AUTHOR'S RESPONSE TO DISCUSSION OF PAPER PUBLISHED IN VOLUME XCI

DISTRIBUTION-BASED PRICING FORMULAS ARE NOT ARBITRAGE-FREE

DAVID L. RUHM

1. INTRODUCTION

I am honored that the paper has drawn interest from my colleagues and that Mr. Wacek has written a discussion of it. This reply addresses some matters that were raised in the discussion.

2. THE PAPER'S TITLE

The discussion criticizes the paper's title, claiming it is "clearly too categorical," but provides no support for this claim other than:

"In the paper, the author himself points out that insurance prices that are based on the probability distribution of outcomes can be arbitrage-free."

The author believes this is a misinterpretation of some part of the paper, since the paper makes the opposite statement in several places, supported by a proof. Perhaps the discussion's claim is referring to Section 6.3, which states that insurance can be priced with distribution-based formulas but does *not* state that those prices are arbitrage-free:

"Insurance almost never covers asset-event combinations that are traded in a liquid market.... As vulnerability to arbitrage does not exist for insurance, formulas that are theoretically not arbitrage-free can be used to price insurance risks without consequent economic penalty."

The paper's formal mathematical proof that distribution-based pricing formulas are not arbitrage-free covers the general case, which supports the title's accuracy.

3. PRICE VERSUS EXPECTED COST AND PRACTICAL RISK PRICING

The discussion highlighted the paper's point that a risk's price and expected value are generally distinct. In practice, risk pricing methods often deal with expected value directly, by charging the expected cost plus a risk load that is based on the risk's distribution so as to provide a margin of safety and an expected gain opportunity.

This is a reasonable, effective approach to risk pricing. The fact that the resulting prices are often technically not arbitrage-free is usually of no practical consequence. Even derivatives could be selectively bought or sold in this way with successful results. Furthermore, such a strategy can be implemented using simple risk pricing methods, without employing exotic formulas. There is interpretive value gained from using simple, transparent formulas, and there is usually not much (if any) benefit to be gained from using more complex formulas, even if they have some theoretical appeal.

With recent advances in risk pricing theory, an actuary could be tempted to use a complex pricing formula that has theoretical connections with arbitrage-free pricing, believing that the resulting prices will be arbitrage-free and, therefore, more economically accurate than prices derived by simpler methods. The point of the paper is that this is generally not true if the complex formula uses only outcome probabilities to calculate the risk load. The pricing formula generally has to incorporate co-variation with underlying events in order to produce genuine arbitrage-free prices.

For example, the single-parameter Wang transform can produce arbitrage-free options prices if the formula's parameter is set to the right value for a particular stock, but the specific parameter value varies widely among stocks and might not even be defined for most insurance risks. Different values of the parameter will give different prices, and it can be unclear how to set the parameter in order to obtain a sensible risk load.

On the other hand, the Wang transform can be used to calibrate a set of interrelated insurance prices to a set of options prices, providing some consistency to the insurance pricing structure. For example, various loss layers for a particular risk can be assigned prices corresponding to analogous option spreads on a stock. The prices for the loss layers would then be additive and generally free of internal inconsistencies, like arbitrage-free option prices.

4. THE ROULETTE WHEEL WITH PAYOFFS THAT VARY BY SPACE

Regarding the example involving the "Ruhm Roulette Wheel" (the name used in the discussion for the metaphorical roulette wheel described in the paper), note that the paper does not claim that a *physical* roulette wheel should have the varying payoffs described. The paper explains in Sections 5.2 and 6.1 that these peculiar roulette-like bets, which all have equal odds but varying payoffs, exist in markets with Black-Scholes pricing. The roulette wheel with varying payoffs is effectively embedded in any such market. One places a bet by buying and selling derivatives in a combination designed to create the particular bet desired, as explained in the paper. (Whether the position is achieved by buying or selling is not relevant, since only the net resulting position determines the economics.)

These surprising bets also exist in other markets. As mentioned above, the paper proves the result in general for markets with arbitrage-free pricing. Black-Scholes pricing is a case that is particularly useful for demonstration, since it is probably a well-known arbitrage-free pricing formula. (The roulette-like bets can also exist in markets that are not arbitrage-free.)

The question that the discussion's example begs is, "If these bets exist in actual financial markets, then why doesn't every participant make only the highest-payoff bets?" The short answer is that probability and risk are distinct concepts, meaning that two events can have the same probability but still differ in risk because of existing risk aggregation. High-payoff bets in markets require assuming risk on just those possibilities for which many parties are already exposed to capital loss (such as a catastrophe). Some market participants can't afford to bet on them because of their existing exposure, even though such a bet offers a positive expected value, since it would expose them to loss when they could least afford it. The high-payoff bets are less attractive to the market as a whole than their simple expected value would suggest because of broad existing risk exposure to the underlying events.

By contrast, a physical roulette wheel offers bets on trivial physical events, all of which have no connection with a participant's existing capital and exposure to risk. Therefore, the spaces are equally preferable, so, logically, payoffs for the spaces are equal.

This key concept—the distinction between probability and risk—is the crucial point underlying the results described in the paper. The difference between probability and risk becomes clearer in the insurance examples presented below, which also make the varying-payoff bets more apparent.

The discussion proposed a different answer to the question, based on the idea that people's opinions differ as to whether the expected return on a stock, E, is higher or lower than the risk-free rate, r. While people do have a variety of opinions on stocks, that explanation does not actually answer the question, because the Black-Scholes pricing theory still works even when the exact value of E is fixed and known by all market participants. (Other pricing theories also work under this condition.) The roulette-like bets with varying payoffs would still exist in such a market, where all participants' opinions about E are the same. The unusual roulette-like bets do not depend on differences of opinion regarding expected value; they depend on exposure of existing capital to loss from potential future events.

The probability/risk distinction is an economic concept based on existing capital and risk exposure that does not rely upon any assumed psychological causes, in contrast to the discussion's characterization of it. By contrast, the "variation of opinion" conjecture offered in the discussion seems more psychologically based than economic.

5. THE DISTINCTION BETWEEN PROBABILITY AND RISK

The insurance market demonstrates the probability/risk concept more clearly than the options market. Differences in exposure, rather than differences in opinion, drive demand for insurance from both personal and commercial customers. Insurance buyers generally do not expect to profit from purchasing insurance, and they do not undertake it as an investment with the prospect of a gain based on expected loss costs. (Those who do might comprise the moral hazard element in the insured population.) Insurance is commonly understood as the cost for hedging risk on assets.

Reinsurance is a clear example. Insurers often accept an expected net cost when buying reinsurance and expect that the reinsurer has an expected profit built into the price. The ceding insurer pays this net cost in order to hedge and manage risk on its book. Insurers are not ignorant in regard to insurance and expected value, yet they often pay more than expected value for reinsurance. They make rational, risk-hedging bets that have negative expected outcomes.

In summary, people and companies that buy insurance effectively make bets having negative expected values in order to obtain reduction of risk, just as some stockholders buy put options on their stocks to reduce risk.

The roulette wheel described in the paper is tied to financial events. The low-numbered spaces come up when a specified asset (such as a stock or a property) suffers a loss of value, exactly when the owners of the asset would require financial relief from the loss. Betting on a low-numbered space is analogous to buying a risk-hedge on the asset, like insurance. The event-driven roulette wheel is a model for representing risk transfer transactions that occur in a variety of forms, such as put options and insurance, but that are all similar in nature: they are wagers on events that impact assets that are valuable to their owners.

6. HURRICANE INSURANCE

Taking another example, the total capital exposed to risk from a hurricane in Florida appears to drive the market price of risk transfer. The more property that is exposed, the greater the demand for this type of coverage. (Variation in people's opinions regarding a hurricane's expected loss cost probably doesn't create most of the demand for coverage.)

While coastal property assets are exposed to the risk of a hurricane's occurrence, parties who do not own coastal property may be economically unaffected by the event, so it poses no risk to them. Although the event's probability remains constant, they have no capital exposure and, therefore, no risk from the event. The noncoastal parties could be in a position to profit by selling insurance to coastal property owners.

Some parties, such as owners of building materials, might actually stand to obtain an economic benefit from a hurricane's occurrence. They would be in a better position to make the positive-value bet of writing insurance, since they own natural hedges to the risk. In insurance parlance, they have more "capacity" to assume risk on such an event.

In summary, insurance buyers are making a negative-expected-value bet on hurricane occurrence during the year, while insurance writers are making a positive expected-value bet on this being a lighter year for hurricane losses. The roulette wheel exists in this market as well, and there are plenty of players willing to bet on the low-payoff spaces because of existing risk exposure.

When risk from hurricane is summed across all parties worldwide—including those who are exposed to loss, those who could gain from such an event, and those not exposed—the net total result is positive risk exposure, because the net economic result of a hurricane is destruction of existing capital. (After a hurricane, there is less total capital than before.)

This positive net risk exposure means that the total demand for insurance risk transfer from those exposed to loss should be stronger than the total supply of insurance risk-hedging capacity, in the absence of a risk load. In other words, if regulations stipulated that coastal hurricane insurance could only be offered at expected value pricing, it's likely that there would be more demand for insurance than supply. This conclusion of the theory coincides with what one would reasonably expect in actual insurance markets.

Risk charges in premiums bring supply and demand into balance. Even if there were a perfect, liquid worldwide market for hurricane coverage, the risk load for hurricane risk transfer would have to be positive, based on these economic forces. This dynamic of profit incentive versus risk reduction on capital makes risk transfer markets possible.

7. NET CAPITAL AT RISK DRIVES THE RISK CHARGE FOR THE EVENT

The hurricane example demonstrates the nature of markets in risk transfer; they are driven by potential loss of current capital. In the hurricane insurance example, the capital assets are coastal properties. In the put options example, the market value of business equity is at stake. In all such cases, there is net capital at risk. The risk of loss to capital carries a net risk charge in a liquid market; the other side of that bet is insurance, which carries a compensating premium. The net risk charge and the compensating premium are mirror images of the same quantity. In practical terms, the risk charge that can be included in insurance sold to a property owner is driven by risk to the owner's property, not just by probability.

8. INTERPRETATION OF THE RISK DISCOUNT FUNCTION $w(s)$

On a technical point raised in the discussion, it is true that the risk discount function $w(s)$ is parameterized by the variable E, as was shown in the paper's derivation of the formula for $w(s)$. The paper's formula for $w(s)$ appears to be simpler than the discussion's.

The discussion states that each person has his own opinion of E's value and challenges the paper's conclusions on that basis. After considering that argument, the author believes that the paper's main conclusions in regard to $w(s)$ still stand: if a person's estimate of E gives $w(s) < 1$, then the price is discounted for risk in the person's estimation and represents an investment with net expected gain. If $w(s) > 1$, the participant is paying a surcharge (in the participant's estimation) and is doing so for insurance (i.e., to hedge risk) or possibly for the entertainment value of a gamble.

9. CONCLUSION

In summary, it is the risk of an event to net current capital that determines whether there is a risk charge and how much it will be in a liquid market. A loss distribution alone is generally not sufficient to establish risk load; risk load also depends on the impacts that the events have on the specific capital base against which risk is to be assumed. The existing exposure of that capital base is central to evaluating the contemplated risk assumption. Distribution-based pricing formulas, by definition, measure potential events but do not measure any relationship between those events and the assuming capital's existing exposure. The nature of the capital base is the reference for defining risk.