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DISTRIBUTION-BASED PRICING FORMULAS ARE NOT ARBITRAGE-FREE

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Abstract

David Ruhm's paper is a welcome addition to the actuarial literature. It illustrates some difficult concepts in a refreshing way. As actuaries are increasingly faced with the need to price non-traditional risks, it is important that they understand how to do so.

One of the paper's main points is to emphasize the important finding from financial economics that the probability distribution of risk outcomes does not always contain enough information to produce arbitrage-free prices for that risk. However, the probability distribution of outcomes can, and indeed must, be used to determine the expected cost of that risk. This discussion uses Ruhm's examples to underscore the distinction between price and cost, and the potential implications for the seller of a derivative.

Ruhm's paper also seeks to generalize about the arbitrage-free prices of calls and puts compared to their expected value payoff. Ruhm concludes that calls are priced at a discount and puts at a premium, at least when the underlying security has an expected return E that is greater than the risk-free rate r. He then seeks to explain why investors would buy puts, given that they are priced at a premium to expected value. He concludes that some risks have a qualitative nature as either insurance or investment. This pattern of discounted calls

and surcharged puts is true ONLY if $E > r$. Under the condition $E < r$, which Ruhm did not discuss, calls are surcharged and puts are discounted. As a result investor behavior can be accounted for in a simpler way than by appealing to investor risk aversion or the "qualitative nature of a risk."

1. INTRODUCTION

David Ruhm's paper is a fascinating attempt to make the paradox of Black-Scholes risk-neutral pricing more comprehensible to actuaries. His mapping of the distribution of stock prices onto a roulette wheel is a brilliant construct that makes plain just how bizarre the arbitrage-free prices that emerge from the risk-neutral framework are.

I have no quarrel with much of the paper. I do have a minor quibble with the title. It is clearly too categorical, and should be something like "Distribution-Based Pricing Formulas Are Not *Always* Arbitrage-Free." In the paper the author himself points out that insurance prices that are based on the probability distribution of outcomes can be arbitrage-free.

I also found it surprising that Ruhm focuses on the derivative buyer's perspective and virtually ignores the seller's perspective. Since actuaries are usually concerned with the pricing problem from the seller's point of view, and particularly since the author does not address it, I am going to weigh in with a discussion of the latter.

In addition, I will show that some of Ruhm's conclusions about his "risk discount" function and the buyer's motivation, which have the *appearance* of generality, depend on certain of his assumptions. He notes these assumptions but does not explore their importance to his conclusions. As a result, some readers might not realize that his conclusions do not hold under some realistic conditions that the author does not discuss.

2. DISTINCTION BETWEEN PRICE AND COST—SELLER'S PERSPECTIVE

Ruhm is correct in saying that the seller of an insurance policy or financial derivative cannot necessarily rely on the probability distribution of outcomes to correctly *price* the risk at the market clearing level unless certain conditions are present (or absent, depending on one's perspective). However, unless the seller takes certain actions that effectively change the applicable probability distribution (about which more later), he *must* use the probability distribution of outcomes to accurately assess the expected *cost* of the risk.

Suppose I decide to open a casino. I acquire the Ruhm Roulette Wheel together with a set of instructions that includes the set of correct arbitrage-free prices (from Exhibit 3 of the paper) to charge for bets on each number from 00 to 36. These prices seem counterintuitive, since they call for varying prices for equally likely outcomes, but I am new to this business, so who am I to question them? There is a section on hedging, but it looks complicated and I ignore it.

I open my casino and charge the prices given in the instructions. For example, for a \$100 payoff on number 30 I charge a "premium" of \$2.08. I monitor the profit and loss on each number, of course, and after some time I notice that my average payoff cost on number 30 is actually \$2.63. One of the features of the Ruhm Roulette Wheel is that I don't have to make the payoff for a year, which allows me to earn 8 cents interest on the premium, but \$2.16 is still 47 cents short of \$2.63. Analyzing the results for the other numbers, I find that, except for number 16, the payoff costs do not match the interest-adjusted premiums. The reason for the mismatch is that while the premiums were determined correctly from the risk-neutral pricing framework, the payoffs continue to be governed by the real world probabilities. The results for each of the numbers 00 through 36 are summarized in Exhibit 1 of this discussion, the core of which is excerpted from Exhibit 3 of Ruhm's paper.

The good news for me as the casino owner is that if bets had been placed in equal proportions over all 38 numbers, the total premiums and interest would match the total payoff costs. The bad news is that the players know that the Ruhm Roulette Wheel is fair, meaning each number is equally likely to come up. Since I charge less for the high numbers than for the low ones, I get more high number bets than low number bets. My casino business is a big loser!

There is a way around this. Along with the arbitrage-free price list, Ruhm's instructions also tell me how to hedge the risk for each number. If I follow that hedging procedure, the sum of my payoff cost for any given number and the associated hedging gain or loss will match the arbitrage-free premiums I collect for that number. For the low numbers the hedging will produce losses. For the high numbers it will produce gains. For the number 30 example, hedging will produce an average gain of 47 cents, which reduces the total expected payoff cost from \$2.63 to \$2.16. The hedging effectively transforms my payoff cost to what it would be if the underlying stock had an expected return equal to the risk-free rate.

If I hedge the bets against each number, then it won't matter whether customers prefer high numbers or low ones.

3. VALUE FOR MONEY—BUYER'S PERSPECTIVE

Ruhm speculates why anyone would place bets on the low numbers, since under the conditions he assumes,[1] they include a surcharge over expected value. He extends the same question to put options generally. He claims a put buyer must be motivated by a desire to hedge the risk, since a speculator would not make an investment with such poor prospects. I don't find his argument

[1]Namely, that the expected return on the stock E exceeds the risk-free rate r.

compelling. There is a simpler explanation that doesn't depend on investor psychology.

Ruhm appears to have overlooked the importance of his assumptions that the expected annual returns on the underlying stock and Treasuries are $E = 10\%$ and $r = 4\%$, respectively. If the expected annual return on the stock is less than the risk-free rate, i.e., $E < r$, then the pattern reverses, and arbitrage-free prices for the put-like low numbers are discounted and the prices for the call-like high numbers are surcharged.

To see this, let's start by grouping the low numbers 00 through 17 and the high numbers 18 through 36. There is a 50% probability associated with each group. The low number group corresponds to a "binary put option" having a fixed payoff (in this case \$100) if the stock price is less than the median of the stock price distribution. The high number group corresponds to a "binary call option" that has a payoff of \$100 if the stock price closes above the median.

In Exhibit 1, which is based on an expected annual stock return of 10%, the sum of the arbitrage-free premiums for the high number group is \$40.95. The sum of the premiums for the low number group is \$55.21. With interest these amounts are \$42.58 and \$57.42, respectively. The puts are priced at a premium to expected cost. The calls are priced at a discount.

Suppose the expected annual stock return is really 0%, in which case $E < r$. Then the median of the stock price distribution is \$95.60. From the price formula for a ray included in the paper, the arbitrage-free price for the binary call option with a strike price of \$95.60 is \$53.08. The corresponding price for the binary put option is \$43.08. With interest these amounts become \$55.20 and \$44.80, respectively. The puts are priced at a discount to expected cost. The calls are priced at a premium.

Remember that no one knows the true parameters of the stock price distribution. If an investor believes that the true expected annual stock return E exceeds the risk-free rate r, then arbitrage-free calls will look attractively priced and puts will not. Such an investor might buy the calls but will shun the puts. On the other hand, if an investor believes that the true stock return parameter is less than the risk-free rate, calls will look expensive and puts will look attractive. That investor will shun the calls and might buy the puts. This is logical profit-maximizing behavior. It is not necessary to appeal to differences in risk aversion or "the qualitative nature of the risk" to explain the behavior.

Meanwhile, the seller of puts and calls can be indifferent to the true stock return parameter, provided he hedges the puts and calls that he sells.

4. THE $w(s)$ FUNCTION

Attempting to generalize his findings from the roulette wheel, Ruhm introduces his $w(s)$ function as a measure of the risk discount for betting on the event $X = s$. Like the roulette wheel concept, this function neatly captures important information about a complex relationship, in this case between the risk-neutral pricing framework and the perceived real world probabilities.

However, as in his analysis of the roulette wheel, the author again overlooks the scenario in which the underlying stock return is less than the risk-free rate. If $E < r$, then the slope of $w(s)$ is positive. Small values of s (i.e., low strike prices) yield large discounts to the expected payoff. High strike prices yield large surcharges. This is the opposite of the behavior of $w(s)$ sketched in the paper, which addressed only the scenario of $E > r$. We give an example of this below. It can be generalized, but it should be clear enough from the example.

The $w(s)$ function is the ratio of the "risk neutral" pdf to the "real world" pdf pertaining to the underlying stock. The following formula is equivalent to Ruhm's. It is not a function of merely s, but of a number of parameters, the most important of which for our purposes is E

$$w(s,E) = \frac{g(d_2)}{g(d_2 + ((\ln(1+E) - \ln(1+r))/\sigma)\sqrt{t})}, \qquad \text{where}$$
$$d_2 = \frac{\ln(P_0/s) + (\ln(1+r) - 0.5\sigma^2)t}{\sigma\sqrt{t}} \tag{1}$$

where $g(x)$ is the standard normal pdf evaluated at x. (Note that d_2 is the well-known Black-Scholes parameter.)

For example, using formula (1) for the scenario involving a strike price s of 90, an initial stock price P_0 of 100, with E = 10%, r = 4%, σ = 30% and t = 1 year, which are the parameters Ruhm used in his main example, we obtain the following value of $w(90, 10\%)$:

$$w(90, 10\%) = \frac{0.3776}{0.3487} = 1.083$$

Contrast this with the value of $w(90, 0\%)$ that we obtain when we change E to zero, leaving all of the other parameters unchanged:

$$w(90, 0\%) = \frac{0.3776}{0.3909} = 0.966$$

Exhibit 2 shows $w(s, 10\%)$ and $w(s, 0\%)$ for the parameter set given above and strike prices ranging from \$10 to \$200 in \$10 increments.

Because $w(s)$ is a function of E, and E is inherently unknowable, $w(s)$ is not unique. Ruhm has a belief about the value of E that might be the same as mine, but it might be different. It is possible to talk about Ruhm's $w(s)$ function or mine, but unless he knows my $w(s)$, he cannot make claims about whether my put or call buying behavior is motivated by investment or insurance considerations. For example, if he believes a particular stock will go up (implying a negatively sloped $w(s)$), then my buying what looks like an expensive put on that stock will strike him as evidence of extremely risk averse behavior, suggesting an insurance orientation on my part. However, if I expect the stock to trade sideways or go down (implying a positively sloped $w(s)$), then my behavior in buying what looks to me to be a cheap put is actually consistent with a profit maximizing investment strategy.

Consequently, Ruhm overreaches in his conclusion about how $w(s)$ can be used. There is no unique value of $w(s)$ independent of E that can tell us whether a risk is viewed as an investment or as insurance. Only if a put buyer is known to believe that $E > r$ could we correctly say that he is acting to "insure" the risk (which the author sees as synonymous with a willingness to pay a surcharge to the risk's expected value). If, on the other hand, he believes that $E < r$, he is "investing" in the risk. I have a hunch that most investors who buy puts believe $E < r$.

5. SUMMARY

There is much to like in Ruhm's paper. His roulette wheel is an excellent metaphor that makes the implications of the Black-Scholes framework more tangible. Likewise, his invention and use of the $w(s)$ function is a laudable attempt to distill important information into a simple measure. While I believe his interpretation of $w(s)$ is flawed, I appreciate his attempt.

Against these positives, I have sought to clarify three points the author chose not to emphasize.

First, from the seller's perspective it is critical to make a distinction between price and cost. While the risk-neutral pricing framework produces arbitrage-free prices in markets where hedging is available, the prices are not necessarily adequate to cover the seller's expected value cost. As we saw in the roulette wheel example with an underlying stock expected return of $E > r$, the seller can expect to lose money on the high numbers if he does not hedge.

Second, it is not correct to say that call options priced in the risk-neutral framework are priced at a discount to their expected value cost and that puts are priced at a premium, without being clear that this is true only if $E > r$. The author noted in passing that his result is true if $E > r$, but did not point out that the opposite is true if $E < r$, in which case calls are priced at a premium and puts are priced at a discount.

Third, I have pointed out that the second point extends to the behavior of the author's $w(s)$ function, which makes it largely useless as a means of categorizing individual behavior as investment or insurance oriented, as Ruhm had hoped.

EXHIBIT 1

Premium, Cost, and Profit or Loss by Number

		(1)	(2)	(3) (1)+(2)	(4)	(5) (3)-(4)
Roulette Wheel Number		Arbitrage-Free Premium	Interest on Premium	Premium with Interest	Expected Payoff ("Cost")	Casino Profit or (Loss)
00		$3.84	$0.15	$3.99	$2.63	$1.36
0		$3.46	$0.14	$3.60	$2.63	$0.97
1		$3.30	$0.13	$3.43	$2.63	$0.80
2		$3.19	$0.13	$3.32	$2.63	$0.69
3		$3.10	$0.12	$3.22	$2.63	$0.59
4		$3.03	$0.12	$3.15	$2.63	$0.52
5		$2.97	$0.12	$3.09	$2.63	$0.46
6		$2.92	$0.12	$3.04	$2.63	$0.41
7		$2.87	$0.11	$2.98	$2.63	$0.35
8		$2.82	$0.11	$2.93	$2.63	$0.30
9		$2.78	$0.11	$2.89	$2.63	$0.26
10		$2.74	$0.11	$2.85	$2.63	$0.22
11		$2.70	$0.11	$2.81	$2.63	$0.18
12		$2.67	$0.11	$2.78	$2.63	$0.15
13		$2.63	$0.11	$2.74	$2.63	$0.10
14		$2.60	$0.10	$2.70	$2.63	$0.07
15		$2.56	$0.10	$2.66	$2.63	$0.03
16		$2.53	$0.10	$2.63	$2.63	($0.00)
17		$2.50	$0.10	$2.60	$2.63	($0.03)
18		$2.47	$0.10	$2.57	$2.63	($0.06)
19		$2.44	$0.10	$2.54	$2.63	($0.09)
20		$2.41	$0.10	$2.51	$2.63	($0.13)
21		$2.38	$0.10	$2.48	$2.63	($0.16)
22		$2.35	$0.09	$2.44	$2.63	($0.19)
23		$2.32	$0.09	$2.41	$2.63	($0.22)
24		$2.29	$0.09	$2.38	$2.63	($0.25)
25		$2.26	$0.09	$2.35	$2.63	($0.28)
26		$2.23	$0.09	$2.32	$2.63	($0.31)
27		$2.19	$0.09	$2.28	$2.63	($0.35)
28		$2.16	$0.09	$2.25	$2.63	($0.39)
29		$2.12	$0.08	$2.20	$2.63	($0.43)
30		$2.08	$0.08	$2.16	$2.63	($0.47)
31		$2.04	$0.08	$2.12	$2.63	($0.51)
32		$1.99	$0.08	$2.07	$2.63	($0.56)
33		$1.94	$0.08	$2.02	$2.63	($0.61)
34		$1.87	$0.07	$1.94	$2.63	($0.69)
35		$1.79	$0.07	$1.86	$2.63	($0.77)
36		$1.62	$0.06	$1.68	$2.63	($0.95)
Total		$96.16	$3.85	$100.00	$100.00	$0.00
00–17	"Put"	$55.21	$2.21	$57.42	$50.00	$7.42
18–36	"Call"	$40.95	$1.64	$42.58	$50.00	($7.42)

EXHIBIT 2

COMPARISON OF $w(s, 10\%)$ AND $w(s, 0\%)$

	Value of $w(s,E)$	
Strike Price, s	$E = 10\%$	$E = 0\%$
10	4.2584	0.3707
20	2.7646	0.5014
30	2.1473	0.5983
40	1.7949	0.6782
50	1.5618	0.7475
60	1.3941	0.8093
70	1.2664	0.8656
80	1.1653	0.9174
90	1.0828	0.9658
100	1.0140	1.0111
110	0.9555	1.0540
120	0.9051	1.0947
130	0.8610	1.1336
140	0.8222	1.1708
150	0.7876	1.2065
160	0.7565	1.2410
170	0.7285	1.2742
180	0.7030	1.3063
190	0.6797	1.3375
200	0.6583	1.3677

$P_0 = 100$, $r = 4\%$, $\sigma = 30\%$, $t = 1$ year