurking confounding variable.

Several methods for the treatment of confounding were discussed including no treatment, experimental design, multivariate analysis, meta-analysis, and use of scaling factors.

The combination of data from more than one year may cause distortion in traditional classification ratemaking techniques if each set of data represents a different base rate adequacy and different exposure distribution by class. The combination of data from more than one state may cause distortion in the traditional pure premium method if the base rate from each state is different and possesses different exposure distributions by class. The combination of data from more than one state may cause distortion in both of the traditional methods if the base rate from each state is different, the base class loss ratio is different, and the state/year data exhibit a different exposure distribution by class. It is more than likely that these conditions will exist within most sets of ratemaking data. These distortions may be remedied by the application of a scaling factor to the data from each year and each state. This scaling factor may address either the exposure distribution or the base rate adequacy. An investigation of the effectiveness of multivariate analysis in comparison with the use of scaling factors reveals that both methodologies reduce the effect of confounding, probably to the same degree.

The authors have encountered the confounding experience numerous times in their work, and it is with this motivation that we introduce Simpson's paradox and the concept of confounding to the actuarial community. We believe that understanding these concepts is a key for actuaries in understanding the "correlation" issue that exists frequently in our actuarial work, and the impact of such "correlation" on analysis results.

REFERENCES

[1] Bailey, Robert A., "Insurance Rates with Minimum Bias," *PCAS* L, 1963, pp. 4–11.

[2] Bailey, Robert A., and LeRoy J. Simon, "Two Studies in Automobile Insurance Ratemaking," *PCAS* XLVII, 1960, pp. 1–19.

[3] Bickel, P. J., E. A. Hammel, and J. W. O'Connell, "Sex Bias in Graduate Admissions: Data from Berkeley," *Science*, Vol. 187, 1975, pp. 398–404.

[4] Brockman, M. J., and T. S. Wright, "Statistical Motor Rating: Making Effective Use of Your Data," *Journal of the Institute of Actuaries*, Vol. 119, 1992, Part III.

[5] Cates, Christopher J., "Simpson's Paradox and Calculation of Number Needed to Treat From Meta-Analysis," *BMC Medical Research Methodology*, 2002, 2:1.

[6] Guszcza, James C., and Cheng-Sheng Peter Wu, "Does Credit Score Really Explain Insurance Losses? Multivariate Analysis from a Data Mining Point of View," Casualty Actuarial Society *Forum*, Winter 2003, pp. 113–138.

[7] Intuiter.com, "Simpson's Paradox—When Big Data Sets Go Bad," http://www.intuitor.com/statistics/SimpsonsParadox.html.

[8] Mildenhall, S. J., "A Systematic Relationship Between Minimum Bias and Generalized Linear Models," *PCAS* LXXXVI, 1999, pp. 393–487.

[9] Montgomery, D. C., *Design and Analysis of Experiments*, John Wiley & Sons, 3rd Version, 1991.

[10] Neter, J., W. Wasserman, and M. H. Kutner, *Applied Linear Statistical Models*, Richard D. Irwin, Inc., 2nd Edition, 1985.

[11] Pearl, Judea, "Simpson's Paradox: An Anatomy," *Causality: Models, Reasoning and Inference*, Chapter 6, Cambridge, 2000.

[12] Pearl, Judea, "Why There Is No Statistical Test For Confounding, Why Many Think There Is, And Why They Are Almost Right," *Technical Report R-256*, Cognitive System Laboratory, UCLA, 1998.

[13] Schield, Milo, "Simpson's Paradox and Cornfield's Conditions," Dept. of Business & MIS, Augsburg College, 2211 Riverside Drive, Minneapolis, MN 55454.

[14] Simpson, E. H., "The Interpretation of Interaction in Contingency Tables," *Journal of the Royal Statistical Society* B, 1951, 2:238–41.

[15] Stenmark, John A., "Sources of Distortion in Classification Ratemaking Data and their Treatment," Casualty Actuarial Society *Discussion Paper Program*, 1990, pp. 85–112.

EXHIBIT 1

PART 1

MULTIPLE STATE—MULTIPLE YEAR SITUATION DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION

Assumptions

Class Factors Underlying Experience			Class 01 Loss Ratio		
	Current	Required		Loss Ratios	
Class	Factor	Factor	State	Year 1	Year 2
01	1.00	1.00	1	50%	75%
02	2.00	2.10	2	60%	90%

Distribution of Exposures

	State 1		State 2		
Class	Year 1	Year 2	Year 1	Year 2	Total
01	10,000	15,000	10,000	15,000	50,000
02	5,000	15,000	15,000	45,000	80,000
Total	15,000	30,000	25,000	60,000	130,000

State 1 Base Rate = \$100
State 2 Base Rate = \$200

(The Derived Loss Experience is shown on the next page.)

Indicated Class 02 Relativity

Loss Ratio Method: $(84.56\%/73.67\%) \times 2.00 = 2.30$
Pure Premium Method: $295.97/110.50 = 2.68$
Modified Loss Ratio Method: $(169.13\%/73.67\%) = 2.30$

EXHIBIT 1

PART 2

MULTIPLE STATE—MULTIPLE YEAR SITUATION
DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION

Derived Loss Experience

	Class	Exposures	Earned Premium	Class Factor	Modified Earned Premium	Incurred Losses	Loss Ratio	Pure Premium	Modified Loss Ratio
State 1									
Year 1	01	10,000	$1,000,000	1.00	$1,000,000	$500,000	50.00%	$50.00	50.00%
	02	5,000	$1,000,000	2.00	$500,000	$525,000	52.50%	$105.00	105.00%
	Total	15,000	$2,000,000		$1,500,000	$1,025,000	51.25%	$68.33	68.33%
Year 2	01	15,000	$1,500,000	1.00	$1,500,000	$1,125,000	75.00%	$75.00	75.00%
	02	15,000	$3,000,000	2.00	$1,500,000	$2,362,500	78.75%	$157.50	157.50%
	Total	30,000	$4,500,000		$3,000,000	$3,487,500	77.50%	$116.25	116.25%
All Years	01	25,000	$2,500,000		$2,500,000	$1,625,000	65.00%	$65.00	65.00%
	02	20,000	$4,000,000		$2,000,000	$2,887,500	72.19%	$144.38	144.38%
	Total	45,000	$6,500,000		$4,500,000	$4,512,500	69.42%	$100.28	100.28%

State 2									
Year 1	01	10,000	$2,000,000	1.00	$2,000,000	$1,200,000	60.00%	$120.00	60.00%
	02	15,000	$6,000,000	2.00	$3,000,000	$3,780,000	63.00%	$252.00	126.00%
	Total	25,000	$8,000,000		$5,000,000	$4,980,000	62.25%	$199.20	99.60%
Year 2	01	15,000	$3,000,000	1.00	$3,000,000	$2,700,000	90.00%	$180.00	90.00%
	02	45,000	$18,000,000	2.00	$9,000,000	$17,010,000	94.50%	$378.00	189.00%
	Total	60,000	$21,000,000		$12,000,000	$19,710,000	93.86%	$328.50	164.25%
All Years	01	25,000	$5,000,000		$5,000,000	$3,900,000	78.00%	$156.00	78.00%
	02	60,000	$24,000,000		$12,000,000	$20,790,000	86.63%	$346.50	173.25%
	Total	85,000	$29,000,000		$17,000,000	$24,690,000	85.14%	$290.47	145.24%

EXHIBIT 1

PART 2

MULTIPLE STATE—MULTIPLE YEAR SITUATION
DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION, *Continued*

Derived Loss Experience									
	Class	Exposures	Earned Premium	Class Factor	Modified Earned Premium	Incurred Losses	Loss Ratio	Pure Premium	Modified Loss Ratio
All States									
Year 1	01	20,000	$3,000,000		$3,000,000	$1,700,000	56.67%	$85.00	56.67%
	02	20,000	$7,000,000		$3,500,000	$4,305,000	61.50%	$215.25	123.00%
	Total	40,000	$10,000,000		$6,500,000	$6,005,000	60.05%	$150.13	92.38%
Year 2	01	30,000	$4,500,000		$4,500,000	$3,825,000	85.00%	$127.50	85.00%
	02	60,000	$21,000,000		$10,500,000	$19,372,500	92.25%	$322.88	184.50%
	Total	90,000	$25,500,000		$15,000,000	$23,197,500	90.97%	$257.75	154.65%
All Years	01	50,000	$7,500,000		$7,500,000	$5,525,000	73.67%	$110.50	73.67%
	02	80,000	$28,000,000		$14,000,000	$23,677,500	84.56%	$295.97	169.13%
	Total	130,000	$35,500,000		$21,500,000	$29,202,500	82.26%	$224.63	135.83%

EXHIBIT 2

MULTIPLE STATE—MULTIPLE YEAR SITUATION
DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION

Derived Loss Experience

	Class	Exposures	Earned Premium	Class Factor	Adjusted Premium	Incurred Losses	Adjusted Loss Ratio	Scaled Losses	Modified Loss Ratio
State 1									
Year 1	01	10,000	\$1,000,000	1.00	\$1,000,000	\$500,000	50.00%	\$1,000,000	100.00%
	02	5,000	\$1,000,000	2.00	\$500,000	\$525,000	105.00%	\$1,050,000	210.00%
	Total	15,000	\$2,000,000		\$1,500,000	\$1,025,000	68.33%	\$2,050,000	136.67%
Year 2	01	15,000	\$1,500,000	1.00	\$1,500,000	\$1,125,000	75.00%	\$1,500,000	100.00%
	02	15,000	\$3,000,000	2.00	\$1,500,000	\$2,362,500	157.50%	\$3,150,000	210.00%
	Total	30,000	\$4,500,000		\$3,000,000	\$3,487,500	116.25%	\$4,650,000	155.00%
All Years	01	25,000	\$2,500,000		\$2,500,000	\$1,625,000	65.00%	\$2,500,000	100.00%
	02	20,000	\$4,000,000		\$2,000,000	\$2,887,500	144.38%	\$4,200,000	210.00%
	Total	45,000	\$6,500,000		\$4,500,000	\$4,512,500	100.28%	\$6,700,000	148.89%

EXHIBIT 2

MULTIPLE STATE—MULTIPLE YEAR SITUATION
DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION, *Continued*

			Derived Loss Experience						
	Class	Exposures	Earned Premium	Class Factor	Adjusted Premium	Incurred Losses	Adjusted Loss Ratio	Scaled Losses	Modified Loss Ratio
State 2									
Year 1	01	10,000	$2,000,000	1.00	$2,000,000	$1,200,000	60.00%	$2,000,000	100.00%
	02	15,000	$6,000,000	2.00	$3,000,000	$3,780,000	126.00%	$6,300,000	210.00%
	Total	25,000	$8,000,000		$5,000,000	$4,980,000	99.60%	$8,300,000	166.00%
Year 2	01	15,000	$3,000,000	1.00	$3,000,000	$2,700,000	90.00%	$3,000,000	100.00%
	02	45,000	$18,000,000	2.00	$9,000,000	$17,010,000	189.00%	$18,900,000	210.00%
	Total	60,000	$21,000,000		$12,000,000	$19,710,000	164.25%	$21,900,000	182.50%
All Years	01	25,000	$5,000,000		$5,000,000	$3,900,000	78.00%	$5,000,000	100.00%
	02	60,000	$24,000,000		$12,000,000	$20,790,000	173.25%	$25,200,000	210.00%
	Total	85,000	$29,000,000		$17,000,000	$24,690,000	145.24%	$30,200,000	177.65%

All States								
Year 1	01	20,000	$3,000,000	$3,000,000	$1,700,000	56.67%	$3,000,000	100.00%
	02	20,000	$7,000,000	$3,500,000	$4,305,000	123.00%	$7,350,000	210.00%
	Total	40,000	$10,000,000	$6,500,000	$6,005,000	92.38%	$10,350,000	159.23%
Year 2	01	30,000	$4,500,000	$4,500,000	$3,825,000	85.00%	$4,500,000	100.00%
	02	60,000	$21,000,000	$10,500,000	$19,372,500	184.50%	$22,050,000	210.00%
	Total	90,000	$25,500,000	$15,000,000	$23,197,500	154.65%	$26,550,000	177.00%
All Years	01	50,000	$7,500,000	$7,500,000	$5,525,000	73.67%	$7,500,000	100.00%
	02	80,000	$28,000,000	$14,000,000	$23,677,500	169.13%	$29,400,000	210.00%
	Total	130,000	$35,500,000	$21,500,000	$29,202,500	135.83%	$36,900,000	171.63%

Indicated Class 02 Relativity
Modified Loss Ratio Method: $(210.00\%/100.00\%)x = 2.10$

EXHIBIT 3

Multiple State—Multiple Year Situation
Different Loss Ratios—Different Distribution

					Derived Loss Experience				
	Class	Exposures	Earned Premium	Class Factor	Incurred Losses	Scaling Factor	Scaled Premium	Scaled Losses	Modified Loss Ratio
State 1									
Year 1	01	10,000	$1,000,000	1.00	$500,000	1.000	$1,000,000	$500,000	50.00%
	02	5,000	$1,000,000	2.00	$525,000	3.200	$1,600,000	$1,680,000	105.00%
	Total	15,000	$2,000,000		$1,025,000		$2,600,000	$2,180,000	83.85%
Year 2	01	15,000	$1,500,000	1.00	$1,125,000	1.000	$1,500,000	$1,125,000	75.00%
	02	15,000	$3,000,000	2.00	$2,362,500	1.600	$2,400,000	$3,780,000	157.50%
	Total	30,000	$4,500,000		$3,487,500		$3,900,000	$4,905,000	125.77%
All Years	01	25,000	$2,500,000		$1,625,000		$2,500,000	$1,625,000	65.00%
	02	20,000	$4,000,000		$2,887,500		$4,000,000	$5,460,000	136.50%
	Total	45,000	$6,500,000		$4,512,500		$6,500,000	$7,085,000	109.00%

State 2									
Year 1	01	10,000	$2,000,000	1.00	$1,200,000	1.000	$2,000,000	$1,200,000	60.00%
	02	15,000	$6,000,000	2.00	$3,780,000	1.067	$3,200,000	$4,032,000	126.00%
	Total	25,000	$8,000,000		$4,980,000		$5,200,000	$5,232,000	100.62%
Year 2	01	15,000	$3,000,000	1.00	$2,700,000	1.000	$3,000,000	$2,700,000	90.00%
	02	45,000	$18,000,000	2.00	$17,010,000	0.533	$4,800,000	$9,072,000	189.00%
	Total	60,000	$21,000,000		$19,710,000		$7,800,000	$11,772,000	150.92%
All Years	01	25,000	$5,000,000		$3,900,000		$5,000,000	$3,900,000	78.00%
	02	60,000	$24,000,000		$20,790,000		$8,000,000	$13,104,000	163.80%
	Total	85,000	$29,000,000		$24,690,000		$13,000,000	$17,004,000	130.80%

EXHIBIT 3

MULTIPLE STATE—MULTIPLE YEAR SITUATION
DIFFERENT LOSS RATIOS—DIFFERENT DISTRIBUTION, *Continued*

Derived Loss Experience

	Class	Exposures	Earned Premium	Class Factor	Incurred Losses	Scaling Factor	Scaled Premium	Scaled Losses	Modified Loss Ratio
All States									
Year 1	01	20,000	$3,000,000		$1,700,000		$3,000,000	$1,700,000	56.67%
	02	20,000	$7,000,000		$4,305,000		$4,800,000	$5,712,000	119.00%
	Total	40,000	$10,000,000		$6,005,000		$7,800,000	$7,412,000	95.03%
Year 2	01	30,000	$4,500,000		$3,825,000		$4,500,000	$3,825,000	85.00%
	02	60,000	$21,000,000		$19,372,500		$7,200,000	$12,852,000	178.50%
	Total	90,000	$25,500,000		$23,197,500		$11,700,000	$16,677,000	142.54%
All Years	01	50,000	$7,500,000		$5,525,000		$7,500,000	$5,525,000	73.67%
	02	80,000	$28,000,000		$23,677,500		$12,000,000	$18,564,000	154.70%
	Total	130,000	$35,500,000		$29,202,500		$19,500,000	$24,089,000	123.53%

Indicated Class 02 Relativity
Modified Loss Ratio Method: (154.70%/73.67%) = 2.10

EXHIBIT 4

PART 1

STOCHASTIC MODEL WITH LOGNORMALLY DISTRIBUTED LOSSES (UNLIMITED)

Iterations: 1000

Class 02 Factor

Iteration	Univariate Method	Scaling Factor 1	Scaling Factor 2	Scaling Factor 3	Scaling Factor 4	Minimum Bias State	Minimum Bias Year	Minimum Bias Class
Expected	2.2657	2.1000	2.1000	2.1000	2.1000	1.1478	1.4403	2.1016
Observed	2.3129	2.1573	2.1149	2.1330	2.1183	1.2071	1.5166	2.1128
MSE	0.1020	0.0643	0.0474	0.0465	0.0554	0.0168	0.0276	0.0461
1	2.3935	2.1936	2.1631	2.1932	2.1736	1.2049	1.5474	2.1476
2	1.8113	1.7371	1.6567	1.6791	1.7141	1.0902	1.6097	1.6514
3	2.2538	2.0191	2.0735	2.0988	1.9792	1.2186	1.5926	2.1182
4	2.4408	2.4355	2.1858	2.1875	2.3471	1.2026	1.4451	2.1536
5	2.1421	2.0335	1.9766	2.0229	1.9670	1.1834	1.4379	1.9186
6	2.4667	2.3157	2.3354	2.3564	2.3083	1.0514	1.4017	2.3349
7	2.5048	2.4576	2.2818	2.3519	2.2324	1.2738	1.5613	2.1804
8	2.3571	2.2147	2.1329	2.1659	2.1591	1.3534	1.3752	2.0514
9	2.1105	1.9180	1.8851	1.8830	1.9294	1.1960	1.7415	1.8736
10	2.4410	2.3869	2.1701	2.1827	2.3078	1.2407	1.4736	2.0896
11	2.1020	1.9224	1.9825	1.9872	1.9249	1.1422	1.3841	2.0751
12	2.0084	1.8736	1.8714	1.9208	1.7754	1.1715	1.5789	1.8671
13	2.2959	2.2104	2.0896	2.1382	2.1279	1.1512	1.6218	1.9925
14	2.1490	1.9574	1.9076	1.9281	1.9514	1.4529	1.3877	1.9140
15	2.2361	1.9854	2.0147	2.0346	1.9838	1.3287	1.5561	2.0263
16	2.1911	2.0321	1.9674	2.0141	1.9646	1.2217	1.7060	1.9186
17	2.2988	2.2033	2.0942	2.0757	2.1698	1.0465	1.7953	2.1426
18	2.5449	2.5903	2.3678	2.4434	2.4114	1.0333	1.5893	2.2433
19	2.3596	2.1615	2.1741	2.1971	2.1472	1.1520	1.6126	2.1583
20	2.2831	2.0866	2.0896	2.1006	2.0688	1.1301	1.5666	2.1466
21	1.9180	1.6937	1.7110	1.7552	1.6477	1.3537	1.6972	1.7397
22	2.1284	2.0116	2.0323	2.0510	2.0127	1.0356	1.3674	2.0325
23	1.9500	1.7895	1.7802	1.7939	1.7911	1.1713	1.4976	1.8193
24	2.5896	2.4719	2.3317	2.3935	2.2911	1.2323	1.6382	2.2866
25	2.5466	2.4214	2.4049	2.4214	2.3956	1.1283	1.3022	2.3517
26	2.2363	2.0273	2.0281	2.0448	2.0267	1.2561	1.5132	2.0196
27	2.4753	2.3979	2.2668	2.3078	2.2603	1.2619	1.3369	2.1870
28	2.3489	2.1869	2.1419	2.1741	2.1485	1.1248	1.6669	2.1070
29	2.1399	2.0637	1.9157	1.9197	2.0193	1.1871	1.5602	1.9278
30	2.4339	2.2187	2.1744	2.1927	2.1946	1.2553	1.7265	2.1142
31	2.6216	2.4300	2.4392	2.4183	2.4331	1.1500	1.4939	2.4109

EXHIBIT 4

PART 2

STOCHASTIC MODEL WITH LOGNORMALLY DISTRIBUTED LOSSES (UNLIMITED)

Loss Ratios								
	State 1				State 2			
	Year 1		Year 2		Year 1		Year 2	
	Class 01	Class 02	Class 01	Class 02	Class 01	Class 02	Class 01	Class 02
Expected	0.5000	0.5250	0.7500	0.7875	0.6000	0.6300	0.9000	0.9450
Observed	0.4985	0.5240	0.7530	0.7865	0.5956	0.6283	0.9019	0.9445
MSE	0.0140	0.0130	0.0150	0.0070	0.0159	0.0057	0.0168	0.0032
1	0.6534	0.5833	0.6450	0.7317	0.4710	0.5502	0.8950	0.9649
2	0.5139	0.4857	1.0362	0.7646	0.7784	0.5104	0.9056	0.8667
3	0.4053	0.4337	0.6989	0.8902	0.5492	0.6797	1.0675	0.9460
4	0.5763	0.4172	0.7978	0.7673	0.6987	0.6395	0.6964	0.9671
5	0.5296	0.5573	0.8452	0.6703	0.4758	0.6336	0.9148	0.8664
6	0.5477	0.6310	0.6757	0.8165	0.4827	0.6095	0.7831	0.8729
7	0.2809	0.5142	0.9039	0.6999	0.3743	0.7287	0.9168	0.9460
8	0.3796	0.4685	0.8075	0.5984	0.5284	0.7062	0.8134	0.8825
9	0.5170	0.5689	0.8034	0.6887	0.5964	0.4380	0.8943	0.9327
10	0.5652	0.3443	0.9224	0.7652	0.5930	0.7358	0.7854	0.9980
11	0.4327	0.4029	0.6272	0.8902	0.7467	0.6580	0.9062	0.8275
12	0.4173	0.4809	0.8150	0.7818	0.4637	0.6358	1.0940	0.8493
13	0.4549	0.6291	0.9469	0.6585	0.4334	0.5548	0.8385	0.9222
14	0.4469	0.3350	0.6642	0.6383	0.7410	0.6557	0.8636	0.8857
15	0.4699	0.5356	0.7201	0.7646	0.6318	0.6411	1.0427	1.0069
16	0.4701	0.4849	0.9358	0.7629	0.4819	0.6214	1.0280	0.9842
17	0.3373	0.3505	0.8378	0.9307	0.7147	0.5039	0.7264	0.8976
18	0.3690	0.5303	1.0123	0.8015	0.3737	0.6576	0.7376	0.8971
19	0.3948	0.6219	0.8550	0.8306	0.5561	0.6195	0.9290	0.9851
20	0.6618	0.4523	0.6090	0.9025	0.5637	0.5814	0.9532	0.9476
21	0.6081	0.4859	0.6171	0.7104	0.5322	0.5148	1.2337	0.9360
22	0.5790	0.6386	0.8475	0.8545	0.6246	0.6556	0.8901	0.8771
23	0.6768	0.4714	0.7298	0.8152	0.7343	0.5626	1.0024	0.9137
24	0.4986	0.4616	0.6781	0.7542	0.3279	0.6220	0.9056	0.9541
25	0.5320	0.7422	0.7481	0.7485	0.5069	0.6984	0.7892	0.9311
26	0.5480	0.5858	0.7793	0.7667	0.6260	0.6374	0.9919	1.0059
27	0.5829	0.5300	0.7134	0.6655	0.4030	0.7254	0.8618	0.9127
28	0.4914	0.5183	0.9033	0.8609	0.4984	0.6278	0.9344	0.9938
29	0.5140	0.3776	0.8174	0.7764	0.7685	0.5537	0.7845	0.9145
30	0.4255	0.6517	0.8581	0.7072	0.4974	0.5526	0.9187	1.0407
31	0.4136	0.8439	0.7312	0.7344	0.5475	0.5574	0.7777	0.9875

EXHIBIT 5

PART 1

STOCHASTIC MODEL WITH TRUNCATED LOGNORMALLY DISTRIBUTED LOSSES ($25,000 LIMIT)

Iterations: 1000

Class 02 Factor

						Minimum Bias		
Iteration	Univariate Method	Scaling Factor 1	Scaling Factor 2	Scaling Factor 3	Scaling Factor 4	State	Year	Class
Expected	2.2657	2.1000	2.1000	2.1000	2.1000	1.1478	1.4403	2.1016
Observed	2.3053	2.1271	2.1088	2.1271	2.1095	1.2030	1.5084	2.1083
MSE	0.0658	0.0249	0.0207	0.0206	0.0228	0.0099	0.0149	0.0213
1	2.3802	2.1588	2.1801	2.2083	2.1394	1.2469	1.4534	2.1771
2	2.0924	1.9451	1.9030	1.9097	1.9427	1.1631	1.5583	1.9315
3	2.1894	1.9774	2.0200	2.0371	1.9742	1.1944	1.5059	2.0514
4	2.2927	2.1755	2.0755	2.0928	2.1518	1.2505	1.3676	2.0510
5	2.3338	2.1908	2.1298	2.1655	2.1426	1.1665	1.5365	2.0773
6	2.3500	2.2127	2.2156	2.2302	2.2132	1.0679	1.3904	2.2182
7	2.4182	2.1886	2.1941	2.2289	2.1514	1.3159	1.5362	2.1689
8	2.2514	2.0719	2.0278	2.0596	2.0368	1.3684	1.4208	1.9732
9	2.0880	1.9078	1.9158	1.9279	1.9166	1.1744	1.5555	1.9314
10	2.3873	2.2889	2.1538	2.1701	2.2455	1.1996	1.4365	2.1102
11	2.2490	2.0481	2.0324	2.0433	2.0486	1.2785	1.4612	2.0641
12	2.3228	2.1178	2.1601	2.1825	2.1109	1.1420	1.5201	2.1736
13	2.3093	2.2093	2.0827	2.1137	2.1556	1.2011	1.5610	2.0053
14	2.0638	1.8534	1.8354	1.8600	1.8543	1.3891	1.4913	1.8448
15	2.1417	1.9027	1.9514	1.9829	1.8830	1.2770	1.6096	1.9699
16	2.0675	1.9028	1.8807	1.9202	1.8672	1.2018	1.5927	1.8508
17	2.2452	2.0472	2.0482	2.0599	2.0556	1.1755	1.5753	2.0882
18	2.4198	2.3042	2.2645	2.3026	2.2484	1.0750	1.5073	2.2190
19	2.3490	2.1635	2.1565	2.1792	2.1507	1.1715	1.5553	2.1383
20	2.3289	2.1199	2.1324	2.1408	2.1210	1.1385	1.5696	2.1678
21	2.1259	1.8874	1.9175	1.9436	1.8794	1.2672	1.6252	1.9396
22	2.1688	2.0283	2.0268	2.0478	2.0267	1.1261	1.3799	2.0264
23	2.2252	2.0744	2.0476	2.0590	2.0720	1.1920	1.4154	2.0694
24	2.4252	2.2689	2.1816	2.2237	2.1666	1.2822	1.5327	2.1346
25	2.3196	2.1454	2.1753	2.1842	2.1531	1.1397	1.3929	2.1718
26	2.1485	1.9262	1.9610	1.9834	1.9253	1.1609	1.7169	1.9640
27	2.4424	2.2630	2.2040	2.2237	2.2470	1.2571	1.4515	2.1820
28	2.1618	2.0127	1.9780	2.0022	1.9958	1.1605	1.5399	1.9492
29	2.2255	2.1499	1.9920	1.9877	2.0999	1.2164	1.5028	2.0116
30	2.4324	2.2201	2.1840	2.1841	2.2225	1.2327	1.6465	2.1610
31	2.7787	2.5628	2.5558	2.5526	2.5620	1.1933	1.4861	2.5268

EXHIBIT 5

Part 2

Stochastic Model with Truncated Lognormally Distributed Losses ($25,000 Limit)

	Loss Ratios							
	State 1				State 2			
	Year 1		Year 2		Year 1		Year 2	
	Class 01	Class 02	Class 01	Class 02	Class 01	Class 02	Class 01	Class 02
Expected	0.5000	0.5250	0.7500	0.7875	0.6000	0.6300	0.9000	0.9450
Observed	0.4973	0.5229	0.7530	0.7890	0.5969	0.6309	0.9005	0.9448
MSE	0.0077	0.0077	0.0083	0.0043	0.0065	0.0024	0.0065	0.0013
1	0.5754	0.6351	0.6206	0.7198	0.5008	0.6076	0.9345	0.9533
2	0.5050	0.4458	0.7667	0.7910	0.7212	0.5417	0.8576	0.8932
3	0.4305	0.4164	0.7654	0.8759	0.6392	0.6960	0.9926	0.9237
4	0.6062	0.4894	0.7510	0.7166	0.6680	0.6652	0.8156	0.9430
5	0.5885	0.6086	0.8329	0.7549	0.4763	0.6185	0.9199	0.9774
6	0.5641	0.6105	0.7519	0.8485	0.5819	0.6475	0.8220	0.9058
7	0.3439	0.5371	0.7703	0.7359	0.5369	0.6851	0.9492	0.9785
8	0.4277	0.4732	0.7943	0.6271	0.5626	0.6907	0.9059	0.9142
9	0.4561	0.5476	0.7536	0.7320	0.6524	0.5204	0.8835	0.8741
10	0.5993	0.3873	0.8301	0.7960	0.5989	0.7115	0.8115	0.9646
11	0.5112	0.4113	0.6639	0.7816	0.6937	0.6421	0.8956	0.9346
12	0.4277	0.5407	0.7578	0.8461	0.5572	0.6457	0.9186	0.9228
13	0.4600	0.5396	0.9565	0.7053	0.5594	0.6241	0.8322	0.9640
14	0.4575	0.3483	0.6963	0.6661	0.6889	0.6096	0.9341	0.8825
15	0.4153	0.5242	0.7279	0.7735	0.5746	0.6100	1.0758	0.9454
16	0.5335	0.5326	0.8806	0.7403	0.5319	0.6038	1.0331	0.9360
17	0.3894	0.3630	0.6886	0.7945	0.6294	0.5661	0.8089	0.8480
18	0.4358	0.5631	0.8784	0.8425	0.4735	0.6724	0.8431	0.9165
19	0.4207	0.5548	0.8551	0.8276	0.5852	0.6575	0.9104	0.9774
20	0.6114	0.5239	0.6711	0.8798	0.5741	0.5867	0.9348	0.9741
21	0.5512	0.5160	0.6732	0.7589	0.5653	0.5557	1.0586	0.9532
22	0.5676	0.5579	0.7885	0.8012	0.6365	0.6578	0.8857	0.8938
23	0.4954	0.5022	0.7312	0.7877	0.7174	0.6319	0.8392	0.9097
24	0.5059	0.4322	0.7337	0.7261	0.4233	0.6761	0.9291	0.9475
25	0.5268	0.7172	0.7390	0.7731	0.6009	0.6251	0.8721	0.9348
26	0.4864	0.5932	0.8073	0.8001	0.5366	0.5285	1.0234	0.9589
27	0.5085	0.4261	0.6857	0.7220	0.5505	0.6438	0.7971	0.9153
28	0.5388	0.5572	0.9476	0.8214	0.6261	0.6596	0.9785	0.9905
29	0.5729	0.4374	0.7486	0.7783	0.7889	0.5675	0.7865	0.9569
30	0.4480	0.5968	0.7583	0.7247	0.5706	0.5350	0.8382	0.9898
31	0.4251	0.7158	0.6981	0.7751	0.5259	0.6301	0.7727	1.0163

APPENDIX A

The symbolic representation of the impact of confounding on class relativity analysis due to the aggregation of more than one year and more than one state.

The Loss Experience Model

Let

e_{iys} = Earned exposures for class i, year y, state s

r_{ys} = Base class loss ratio for year y, state s

B_{ys} = Base Rate for year y, state s

b = Base class subscript

c_i = Current class factor for class i ($c_b = 1$)

f_i = Required factor for class i ($f_b = 1$)

g_i = Factor yielded by method for class i

E_i = Total earned exposures for class $i = \sum_y \sum_s e_{iys}$

P_i = Total earned premiums for class i on present rates

$= \sum_y \sum_s e_{iys} B_{ys} c_i$

L_i = Total incurred losses for class $i = \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i$

An "O" superscript indicates that the variable is relative to overall (all class) rather than the base class. For example:

r^O_{ys} = Overall class loss ratio for year y, state s

f^O_{iys} = Required factor for class i where overall class factor for year y, state s is unity

Special Case

If each class's loss ratio is related to the base class loss ratio, use the Special Case below to determine relativities. The bias resulting from the Loss Ratio Method, the Pure Premium Method, and the Modified Loss Ratio Method have been derived.

Pure Premium Method

$$g_i = \frac{\dfrac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys}}}{\dfrac{\sum_y \sum_s e_{bys} r_{ys} B_{ys} f_b}{\sum_y \sum_s e_{bys}}}$$

The bias in the method is

$$\frac{g_i}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys}} \bullet \frac{\sum_y \sum_s e_{bys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys} f_i} - 1$$

$$= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{E_i} \bullet \frac{E_b}{L_b} - 1.$$

Loss Ratio Method

$$g_i = \frac{\dfrac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys} c_i}}{\dfrac{\sum_y \sum_s e_{bys} r_{ys} B_{ys} f_b}{\sum_y \sum_s e_{bys} B_{ys} c_b}}$$

The bias in the method is

$$\frac{g_i}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_i} \bullet \frac{\sum_y \sum_s e_{bys} B_{ys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1$$

$$= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{P_i} \bullet \frac{P_b}{L_b} - 1.$$

Modified Loss Ratio Method

$$g_i = \frac{\dfrac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys} c_b}}{\dfrac{\sum_y \sum_s e_{bys} r_{ys} B_{ys} f_b}{\sum_y \sum_s e_{bys} B_{ys} c_b}}$$

The bias in the method is

$$\begin{aligned}
\frac{g_i}{f_i} - 1 &= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{\sum_y \sum_s e_{bys} B_{ys}}{\sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1 \\
&= \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{P_b}{L_b} - 1. \qquad \text{(A.1)}
\end{aligned}$$

General Case

If each class's loss ratio is related to the overall loss ratio rather than the base class loss ratio, to determine relativities use the General Case below. Only the error resulting from the modified loss ratio method has been derived.

Modified Loss Ratio Method

$$g_i^O = \frac{\dfrac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys} c_b}}{\dfrac{\sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_i \sum_y \sum_s e_{iys} B_{ys} c_b}}$$

The bias in the method is

$$\frac{g_i^O}{f_{iys}^O} - 1 = \frac{1}{f_{iys}^O} \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys} c_b}{\sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i} - 1.$$

APPENDIX B

Derivation of Scaling Factors

Criterion 1: The scaling factor should maintain the relationship between class loss ratios by year and state.

Criterion 2: The scaling factor should reduce the method error to zero.

Let primed variables indicate variables after the application of a scaling factor (e.g., g_i' is the factor yielded by a method after the application of a scaling factor).

First Special Scaling Factor—Scaling Factor 1

Consider Equation (A.1) (from Appendix A):

The bias in the method is

$$\begin{aligned}\frac{g_i}{f_i}-1 &= \frac{\sum_y\sum_s e_{iys}r_{ys}B_{ys}}{\sum_y\sum_s e_{iys}B_{ys}c_b}\bullet\frac{\sum_y\sum_s e_{bys}B_{ys}}{\sum_y\sum_s e_{bys}r_{ys}B_{ys}}-1\\ &= \frac{\sum_y\sum_s s_{ys}e_{iys}r_{ys}B_{ys}}{\sum_y\sum_s e_{iys}B_{ys}c_b}\bullet\frac{\sum_y\sum_s e_{bys}B_{ys}}{\sum_y\sum_s s_{ys}e_{bys}r_{ys}B_{ys}}-1=0.\end{aligned}$$

If each $r'_{ys}=1$, then error in method $=0$; therefore $1/r_{ys}$ is a scaling factor and $S_{ys}=1/r_{ys}$ and will be applied to each loss.

Second Special Scaling Factor—Scaling Factor 2

Consider a scaling factor S, to be applied to premiums and losses:

$$\text{Let}\quad S_{iys}=\frac{\sum_y\sum_s e_{iys}}{\sum_y\sum_s e_{bys}}\bullet\frac{e_{bys}}{e_{iys}}.$$

Also (for convenience)

$$\text{Let}\quad e'_{iys}=\frac{\sum_y\sum_s e_{iys}}{\sum_y\sum_s e_{bys}}\bullet e_{bys}.$$

The bias in the method is

$$\begin{aligned}
\frac{g_i'}{f_i} - 1 &= \frac{\sum_y \sum_s S_{iys} e_{iys} r_{ys} B_{ys}}{\sum_y \sum_s S_{iys} e_{iys} B_{ys} c_b} \bullet \frac{\sum_y \sum_s S_{iys} e_{iys} B_{ys}}{\sum_y \sum_s S_{iys} e_{iys} r_{ys} B_{ys}} - 1 \\
&= \frac{\sum_y \sum_s e_{iys}' r_{ys} B_{ys}}{\sum_y \sum_s e_{iys}' B_{ys} c_b} \bullet \frac{\sum_y \sum_s e_{bys}' B_{ys}}{\sum_y \sum_s e_{bys}' r_{ys} B_{ys}} - 1 \\
&= \frac{e_i' \sum_y \sum_s e_{bys} r_{ys} B_{ys}}{e_i' \sum_y \sum_s e_{bys} B_{ys} c_b} \bullet \frac{e_b' \sum_y \sum_s e_{bys} B_{ys}}{e_b' \sum_y \sum_s e_{bys} r_{ys} B_{ys}} - 1 = 0,
\end{aligned}$$

where $e_i' = \sum_y \sum_s e_{iys} / \sum_y \sum_s e_{bys}$.

So this scaling factor satisfies Criterion 2.

Since this scaling factor is applied to premiums *and* losses by class, each class loss ratio remains unchanged satisfying Criterion 1.

Second Special Scaling Factor:

$$S_i = \frac{\sum_y \sum_s e_{iys}}{\sum_y \sum_s e_{bys}} \bullet \frac{e_{bys}}{e_{iys}}.$$

General Scaling Factors

If each class's loss ratio related to the overall loss ratio is used rather than the base class loss ratio, another set of scaling factors (generalized scaling factors) is used. First it is necessary to establish some relationships:

Define

$$f_{iys}^O = \frac{f_i \sum_i e_{iys}}{\sum_i e_{iys} f_i}.$$

Then

$$r_{ys}^O = \frac{\sum_i e_{iys} r_{ys} B_{ys} f_i}{\sum_i e_{iys} B_{ys} c_b} = \frac{r_{ys} \sum_i e_{iys} f_i}{\sum_i e_{iys}} = \frac{r_{ys} f_i}{f_{iys}^O}$$

and $r^O_{ys} f^O_{iys} = r_{ys} f_i$. Also

$$r^O_{ys} = \frac{\sum_i e_{iys} r_{ys} B_{ys} f_i}{\sum_i e_{iys} B_{ys} c_b} = \frac{\sum_i e_{iys} r^O_{ys} B_{ys} f^O_{iys}}{\sum_i e_{iys} B_{ys} c_b}$$

$$= \frac{r^O_{ys} B_{ys} \sum_i e_{iys} f^O_{iys}}{B_{ys} \sum_i e_{iys} c_b} = \frac{r^O_{ys} \sum_i e_{iys} f^O_{iys}}{\sum_i e_{iys}},$$

therefore $\sum_i e_{iys} f^O_{iys} = \sum_i e_{iys}$.

First General Scaling Factor—Scaling Factor 3

Consider a scaling factor, to be applied to losses only.

$$S_{ys} = \frac{\dfrac{\sum_i e_{iys} c_b B_{ys}}{\sum_i e_{iys}}}{\dfrac{\sum_i e_{iys} r_{ys} f_i B_{ys}}{\sum_i e_{iys}}} = \frac{\sum_i e_{iys} c_b}{\sum_i e_{iys} r_{ys} f_i} = \frac{\sum_i e_{iys}}{r_{ys} \sum_i e_{iys} f_i} = \frac{f^O_{iys}}{r_{ys} f_i} = \frac{1}{r^O_{ys}}.$$

$$\frac{g_i}{f_i} - 1 = \frac{\dfrac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys} c_b} \bullet \dfrac{\sum_i \sum_y \sum_s e_{iys} B_{ys} c_b}{\sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}}{f^O_{iys}} - 1$$

$$= \frac{1}{f^O_{iys}} \bullet \frac{\sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys} f_i} - 1.$$

$$\frac{g'_i}{f_i} - 1 = \frac{1}{f^O_{iys}} \bullet \frac{\sum_y \sum_s e_{iys} S_{ys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_i \sum_y \sum_s e_{iys} S_{ys} r_{ys} B_{ys} f_i} - 1.$$

$$\frac{g'_i}{f_i} - 1 = \frac{1}{f^O_{iys}} \bullet \frac{\sum_y \sum_s e_{iys} \left(\dfrac{f^O_{iys}}{r_{ys} f_i} \right) r_{ys} B_{ys} f_i}{\sum_y \sum_s e_{iys} B_{ys}}$$

$$\bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_i \sum_y \sum_s e_{iys} \left(\dfrac{f^O_{iys}}{r_{ys} f_i} \right) r_{ys} B_{ys} f_i} - 1.$$

$$\frac{g_i'}{f_i} - 1 = \frac{1}{f_{iys}^O} \bullet \frac{\sum_y \sum_s e_{iys} f_{iys}^O B_{ys}}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_i \sum_y \sum_s e_{iys} f_{iys}^O B_{ys}} - 1.$$

$$\frac{g_i'}{f_i} - 1 = \frac{1}{f_{iys}^O} \bullet \frac{f_{iys}^O \sum_y \sum_s e_{iys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_y \sum_s B_{ys} \sum_i e_{iys} f_{iys}^O} - 1.$$

$$\frac{g_i'}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_y \sum_s B_{ys} \sum_i e_{iys}} - 1.$$

$$\frac{g_i'}{f_i} - 1 = \frac{\sum_y \sum_s e_{iys} B_{ys}}{\sum_y \sum_s e_{iys} B_{ys}} \bullet \frac{\sum_i \sum_y \sum_s e_{iys} B_{ys}}{\sum_i \sum_y \sum_s e_{iys} B_{ys}} - 1 = 0.$$

Second General Scaling Factor—Scaling Factor 4

Consider a scaling factor, to be applied to premiums and losses.

$$S_{iys} = \frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \frac{\sum_i e_{iys}}{e_{iys}}.$$

$$\text{Let} \quad e_{iys}' = \frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \sum_i e_{iys}.$$

The bias in the method is

$$\frac{g_i'^O}{f_{iys}^O} - 1 = \frac{1}{f_{iys}^O} \bullet \frac{\sum_y \sum_s e'_{iys} r_{ys} B_{ys} f_i}{\sum_y \sum_s e'_{iys} B_{ys} c_b} \bullet \frac{\sum_i \sum_y \sum_s e'_{iys} B_{ys}}{\sum_i \sum_y \sum_s e'_{iys} r_{ys} B_{ys} f_i} - 1$$

$$= \frac{1}{f_{iys}^O} \bullet \frac{\sum_y \sum_s \left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \sum_i e_{iys} \right) r_{ys}^O B_{ys} f_{iys}^O}{\sum_y \sum_s \left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \sum_i e_{iys} \right) B_{ys} c_b}$$

$$\bullet \frac{\sum_i \sum_y \sum_s \left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \sum_i e_{iys} \right) B_{ys}}{\sum_i \sum_y \sum_s \left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \sum_i e_{iys} \right) r_{ys}^O B_{ys} f_{iys}^O} - 1$$

$$= \frac{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_y \sum_s \left(\sum_i e_{iys}\right) r^O_{ys} B_{ys}}{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_y \sum_s \left(\sum_i e_{iys}\right) B_{ys} c_b}$$

$$\bullet \frac{\left(\frac{1}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \left(\sum_y \sum_s e_{iys}\right) \sum_y \sum_s \left(\sum_i e_{iys}\right) B_{ys}}{\left(\frac{1}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \left(\sum_y \sum_s e_{iys}\right) \sum_y \sum_s \left(\sum_i e_{iys} f^O_{iys}\right) r^O_{ys} B_{ys}} - 1$$

$$= \frac{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys}}{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \sum_y \sum_s e_{iys} B_{ys} c_b}$$

$$\bullet \frac{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \sum_y \sum_s e_{iys} B_{ys}}{\left(\frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}}\right) \sum_i \sum_y \sum_s e_{iys} r_{ys} B_{ys}} - 1 = 0.$$

So this scaling factor satisfies Criterion 2.

Since this scaling factor is applied to premiums *and* losses by class, each class loss ratio remains unchanged, satisfying Criterion 1.

Second General Scaling Factor:

$$S_{iys} = \frac{\sum_y \sum_s e_{iys}}{\sum_i \sum_y \sum_s e_{iys}} \bullet \frac{\sum_i e_{iys}}{e_{iys}}.$$