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DISTRIBUTION-BASED PRICING FORMULAS ARE NOT ARBITRAGE-FREE

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Abstract

David Ruhm is entirely correct that risk load formulas based on transforming probability distributions of contract outcomes cannot guarantee arbitrage-free prices. This is what he illustrates by a clever and entertaining example. But the title of the paper seems to assert that no method of transforming distributions is arbitrage-free. This is not the case, as transforms of the probabilities of the underlying events that generate the outcomes are well known to produce arbitrage-free prices. In fact, Ruhm illustrates this by showing that the Black-Scholes formula arises from such a transform. He also shows that this formula builds in risk-adjustments to prices, thus addressing the misapprehension that since the options prices come from a risk-neutral valuation they do not incorporate risk adjustments.

To illustrate the application of probability transforms to fundamental events in insurance, this discussion provides an example of using an alternative transform of underlying frequency and severity distributions to price loss layers.

Arbitrage pricing theory is often described as showing that prices are arbitrage-free if and only if they are based on transformed probabilities. This is an over-simplification. As David Ruhm's paper shows, it is possible to create examples where no transformations of the probabilities of the outcomes produce arbitrage-free prices. What arbitrage-pricing theory actually requires is transforms of the probabilities of the underlying events that generate the outcomes. This is the fundamental result of arbitrage pricing theory, but is often stated more abstractly. For example, see Brigo and Mercurio (2001) page 25. Also Panjer (1998) page 180 makes it clear that probabilities are applied to states of nature, and the prices of securities are functions of the states. Thus any probability transform will apply to the underlying states. Furthermore, the impossible events (those with zero probability) have to be the same under the original and transformed probabilities. Two probability measures with the same set of possible outcomes are sometimes called "equivalent."

Ruhm's paper illustrates this effect for the pricing of stock options. Although he does not present it this way, his results show that applying transforms to options prices will not be arbitrage-free, but applying them to the prices of the underlying securities will be. For a stock that has a lognormal distribution in μ and σ with expected return E and risk-free rate r Ruhm states:

"The Black-Scholes price of an option is equal to the option's discounted expected value, under a risk-neutral lognormal density function that is parameterized by μ^* and σ^*:

$$\mu^* = \mu - \ln[(1+E)/(1+r)]$$

$$\sigma^* = \sigma."$$

This makes it clear that it is the probability distribution of the stock price that is transformed in the Black-Scholes model. The expected value of an option's outcome under the transformed probability distribution is the option price, and these prices are known to be arbitrage-free in this model. Ruhm's roulette wheel example shows further that transforming the probabilities of the outcomes of the options themselves will not give the same answer. Ruhm does not claim that Black-Scholes prices contain arbitrage, as one might think from the paper's title. Thus the paper effectively distinguishes between transforming event probabilities and transforming the probabilities of contract outcomes, even though it does not strongly emphasize this distinction.

For insurance pricing, the comparable underlying events are the primary insurance claim counts and loss sizes. Pricing based on transforming these frequency and severity probabilities (keeping the same zero probability events—negative losses, perhaps) is what is required by arbitrage-pricing theory. Ruhm comes close to this conclusion when he states: "The value of the insurance is determined by the stochastic process of the covered perils; the value of the derivative is driven by the stochastic process of the asset's market price. If insurance could be thought of as a derivative at all, it would be as a derivative of hurricane occurrence and severity, auto accident occurrences and severities, etc."

What will not work is transforming the probabilities of outcomes of contracts—such as aggregate losses, reinsurance layers, etc.—which would be like applying transforms to option prices. This is not entirely new to the CAS literature. Arbitrage-free pricing provides a completely additive allocation of the overall company risk load to line and contract. Wang (1998) gives examples where transforming the probabilities of the results of aggregate covers produces strictly sub-additive allocations, which are thus not arbitrage-free.

However, this does not automatically mean the sub-additive allocations are wrong. For one thing, there is a tension in the pricing literature between calculating actual market prices and the prices a company would ideally like to achieve, which might contain arbitrage possibilities. For another, there are issues of incompleteness in insurance markets that some observers feel permit a degree of theoretical arbitrage possibilities that can never in fact be realized. The arbitrage possibilities that can actually exist in the insurance market is not a settled issue. However some lines of business are very competitive, and if a company has pricing structures that would allow arbitrage against it in a complete market, it could end up with competitive disadvantages.

As an example, suppose a company would like to make the profit load on a fleet of 100 cars 10 times the load on a single car, so it is 10% of the load on a per car basis. It wants to do this because the fleet is more stable. In a complete market, an arbitrageur might sell 100 individual policies and cede them bundled to the company at the fleet price, thus ending up with 90% of the profit and no risk. But barriers to entry, etc. might prevent this from taking place in the real market. Nonetheless, a competitor could decide that since it is doing the diversification internally, it can sell the individual policies at the fleet price. If the fleet price has the right risk and return characteristics, then the competitor ends up with this risk profile on its book of individual policies, and the first company loses a book of business that as a whole it would find desirable. Thus arbitrage opportunities can be competed away, even in an incomplete insurance market without arbitrageurs.

Supposing that a company does want to set arbitrage-free prices, what transforms would be appropriate for claim frequency and severity distributions? The basic result for arbitrage-free pricing is that prices are expected values from a transformed process that is a martingale. This criterion requires that there is no expected upward or downward trend in the transformed process. For insurance this means that the aggregate transformed frequency and severity processes have a mean equal to that of the overall loaded premium. Then premium minus transformed losses has an expected value of zero, although perhaps a great deal of volatility. Unlike the Black-Scholes case, however, there is not a unique transform in the insurance market. This is typical of incomplete markets—that is markets where not every instrument can be subdivided and hedged at will.

A recent paper, Møller (2003), summarizes much of the literature on probability-transform pricing for the compound Poisson process with risk-loaded premium. The fundamental result he presents, based on Girsanov's Theorem (a basic element of arbitrage-pricing theory), describes a procedure for producing arbitrage-free transforms of frequency and severity distributions.

The starting point is selecting a function $\phi(y)$, where the loss size variable is Y, with the only restriction being that $\phi(y) > -1$ for all positive losses y. The frequency parameter λ is transformed to $\lambda[1 + E\phi(Y)]$. The severity density $g(y)$ gets transformed to $g(y)[1 + \phi(y)]/[1 + E\phi(Y)]$.

Møller introduces a ranking order for such transforms, based on specific pricing impacts. He provides several examples, three of which are reasonable in terms of being in the middle of the ranking order. The transforms are calibrated by a parameter $\theta = E[Y\phi(Y)]/EY$ which is the loading in the primary rates, so that the primary loaded pure premium is $(1 + \theta)\lambda E(Y)$.

The first example, from Delbaen and Haezendonck (1989), sets $\phi(y) = \theta(y - EY)EY/\text{Var}(Y)$. This is > -1 $\forall y$ as long as $\theta < \text{CV}^2$, where CV is the severity coefficient of variation—its ratio of standard deviation to mean. Since $E\phi(Y) = 0$, the transformed frequency parameter is just the actual parameter λ and the transformed severity density is $g(y)[1 + \phi(y)]$.

An example Møller introduces, which he calls the minimum martingale measure, takes $\phi(y) = (y/EY)\theta/[1 + \text{CV}^2]$, with $E\phi(Y) = \theta/[1 + \text{CV}^2]$. The transformed frequency is $\lambda[1 + \theta/(1 + \text{CV}^2)]$, and the transformed severity density is $g(y)[1 + \text{CV}^2 + \theta y/EY]/[1 + \text{CV}^2 + \theta]$. In this and the previous transform, severity probabilities are reduced for losses below the mean, and increased for losses above it. This is minimal in its squared distance from the actual probability measure.

The third example Møller calls the minimum entropy martingale measure. It starts with $\phi(y) = e^{\eta y} - 1$. Then the transformed frequency is $\lambda E e^{\eta Y}$ and the severity is $g(y)e^{\eta y}/Ee^{\eta Y}$. If you want this to match a pre- existing premium load θ, you need to find $\eta > 0$ so that $E[Ye^{\eta Y}] = (1 + \theta)E(Y)$. This is not possible for some severity distributions, but if the severity has policy limits or is light tailed, like a mixed exponential, the expectation will exist. Usually η will be quite small, like maybe 10^{-10}.

The relative entropy between two measures P and Q is $EP[dQ/dP \log(dQ/dP)]$. This is a distance of a sort, as it is zero if $P = Q$ and is otherwise positive. However it is not symmetric in P and Q. Minimizing the relative entropy is a popular fitting method and is related to optimizing a fit given the information available, according to principles of information theory. In the insurance pricing case, P is the real-world measure and Møller shows that the transform above gives the martingale Q that minimizes the relative entropy. Q is then the martingale closest to the actual probability measure P in the sense of relative entropy. The minimum entropy is usually realized by the Esscher transform. In fact Ballotta (2004) shows that the minimum entropy transform above is the Esscher transform applied to frequency and severity combined. However, Møller shows that applying the Esscher transform to severity alone, which would be the above transform for severity but with no change to frequency, gives less satisfactory results by his criteria. This transform uses $\phi(y) = e^{\eta y}/Ee^{\eta Y} - 1$. Then the transformed frequency is just λ and the severity is still $g(y)e^{\eta y}/Ee^{\eta Y}$.

For an example of the minimum martingale measure, consider a book of business with 2,500 expected claims, a Pareto severity $G(y) = 1 - (1 + y/10{,}000)^{-1.2}$, a policy limit of 10,000,000, and a loading of $\theta = 20\%$. The severity mean and CV^2 are about 37,443 and 43.11. This makes the λ load factor $(1 + 0.2/44.11) = 1.00453$. The factor on $g(y)$ is $(44.11 + y/187{,}215)/44.31$. This can be applied numerically to a discretization of the severity distribution. The maximum severity has to stay at 10,000,000 in order to keep the zero-probability events the same. The original probability mass at 10,000,000 is 0.025% which gets transformed to 0.055%. The severity mean is increased by 19.46%, which together with the frequency transform gets the 20% load.

The transformed probabilities can be used to price any type of contract on this business as the expected value of the contract using the transformed probabilities. In this case that has to be done numerically with the discretized severity. The risk load for any contract is its expected value from the transformed probabilities less its expected value from the actual probabilities. For instance, a 4,000,000 xs. 1,000,000 contract ends up with a risk load of 62.3%, and a 5,000,000 xs. 5,000,000 gets 112.8%. The total amount of those loads is 13,730,500, which is the risk load for the layer 9,000,000 xs. 1,000,000 calculated separately. This is 73.3% of the entire loading on the primary business—as most of the risk is attributed to the higher layers by this method.

It is also interesting to apply this example to a difficult test case for pricing methods attributed to Thomas Mack, which is to price a buy-back of a franchise deductible. For example, for a deductible of 1,000, this contract would pay the full loss if it is less than or equal to 1,000, but nothing if it is greater. Venter (1998) tries a number of pricing transforms on such contracts, and they all give negative risk loads in some cases. For the book of business outlined above and a range of deductibles, minimum martingale pricing gives a (barely) positive risk load. In fact the severity-only risk loads are negative in all cases tested, but the frequency load, small as it is, is enough to compensate and make the total load positive. The combined frequency-severity increment $\lambda g(y)$ can be seen to transform by a factor of $1 + \theta y/[(1 + \mathrm{CV}^2)EY]$, which is > 1 for any positive y. Thus any combination of losses will get a positive risk load. This will hold for the minimum entropy martingale as well.

The reason Mack's example is difficult is that transforms of severity have to produce a density that integrates to 1, so giving more probability to large losses must take it away from small losses. Thus contracts that cover only small losses tend to get negative loads. But as these examples show, that problem can be alleviated by making the percentage load on frequency greater than the largest reduction in severity probability. The transforms that do not do this, such as the one of Delbaen and Haezendonck and the Esscher transform of severity only, would be subject to Mack's problem.

The minimum entropy transform, with its exponential moment, gives higher loads to higher layers. In some reinsurance contracts tested, this was better than the minimum martingale transform at pricing low-mean high-variance layers, like top layers of cat programs. For example, see Venter, Barnett, and Owen (2004). Quadratic transforms, like the minimal martingale measure, appear to be less capable of matching market pricing of higher cat layers. The minimum entropy transform also has more theoretical strength, in that it is the closest martingale to the actual probabilities in the quasi-distance measure from information theory.

The minimum entropy and minimum martingale measures provide reasonable candidates for probability transforms for pricing insurance and reinsurance contracts. The key is that the price for a contract is the expected value of the contract outcomes under the transformed primary frequency and severity probabilities, and not, as Ruhm emphasizes, the mean of any transformation of the probabilities of the possible outcomes of the contract itself.

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