DISTRIBUTION-BASED PRICING FORMULAS ARE NOT ARBITRAGE-FREE

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Abstract

A number of actuarial risk-pricing methods calculate risk-adjusted price from the probability distribution of future outcomes. Such methods implicitly assume that the probability distribution of outcomes contains enough information to determine an economically accurate risk adjustment.

In this paper, it will be shown that distinct risks having identical distributions of outcomes generally have different arbitrage-free prices. This is true even when the outcomes are completely determined by the same underlying contingent events. Risk-load formulas that use only the risk's outcome distribution cannot produce arbitrage-free prices and, in that sense, are not economically accurate for risks traded in markets where arbitrage is possible. In practice, most insurance underwriting risks are not traded in such markets. Distribution-based pricing usually does not carry a direct arbitrage penalty for insurance and can reflect an insurer's risk preferences.

A ratio is used to measure the implicit discount or surcharge for risk that is present in a price: the ratio of price density to discounted probability density. This ratio can be used to identify the qualitative nature of a risk as investment or insurance: a risk discount factor less than unity indicates investment, whereas a risk surcharge factor above unity indicates insurance.

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1. INTRODUCTION

Risk, in a financial context, can be considered exposure to potential financial loss. Pricing for risk is a central problem in casualty actuarial science. Casualty actuaries have developed several mathematical pricing methods intended to compensate for risk equitably and adequately.

Recently, a number of authors have sought to make actuarial pricing methods consistent with Black-Scholes options pricing theory, or (more generally) arbitrage-free pricing theory, notably Wang [4] and Venter [3]. This is an appealing goal, since such methods would price a variety of risks in a consistent, "universal" way, regardless of whether the risk arose from insurance or from a financial market. For both insurance risk and capital risk, such methods would produce the correct charge, based on the philosophy that "risk is risk," irrespective of context.

A second benefit of actuarial pricing that is consistent with arbitrage-free pricing is integrity. In an ideal market, an option's price would have to be the arbitrage-free price, otherwise an arbitrage opportunity would exist. In such a market, arbitrage opportunities would not exist for any appreciable length of time, so market forces would actually drive the market price to the arbitrage-free price. This lends an integrity to arbitrage-free pricing that is not found in other risk-pricing methods. In this sense, arbitrage-free pricing is the "natural" risk-adjusted price. Furthermore, no additional assumptions about the cost of risk are needed, other than the market's implicit pricing of risk in the security.

Several actuarial methods for pricing risk use only the probability distribution of the risk's economic outcomes to determine the risk charge. In this paper, the Black-Scholes options pricing formula is used to price some derivatives that have simple outcome distributions. A surprising result of the analysis is that two risks with identical outcome distributions generally have different Black-Scholes prices, even when they are both derivatives of the same security. The reasons that underlie this phenomenon are discussed. The result is shown to be true in general: arbitrage-free pricing cannot be produced by any formula that uses only the distribution of economic outcomes.

2. DEFINITIONS AND TERMINOLOGY

Throughout this paper, the term "security" will be used to refer to a hypothetical ideal stock that satisfies the hypotheses of the Black-Scholes model. This security will be the basis for most of the theoretical development that follows.

Also, the term "derivative" refers to any financial instrument whose value at some fixed time in the future is a function of the security's price at that time. The term "option" refers to a European call or put option. All options discussed in this paper have the same expiration date, which can be any date. "Price," as applied to an option or portfolio of options, refers to the Black-Scholes formula price.

The variable p represents the current price of the security, and X denotes the future price of the security on the options expiration date. From the Black-Scholes hypotheses, X is a lognormal random variable. The positive number line $(0,\infty)$ contains X and can be thought of as the space of possible expiration prices.

To facilitate the analysis, special derivatives will be constructed from call options. These derivatives, which will be referred to as "binary risks," are worth one unit at expiration if X is in a specified price range and zero if X is outside of the range. Note that the expected value of a binary risk at expiration equals the probability that X will be within the specified price range.

3. BACKGROUND

Define R, the return on the security at the options expiration date, by the formula:

$$R = (X/p) - 1.$$

Then, since X is lognormal and p is a constant, $(1 + R)$ also has a lognormal distribution. A lognormal variable can be parameterized in terms of its expectation and sigma parameter, so the distribution of R is completely determined by the expected return $E = \mathrm{E}[R]$, and the return volatility σ:

$$\ln(1 + R) \sim N(\mu_R, \sigma^2), \qquad \text{and}$$
$$\mu_R = \ln(1 + E) - \sigma^2/2.$$

Note the absence of a time variable. Many options pricing formulas define the volatility parameter with respect to an annual time horizon, and then apply a square-root time factor to adjust for the time to expiration. In this formulation, the time factor is implicitly included within the volatility parameter for notational convenience.

The distribution of X can be expressed in terms of current price, expected return, and volatility of return:

$$X = p(1 + R),$$
$$\ln(X) = \ln(p) + \ln(1 + R), \qquad \text{and}$$
$$\ln(X) \sim N(\mu, \sigma^2), \qquad \text{where}$$
$$\mu = \ln(p) + \mu_R = \ln(p) + \ln(1 + E) - \sigma^2/2.$$

The Black-Scholes price of an option is equal to the option's discounted expected value, under a risk-neutral lognormal density function that is parameterized by μ^* and σ^*:

$$\mu^* = \mu - \ln[(1 + E)/(1 + r)], \qquad \text{and}$$
$$\sigma^* = \sigma,$$

where μ and σ are the parameters given above in the probability density function of X, E is defined as above, and r is the risk-free return for the time period to expiration. This risk-neutral density will be referred to as the "price density" function. Arbitrage-free pricing implies price additivity, which means the price density function can be used to price derivatives that are equivalent to combinations of options.

Using the formula given above for μ, the risk-neutral distribution's μ^* parameter is given by:

$$\mu^* = \ln(p) + \ln(1 + r) - \sigma^2/2.$$

The graph of a call option's value at expiration is shown in Exhibit 1. The call's value is zero when the security's expiration price is below the strike price. Above the strike, the call's value increases linearly with a slope of one.

By buying one call and selling another call, an investor can create what is commonly known as a "spread." For example, the investor could buy a call option with a December expiration and a strike price of 50 and sell a call option with a December expiration and a strike price of 60. In this example, the expiration date is the same for both calls, and the purchased call has a lower strike price than the sold call. The graph of this spread's value at expiration is shown in Exhibit 2. The value starts at zero for expiration prices at or below 50, then increases dollar-for-dollar from 50 to 60, and finally remains constant at 10 for expiration prices above 60. With a few arithmetic calculations, the reader can verify that the graph accurately represents the spread's value at expiration as a function of X, the security's expiration price.

This spread would commonly be referred to as a "bull spread," because the spread's value at expiration is positively related to the underlying security's price. Bull spreads can be constructed from either call or put options; a consequence of arbitrage-free pricing is that the price of the spread is the same under either construction. A bull spread is a combination of options (one long and one short), so its price is found by calculating its discounted expected value at expiration using the price density function.

4. THEORETICAL DEVELOPMENT

In the following sections, pricing theory is developed for particular derivatives. This will provide the basis for comparisons between derivatives and games of chance.

4.1. Rays

As defined earlier, X represents the future price of the security on a specific options expiration date. For the purpose of the construction that follows, it is assumed that call and put options are available for all strike prices and in any amount, including fractional amounts. Let $A(s,e)$ denote a position consisting of $(1/e)$ bull spreads from expiration prices s to $(s+e)$. For example, $A(50,10)$ represents 0.10 bull spreads on the expiration price range $[50,60]$. Then the value at expiration of $A(s,e)$ is:

$$\text{Value}[A(s,e)] = \begin{cases} 0, & \text{when} \quad X < s; \\ (X-s)/e, & \text{when} \quad X \in [s, s+e]; \\ 1, & \text{when} \quad X > s+e. \end{cases}$$

As the variable e approaches zero, $A(s,e)$ converges pointwise to a limiting bull spread, denoted by $A^*(s)$, which has a binary payoff at expiration:

$$\text{Value}[A^*(s)] = \begin{cases} 0, & \text{when} \quad X \leq s, \quad \text{and} \\ 1, & \text{when} \quad X > s. \end{cases}$$

In other words, $A^*(s)$ pays one unit if $X \in (s,\infty)$, and zero otherwise. In this way, $A^*(s)$ can be viewed as corresponding to the set (s,∞). The graph of (s,∞) on a number line is a ray with open endpoint at s, extending to the right. In this paper, the limiting bull spread $A^*(s)$ will be referred to as a "ray," after its geometric representation.

A ray can be thought of as a gamble. $A^*(s)$ is effectively a bet on whether the future expiration price will be higher than the strike price, $X > s$. If $X \leq s$, then $A^*(s)$ is worth nothing and the purchase price is lost, just as a wager can be lost in a bet. The price of $A^*(s)$ is the amount wagered. If $X > s$, then $A^*(s)$ is worth one unit, which has present value $v = 1/(1+r)$. This amount can be decomposed into two parts: the return of the purchase price (the wager), plus a "payoff" of v minus the purchase price:

$$v = (\text{return of purchase price}) + (\text{payoff of } v - \text{purchase price}).$$

This wager perspective makes it possible to compare probabilities and payoffs for rays with those for games of chance, which creates some surprising results discussed below.

$A^*(s)$ is worth either 0 or 1 at expiration, so it is a binary risk. The expected value of $A^*(s)$ at expiration equals the probability that $X > s$ (the "Value" operator on $A^*(s)$ is omitted for notational convenience):

$$\mathrm{E}[A^*(s)] = \mathrm{P}(X > s).$$

Since X is lognormal, the "payoff probability" $\mathrm{P}(X > s)$ can be expressed in terms of the cumulative normal distribution function:

$$\begin{aligned}\mathrm{P}(X > s) &= \mathrm{P}(\ln(X) > \ln(s)) \\ &= 1 - \Phi[(\ln(s) - \mu)/\sigma] = \Phi[(\mu - \ln(s))/\sigma].\end{aligned}$$

The μ parameter was given previously in the density function for X:

$$\mu = \ln(p) + \ln(1+E) - \sigma^2/2.$$

Substituting yields the formula for payoff probability and $\mathrm{E}[A^*(s)]$:

$$\mathrm{E}[A^*(s)] = \mathrm{P}(X > s) = \Phi[\ln(p(1+E)/s)/\sigma - \sigma/2].$$

As discussed above, the price of $A^*(s)$ equals the discounted expected value under the price density function. The price density has the same formula as the probability density with the μ^*

parameter in place of μ. With $v = 1/(1 + r)$, the risk-free discount factor:

$$\text{Price}[A^*(s)] = v\Phi[(\mu^* - \ln(s))/\sigma],$$

$$\mu^* = \ln(p) + \ln(1 + r) - \sigma^2/2, \qquad \text{and}$$

$$\text{Price}[A^*(s)] = v\Phi[\ln(p(1 + r)/s)/\sigma - \sigma/2].$$

An example will illustrate how these formulas can be used. Suppose the following:

p = Current price of underlying security = 100,

R = Expected return on underlying security = 10%,

σ = Volatility of return = 30%,

r = Risk-free rate = 4%,

$v = 1/(1 + r) = (1.04)^{-1}$, and

s = Strike price = 120.

Then, the price of $A^*(120)$ and the payoff probability can be calculated:

$$\text{Price} = v\Phi[\ln(p(1 + r)/s)/\sigma - \sigma/2];$$

$$\text{Price} = (1.04)^{-1}\Phi[\ln(104/120)/0.30 - 0.15];$$

$$\text{Price} = 0.2551.$$

$$\text{Probability} = \Phi[\ln(p(1 + E)/s)/\sigma - \sigma/2];$$

$$\text{Probability} = \Phi[\ln(110/120)/0.30 - 0.15];$$

$$\text{Probability} = 0.3300.$$

$A^*(120)$ has about a 1/3 chance of paying one unit, and about a 2/3 chance of zero payment. Suppose a gambler purchases 100 units of $A^*(120)$ and views it as a bet. Then the amount wagered is \$25.51 (the purchase price, which is the amount placed at risk), the odds of winning are 33%, and the payoff at present value is:

$$\text{Payoff} = \$100(v) - \text{Wager} = \$96.15 - \$25.51 = \$70.64.$$

This is a gamble with better-than-breakeven prospects. The amount of the advantage can be quantified by calculating expected return. Since the value of $A^*(120)$ at expiration is one or zero, the expected return equals the ratio of the probability to the price, minus unity:

$$\text{Expected Return}[A^*(120)]$$
$$= [(\$100)(.3300) + (\$0)(.6700)]/\$25.51 - 100\%;$$
$$\text{Expected Return}[A^*(120)]$$
$$= 0.3300/0.2551 - 100\% = 29.33\%.$$

An interpretation of this result is that the $A^*(120)$ derivative is riskier than the underlying security, and therefore commands a higher expected return: 29.33% versus 10%.

The high expected return for $A^*(120)$ implies that the price of $A^*(120)$ contains a large discount beyond risk-free discounting. The implicit discount factor is the price divided by the risk-free-discounted expected value (which equals the discounted probability):

$$\text{Discount Factor} = \text{Price}/\text{Discounted Probability}$$
$$= 0.2551/0.3300v,$$
$$\text{Discount Factor} = 80.41\%.$$

This factor can be interpreted as a "risk discount" within the price of 80.41%. For an even gamble with no statistical advantage, the price would be equal to the discounted expected value of the payoff, so that the expected net outcome at present value would be zero. This 80.41% risk discount factor is the extent to which $A^*(120)$ deviates from an even gamble—the derivative $A^*(120)$ sells for 80.41% of the even-gamble price.

As it turns out, $A^*(s)$ is never an even gamble, except for one particular value of s. By calculating prices and probabilities for various values of s, one finds a wide range of gambles, with both positive and negative expected net outcomes.

The risk discount factor is inversely proportional to the expected return:

$$\text{Risk Discount Factor} = (1 + r)/(1 + \text{Expected Return}).$$

The right-hand-side expression is the inverse of the risk premium in the derivative's expected return, expressed as a ratio.

The risk discount factor will be used frequently in the discussion below. For convenience, it will be denoted by w:

$$\begin{aligned} w &= \text{Risk Discount Factor} \\ &= \text{Price}/\text{Discounted Expected Value at Expiration.} \end{aligned}$$

When applied to binary risks, this reduces to:

$$w = \text{Price}/\text{Discounted Payoff Probability.}$$

4.2. *Segments*

Just as $A^*(s)$ represents a derivative that corresponds to a geometric ray, we can construct a derivative that corresponds to a line segment. As before, let $A(s,e)$ represent a portfolio consisting of $(1/e)$ bull spreads from s to $(s+e)$. Then, define a "segment" derivative $D(s,t)$ as the limit of a long position in $A(s,e)$ and a short position in $A(t,e)$, where $s < t$, as e approaches zero. Informally, a segment is the difference of two rays:

$$D(s,t) = A^*(s) - A^*(t), \qquad \text{when} \quad s < t.$$

It is easily verified that the value at expiration of a segment is binary:

$$\text{Value}[D(s,t)] = \begin{cases} 1, & \text{when} \quad X \in (s,t], \quad \text{and} \\ 0, & \text{when} \quad X \notin (s,t]. \end{cases}$$

In other words, $D(s,t)$ pays one unit if the expiration price of the security is contained in the segment $(s,t]$, and zero otherwise. Geometrically, $D(s,t)$ corresponds to the half-open line segment

$(s,t]$. Like a ray, a segment is a binary risk with expected value equal to its probability.

Just as the ray $A^*(s)$ can be thought of as a bet on the set of events $X > s$, the segment $D(s,t)$ is effectively a bet on $X \in (s,t]$. If $X \leq s$ or $X > t$, then $D(s,t)$ is worth zero and the purchase price is lost. If $X \in (s,t]$ then $D(s,t)$ is worth one unit at expiration.

Each segment has a payoff probability and a price. In the section above, the price and probability were calculated for the ray $A^*(120)$:

$$\text{Price}[A^*(120)] = 0.2551, \qquad \text{and}$$
$$\text{Probability}[A^*(120)] = 0.3300.$$

These values implied a risk discount factor for the ray:

$$w[A^*(120)] = \text{Price/Discounted Probability} = 0.2551/0.3300v;$$
$$w[A^*(120)] = 80.41\%.$$

Continuing with this example, we can calculate the price and probability for the ray $A^*(150)$, and then subtract the results to calculate values for the segment $D(120,150)$:

$$\text{Price}[A^*(150)] = v\Phi[\ln(p(1+r)/s)/\sigma - \sigma/2];$$
$$\text{Price}[A^*(150)] = (1.04)^{-1}\Phi[\ln(104/150)/0.30 - 0.15];$$
$$\text{Price}[A^*(150)] = 0.0819.$$
$$\text{Probability}[A^*(150)] = \Phi[\ln(p(1+E)/s)/\sigma - \sigma/2];$$
$$\text{Probability}[A^*(150)] = \Phi[\ln(110/150)/0.30 - 0.15];$$
$$\text{Probability}[A^*(150)] = 0.1182.$$

Values for the segment can now be found by subtracting values for the rays:

$$\text{Price}[D(120,150)] = 0.2551 - 0.0819 = 0.1732;$$
$$\text{Probability}[D(120,150)] = 0.3300 - 0.1182 = 0.2117.$$

$D(120, 150)$ has about a 21% chance of paying one unit, and about a 79% chance of expiring worthless. If a gambler purchased 100 units of $D(120, 150)$ and viewed it as a bet, the amount wagered would be \$17.32 (the purchase price), the odds of winning would be 21.17%, and the payoff at present value would be:

$$\text{Payoff} = \$100(v) - \text{Wager} = \$96.15 - \$17.32 = \$78.83.$$

In comparison with $A^*(120)$, this gamble has lower odds of winning, but also has a lower wagered amount and a higher payoff.

As with a ray, the segment's statistical advantage can be quantified by calculating expected return. Since the expected value at expiration equals the payoff probability:

$$\text{Expected Return}[D(120, 150)] = 0.2117/0.1732 - 1 = 22.25\%.$$

The risk discount factor for the segment can also be calculated:

$$w[D(120, 150)] = 0.1732/0.2117v = 85.07\%.$$

This segment's calculated price is 85.07% of the price that would offer an even gamble. It is not as strongly discounted as the ray $A^*(120)$, so it has a lower expected return.

5. CONSTRUCTING A ROULETTE WHEEL FROM SEGMENTS AND RAYS

Using the pricing theory developed in the previous section, it is now possible to directly compare the performance of derivatives with the results from a roulette gamble.

5.1. The Game of Roulette

Roulette is a casino game that uses a wheel with a ring around its perimeter. The ring is evenly divided into 38 spaces, numbered

"00" and 0 through 36. Players can bet on any numbered space except 0 and 00, or on several spaces at once. After bets are placed, the wheel is spun and a winning number is determined. The payoff for betting on the winning number is 35:1, meaning that the bettor's wager is returned plus a payoff from the casino of 35 times the wager. Wagers on losing numbers are lost to the casino.

Roulette does not offer an even gamble—the casino has a constant advantage. For the gambler, $1 bet on a number returns $36 total in the event of a win, which has probability 1/38, or zero in the event of a loss, which has probability 37/38. The gambler's expected net outcome is therefore $-2/38$:

$$\begin{aligned}\text{Expected Net Outcome} &= \text{E[Outcome]} - \text{Wager}\\ &= 36(1/38) + 0(37/38) - 1 = -2/38.\end{aligned}$$

The term "binary risk" was defined earlier as a financial instrument that pays either zero or one at expiration. Rays and segments were shown to be examples of binary risks. A roulette wager of 1/36 on a numbered space is also a binary risk. The player is effectively paying the casino a "price" of 1/36 in the form of a bet, and receives either zero if the wager is lost, or one (return of amount bet, plus payoff) if the wager is won.

The "expiration date" for a roulette spin is the time at which the outcome is finalized, the moment at which the wheel's spin is completed and the winning number is determined. Since this occurs only seconds after the wager, a time value factor of one can be used with no significant loss of accuracy.

The risk discount factor for a roulette gamble from the gambler's perspective can be calculated:

$$\begin{aligned}\text{Risk Discount Factor} &= \text{Price}/\text{Discounted Probability}\\ &= (1/36)/(1/38) = 105.56\%.\end{aligned}$$

Since this factor is greater than one, it is actually not a discount but rather a 5.56% surcharge to the gambler for the entertainment value of assuming the gambling risk.

The risk discount factor can also be calculated from the casino's perspective. The "price" is the amount that the casino might have to pay the gambler, which is 35/36. The probability of the casino winning is 37/38. When the gambler bets 1/36, it is the same as if the casino pays the gambler 35/36 up front, spins the wheel, and collects 1 from the gambler if he loses, or nothing if he wins. The casino's "price" is thus 35/36 and the success probability is 37/38. Then,

$$\begin{aligned}\text{Risk Discount Factor} &= \text{Price}/\text{Discounted Probability}\\ &= (35/36)/(37/38) = 99.85\%.\end{aligned}$$

This discount represents a small edge for the casino, but more than sufficient given the frequency of play. It is interesting that the casino's discount is far smaller than the gambler's surcharge, even though these are just opposite sides of the same gamble. The reason is that the casino's expected return on the amount it places at risk is small but positive; the gambler's expected loss is a much larger percentage of the amount risked.

All numbered spaces on the wheel have the same probability (1/38), and the same payoff ratio (35:1). In regard to probability and payoff the spaces are completely identical to each other. The bettor's expected net outcome has the same value ($-2/38$) for every space; the gambler's prospects are the same no matter which space he chooses to bet.

5.2. Mapping Expiration Prices onto the Roulette Wheel

Recall that a ray $A^*(s)$ is equivalent to a bet on whether $X \in (s, \infty)$, and a segment $D(s,t)$ is equivalent to a bet on whether $X \in (s,t]$. As such, rays and segments are similar to bets on roulette numbers, just with different probabilities and payoffs. A correspondence between these derivatives and the roulette wheel can be constructed as follows:

The space of all possible expiration prices is the positive number line $(0, \infty)$. Segments and rays will be referred to collectively as "sections." Partition this space into 38 sections, consisting of 37 segments and 1 ray:

$$
\begin{aligned}
&(0, s_1] \\
&(s_1, s_2] \\
&(s_2, s_3] \\
&\dots \\
&(s_{37}, +\infty),
\end{aligned}
$$

where the s_n are specifically chosen so that each section has a 1/38 probability of containing the expiration price for the security. Then, each of the 38 sections has the same probability distribution as a space on a roulette wheel: 1/38 probability of payoff, 37/38 probability of no payoff. Taken as a group, these sections cover the entire space of expiration prices $(0, \infty)$ with no overlap, like the 38 spaces that cover the roulette wheel. The space of expiration prices $(0, \infty)$ can thus be viewed as a roulette wheel, divided into these 38 section-spaces. Just as a roulette spin produces a single winning number, exactly one of these sections will contain the expiration price and will have a value of one on the expiration date. In summary, purchasing one of these section derivatives is almost exactly like making a bet on a roulette number.

The major difference between this roulette-like partitioning of $(0, \infty)$ and an actual roulette wheel is the payoff ratio. A roulette bet pays $35:1$ regardless of which space is selected. A bet of 1/36 produces a payoff of 1 for winning. The 38 sections on the space $(0, \infty)$ have equal probabilities (1/38) and equal payoffs (1 unit), but they have varying prices. This means that one can "bet" at a discount or a surcharge, depending on which "space" one selects.

To demonstrate this tangibly, let's consider the example used earlier. As before, the parameters are:

$$p = \text{Current price of underlying security} = 100,$$

$$R = \text{Expected return on underlying security} = 10\%,$$

$$\sigma = \text{Volatility of return} = 30\%,$$

$$r = \text{Risk-free rate} = 4\%, \qquad \text{and}$$

$$v = 1/(1+r) = (1.04)^{-1}.$$

Exhibit 3 shows roulette spaces mapped to sections on $(0,\infty)$ for this example. The mapping starts with the "00" space, which corresponds to the segment $D(0,58.80)$. Next is the "0" space representing the adjacent segment $D(58.80,64.68)$, and so on up to the "36" space representing the ray $A^*(188.08)$.

Probabilities and prices for each space are shown in the exhibit. These are calculated by using the probability and price formulas for rays and segments derived above. Each space has the same 1/38 probability of a one-unit payoff at expiration. The prices vary by space, decreasing for the higher numbered spaces that correspond to higher prices for the security. Expected returns and corresponding risk factors (both discounts $< 100\%$ and surcharges $> 100\%$) are also shown.

As the exhibit indicates, some section-spaces are more favorable than others. On a wheel of derivatives such as this, each number has a different payoff ratio even though all numbers are equally likely. The right-hand column, "Ratio to Roulette Payoff," shows what the payoffs are compared to a real roulette wheel. For example, the "00" space has an equivalent roulette payoff of 69%, meaning that buying the "00" segment is like placing a bet on a roulette number and receiving just 69% of the usual payoff in the event of a win. The "36" space is the best choice on this wheel of derivatives, since it pays 165% of the standard roulette payoff if you win.

Two of the spaces are of particular interest. Space "12" has a risk factor of 105.33%, which is approximately equal to the 105.56% factor for an actual roulette wager. Purchasing the "12" segment $D(93.08, 95.08)$ is about equivalent to placing an actual roulette bet. This is also shown in the "Ratio to Roulette Payoff" column, where a value of 100% appears for space 12.

The other interesting space is "16," which is the segment $D(101.07, 103.10)$. The risk factor for "16" is 100.11%, meaning that this segment is approximately an even gamble. Note that all spaces numbered less than "16" are disadvantageous, while those numbered higher than "16" offer an advantage. Space "16" is located just above the current security price (100) on a nominal basis, but since strike prices represent future values at expiration, it is actually just below the current price at future value (104).

In summary, all 38 sections are identically distributed, and their outcomes are determined by the same underlying event, but they have different risk factors. Higher spaces have stronger risk discounts, meaning that there is a positive risk load paid to the purchaser for accepting the distribution of outcomes (the "risk"). Lower spaces have risk surcharges, meaning that the purchaser actually pays a charge to assume the risk.

6. PROBABILITY DISTRIBUTION, RISK LOAD, AND EXPECTED RETURN

The results of the roulette mapping are somewhat counterintuitive, but some conclusions can be drawn about distributions, risk load, and return from the roulette analysis.

6.1. Probability Distribution Does Not Imply Risk Load in an Arbitrage-Free Environment

The surprising result of the roulette wheel mapping is that the 38 risks have different risk loads under arbitrage-free Black-Scholes pricing, even though their distributions are identical and their outcomes are completely determined by the same underlying event (the expiration price of the underlying security). It is therefore not possible to price these risks with any pricing formula that uses only the probability distribution of outcomes. This result can be generalized to other derivatives of a security in an arbitrage-free market. The principal result is that, if the risk load for any of a security's derivatives can be calculated from the probability distribution of the derivative's outcomes, then it must be the trivial risk load and the price must be the derivative's discounted expected value. The formal reasoning is as follows.

Let F be the cumulative distribution function of the underlying security at expiration, with $F(x) = 0$ for $x \leq 0$ and $F(x)$ smooth on $[0,\infty)$. For $E \subset [0,\infty)$ define $\mu(E) = \int_E dF(x)$. Given any measurable function $g : [0,\infty) \to (-\infty,\infty)$, let $p(g) =$ the price of the derivative with payoff function g. We assume that $p(\bullet)$ is a linear operator, and that $h(x) \geq j(x)$ for all x implies $p(h) \geq p(j)$ (the arbitrage-free condition). Let $v = p(h)$, where $h = 1$ on $[0,\infty)$. (The constant v represents the present value of one unit certain at expiration.) If $p(g)$ is completely determined by the probability distribution of the payoff values $\{g(x)\}$ for any g, then $p(g) = v \int g\, dF$.

The proof is in three parts:

A) If $g(x) = X_E(x)$, where $E \subset [0,\infty)$ is measurable and X_E is the indicator for E, then $p(g) = v\mu(E) \equiv v \int g\, dF$.

B) If $g(x)$ is a linear combination of indicator functions (i.e., if g is a simple function), then $p(g) = v \int g\, dF$.

C) $p(g) = v \int g\, dF$ for measurable $g(x)$.

Proof of "A" First, the result is shown for $\mu(E) = 2^{-n}$, $n \geq 0$. If $n = 0$, then by the definition of v, $p(g) = v = v\mu(E)$. By induction on n, assume that the result is true for $n = k$. If $\mu(E) = 2^{-(k+1)}$ then $\mu(E^c) = 1 - 2^{-(k+1)} \geq 2^{-(k+1)}$. Since F is smooth and therefore continuous, there exists x such that $\mu(E^c \cap [0,x)) = 2^{-(k+1)}$ (with $x = \infty$ if $k = 0$). Let $E_x = E^c \cap [0,x)$. Then, $\mu(E) = 2^{-(k+1)}$ and $\mu(E \cup E_x) = 2^{-k}$, since E and E_x are disjoint. Since the values of X_E and X_{E_x} have the same distribution function (0 with probability $1 - 2^{-(k+1)}$ and 1 with probability $2^{-(k+1)}$), $p(X_E) = p(X_{E_x})$. Let $G = E \cup E_x$. Then, by linearity of $p(\bullet)$, $p(X_G) = p(X_E) + p(X_{E_x}) = 2p(X_E)$. By the induction hypothesis, $p(X_G) = v2^{-k}$, so $p(X_E) = (1/2)(v2^{-k}) = v2^{-(k+1)} = v\mu(E)$, completing the induction.

Next, if $\mu(E) = 0$, then letting $F = E^c$, $p(X_E) + p(X_F) = p(X_E + X_F) = p(X_{E\cup F}) = v$. Since $\mu(F) = 1 - \mu(E) = 1$, $p(X_F) = v$ and $p(X_E) = 0 = v\mu(E)$.

If $0 < \mu(E) < 1$, then for any $\varepsilon > 0$ there exist positive integers k,n such that $2^{-n} < \varepsilon/v$ and $k2^{-n} \leq \mu(E) < (k+1)2^{-n} < 1$. By continuity of $F(x)$, there exist $\{x_i\}$, $1 \leq i \leq k$, such that $\mu([0,x_i) \cap E) = i2^{-n}$ (with $x_k = \infty$ if $\mu(E) = k2^{-n}$), and y such that $\mu([0,y) \cap E^c) = (k+1)2^{-n} - \mu(E)$. Defining $x_0 = 0$, let $D_{i,\varepsilon} = [x_{i-1},x_i) \cap E$ for each i. Then, the $D_{i,\varepsilon}$ are mutually disjoint subsets of E, with $\mu(D_{i,\varepsilon}) = 2^{-n}$ for all i. Define $D_\varepsilon = \cup D_{i,\varepsilon}$ and let $F_\varepsilon = (E \backslash D_\varepsilon) \cup ([0,y) \cap E^c)$. Then, $\mu(D_\varepsilon) = k2^{-n}$, $\mu(F_\varepsilon) = (\mu(E) - k2^{-n}) + ((k+1)2^{-n} - \mu(E)) = 2^{-n}$, and $D_\varepsilon \cap F_\varepsilon = \emptyset$. Therefore, $\mu(D_\varepsilon \cup F_\varepsilon) = \mu(D_\varepsilon) + \mu(F_\varepsilon) = (k+1)2^{-n}$. Also, $D_\varepsilon \subseteq E \subset (D_\varepsilon \cup F_\varepsilon)$, so by inclusion, $X_{D_\varepsilon}(x) \leq X_E(x) \leq X_{(D_\varepsilon \cup F_\varepsilon)}(x)$ for all x. Because this price functional is arbitrage-free, we also know that $p(X_{D_\varepsilon}) \leq p(X_E) \leq p(X_{(D_\varepsilon \cup F_\varepsilon)})$ or $vk2^{-n} \leq p(X_E) \leq v(k+1)2^{-n}$. From above, $k2^{-n} \leq \mu(E) < (k+1)2^{-n}$ and $vk2^{-n} \leq v\mu(E) < v(k+1)2^{-n}$, so $|p(X_E) - v\mu(E)| \leq v2^{-n}$, the length of the interval containing both quantities. As $2^{-n} < \varepsilon/v$, $|p(X_E) - v\mu(E)| < \varepsilon$, and since ε is arbitrarily small, $p(X_E) = v\mu(E) \equiv v\int g\,dF$.

Proof of "B" This follows immediately from "A," as $p(\bullet)$ is a linear operator.

Proof of "C" For any $\varepsilon > 0$, let $h_\varepsilon(x)$ be a simple function such that $|g(x) - h_\varepsilon(x)| < \varepsilon$ for all x. (For example, let

$h_\varepsilon(x) = \Sigma_{i \in Z}(i\varepsilon)X_{i,\varepsilon}(x)$, where $X_{i,\varepsilon}(x)$ is the indicator function for the set $g^{-1}\{[i\varepsilon,(i+1)\varepsilon)\}$.) Then, $-\varepsilon < g(x) - h_\varepsilon(x) < \varepsilon$ for all x. By the absence of arbitrage, $p(-\varepsilon) \leq p(g - h_\varepsilon) \leq p(\varepsilon)$ and, equivalently, $-v\varepsilon \leq p(g) - p(h_\varepsilon) \leq v\varepsilon$. Therefore, $|p(g) - p(h_\varepsilon)| \leq v\varepsilon$. As h_ε is simple, $p(h_\varepsilon) = v\int h_\varepsilon dF$, so $|p(g) - v\int h_\varepsilon dF| \leq v\varepsilon$. Also, $|v\int g\,dF - v\int h_\varepsilon dF| = |v\int(g - h_\varepsilon)dF| \leq v\int|(g - h_\varepsilon)|dF \leq v\int \varepsilon\,dF = v\varepsilon$. Applying the triangle inequality, $|p(g) - v\int g\,dF| = |[p(g) - v\int h_\varepsilon dF] - [v\int g\,dF - v\int h_\varepsilon dF]| \leq 2v\varepsilon$. Since ε is arbitrary, $p(g) = v\int g\,dF$.

This proof relies on the continuity of $F(x)$. A counterexample for discontinuous $F(x)$ is as follows: $F(x) = 0$ on $(-\infty, 1)$, $F(x) = 0.2$ on $[1,2)$, $F(x) = 1.0$ on $[2,\infty)$. In other words, the underlying security's future value is 1 with probability 20% and 2 with probability 80%. Then, suppose $v = 1$, $p(X_{\{1\}}) = 0.30$ and $p(X_{\{2\}}) = 0.70$. For this example, the price-integral equality does not hold; yet the probability distribution of any derivative's outcomes uniquely determines its price, and there is no arbitrage possibility.

In general, methods that calculate risk loads based only on the probability distribution of outcomes will produce prices that are not arbitrage-free. Offering such prices could produce economically disadvantageous transactions in an ideal arbitrage-free market, through a process akin to adverse selection.

6.2. The Insurance Risk Load in the Form of Negative Expected Real Return

In the "roulette wheel" construction above, the surcharges are most punitive for the bets on low expiration prices, and the most advantageous discounts are available for bets on high expiration prices. It can be shown that this is the case for any security with positive expected real return, and it applies to options on stocks as well: high-strike calls sell at a discount to expected present value, while low-strike puts sell at a surcharge to expected present value.

This suggests a question: Why would anyone purchase one of the lower-numbered spaces or lower-strike put options that offers a negative expected real return?

High-numbered spaces and high-strike calls, which have strong risk discounts and corresponding high expected returns, are speculative purchases. With their high expected returns, they would be attractive to speculators, even some who are mildly risk-averse.

The lower-numbered spaces and low-strike puts, which carry a charge for their purchase and have expected returns below the risk-free rate, are hedges that function like insurance. A buyer would purchase such a derivative only in order to hedge risk, not to speculate. Even the most extreme risk-seeking gambler would never pay a premium for a lower-numbered space, when a higher-numbered space offers identical odds and a better payoff ratio.

Consider investors in a company. Shareholders own both the expected positive return and the risk of loss. They have two possible transactions with external parties to reduce risk: 1) Purchase some form of insurance against the possibility of an adverse outcome, and 2) Sell participation in any or all favorable outcome possibilities.

For either transaction, any risk-averse external party will require a fee for engaging in the transaction and assuming risk to their own capital. For insurance, the shareholders will have to pay a surcharge above discounted expected value for the derivatives that pay off in the event of an adverse outcome. They might be willing to do so in order to reduce their loss exposure. On the investment side, the owners will have to offer participation in the favorable outcomes at a discount to expected present value. They might be willing to, since funds from the derivatives' sale will offset loss in the event of an adverse outcome, again reducing risk.

In summary, gambles in the direction of capital growth will be priced at a discount by the owners of the capital, and by the potential investors. Gambles in the direction of capital loss will be priced at a surcharge by the same parties. In general, instruments with $\mathrm{E}[R] < r$, which is a risk surcharge, are insurance. Instruments with $\mathrm{E}[R] > r$, which is a risk discount, are investments in the broad sense of the word (some are very speculative).

This explains why the distribution of a derivative's outcomes does not imply the risk load for the derivative. The distribution contains information only about variations in future value for that derivative. It provides no information about the relationship between that particular derivative's outcomes and the risk to investments in the underlying security. The correspondence between the future value of the derivative and the future value of participants' total capital could be the most significant factor in determining the risk load that will be set by the parties to the risk transfer.

6.3. Insurance Can Still be Priced with Distribution-Based Risk Loads

These results still do not invalidate distribution-based pricing of insurance risks, for the following reasons.

Insurance protects assets against loss from specific destructive perils. Insurance does not generally respond to a decline in the value of the insured asset, unless the decline is specifically attributable to a covered peril. In particular, it does not respond to a decline in the market value of an asset the way that a put option does.

In general, the value of an insurance policy on an asset is very different than the value of a derivative on the asset's market price. The value of the insurance is determined by the stochastic process of the covered perils; the value of the derivative is driven by the stochastic process of the asset's market price. If insurance could be thought of as a derivative at all, it would be as a derivative of hurricane occurrence and severity, auto accident occurrences and severities, etc.

Insurance almost never covers asset-event combinations that are traded in a liquid market. Those insured assets that are somewhat liquid (such as property and vehicles) do not usually have traded derivatives. Even if they did, the derivatives would correlate with insurance only on losses due to covered perils, providing a very incomplete hedge. As vulnerability to arbitrage does not exist for insurance, formulas that are theoretically not arbitrage-free can be used to price insurance risks without consequent economic penalty.

While a formula that uses only the outcome distribution cannot produce arbitrage-free prices, such a formula can accurately represent the risk evaluation of a market participant, such as an insurer in an insurance market. The potential for correlations among risks within the insurer's portfolio would still have to be handled, possibly through limiting the aggregate accumulation of particular risks or by using an additional covariance load.

7. THE RISK DISCOUNT FUNCTION

We can follow the construction used in the roulette example above to partition the space of expiration prices into more than 38 sections. In the limit, this leads to a continuous function that shows the variation in the implicit risk discount for equally probable events.

Choose a large positive integer M and partition the positive number line $(0,\infty)$ into M adjacent sections ($M-1$ half-open line segments and one ray), so that the probability of any given section containing the expiration price is $1/M$. Each of the M sections has a corresponding binary derivative with expiration value of one, if the section contains the expiration price, and zero otherwise.

Each of these M section-derivatives has a risk discount or surcharge factor, which can be calculated using the formulas derived earlier:

$$\text{Risk Factor} = \text{Price}/\text{Discounted Probability}.$$

For a segment $D(s,t)$, the price is given by:

$$\text{Price}[D(s,t)] = \text{Price}[A^*(s)] - \text{Price}[A^*(t)];$$

$$\text{Price}[A^*(s)] - \text{Price}[A^*(t)] = v\Phi[\ln(p(1+r)/s)/\sigma - \sigma/2] - v\Phi[\ln(p(1+r)/t)/\sigma - \sigma/2];$$

$$\text{Price}[D(s,t)] = v\{\Phi[\ln(p(1+r))/\sigma - \ln(s)/\sigma - \sigma/2] - \Phi[\ln(p(1+r))/\sigma - \ln(t)/\sigma - \sigma/2]\}.$$

For sufficiently large M, the difference in cumulative probabilities (shown in braces) is approximated by the normal density function times the interval width:

$$\text{Price}[D(s,t)] \approx vg[\ln(p(1+r))/\sigma - \ln(s)/\sigma - \sigma/2] \cdot [\ln(t) - \ln(s)]/\sigma,$$

where g represents the standard normal density. Similarly,

$$\text{Probability}[D(s,t)] = \Phi[\ln(p(1+E))/\sigma - \ln(s)/\sigma - \sigma/2] - \Phi[\ln(p(1+E))/\sigma - \ln(t)/\sigma - \sigma/2], \qquad \text{and}$$

$$\text{Discounted Probability}[D(s,t)] \approx vg[\ln(p(1+E))/\sigma - \ln(s)/\sigma - \sigma/2] \cdot [\ln(t) - \ln(s)]/\sigma.$$

Then,

$$\text{Risk Factor} \approx g[\ln(p(1+r))/\sigma - \ln(s)/\sigma - \sigma/2] / g[\ln(p(1+E))/\sigma - \ln(s)/\sigma - \sigma/2].$$

As M becomes infinitely large, the risk factor approaches this ratio of the normal densities for price and probability, the "risk discount function" $w(s)$:

$$w(s) = g[\ln(p(1+r))/\sigma - \ln(s)/\sigma - \sigma/2]/$$
$$g[\ln(p(1+E))/\sigma - \ln(s)/\sigma - \sigma/2].$$

Substituting the standard normal density for $g(x)$, this expression can be simplified greatly, so that $w(s)$ is found to be a monomial (derivation given in Appendix A):

$$w(s) = (s/s_n)^{-k}, \qquad \text{where}$$
$$k = [\ln(1+E) - \ln(1+r)]/\sigma^2, \qquad \text{and}$$
$$s_n = \text{the "neutral" strike price satisfying } w(s_n) = 100\%.$$

This function $w(s)$ represents the risk discount for betting on the event $X = s$. The risk discount function is equivalent to Bühlmann's "pricing density" [1]. The graph of $w(s)$ is shown in Exhibit 4 for the parameters used in the examples. As the graph indicates, the risk surcharge factor increases without bound as the strike price approaches zero. As the strike price increases, the risk discount factor decreases toward zero but at a very gradual rate: for a strike price of 212, the risk discount is still relatively mild, at 63%. This strike price is just above the 99th percentile of the future price distribution for the security, where one might expect a more substantial discount.

The graph of $w(s)$ (Exhibit 4) also shows that the continuum of expiration price events splits into discount and surcharge zones. Strike prices below s_n (which is 102.25 in this example) can be considered within the "insurance zone" of potential outcomes, while prices above 102.25 are in the "speculation zone."

Any derivative's risk discount factor can be calculated by averaging $w(s)$ against probability density times expiration value:

Derivative's Risk Discount Factor

$$= \int w(s)p(s)x(s)ds \Big/ \int p(s)x(s)ds,$$

where $x(s)$ is the expiration value of the derivative as a function of the underlying security's price, and $p(s)$ is the probability density of the expiration price.

For almost any finite payoff density function, one can construct an unlimited number of distinct derivatives having the same specified payoff density and different prices.

8. CONCLUSION

The probability distribution of a risk's outcomes does not imply the price of the risk under arbitrage-free pricing in an ideal market. Distinct, identically distributed risks generally have different arbitrage-free prices. In practice, this result does not invalidate distribution-based pricing for most insurance risks, but it should be considered when insurance pricing is related to financial theory involving arbitrage-free prices.

It is possible for a formula to produce arbitrage-free prices from a risk's probability distribution, if the formula contains an adjustment parameter that varies by risk. In one recent theory, this adjustment parameter indicates the correlation (in the general sense) between the underlying security's risk and overall market risk [5].

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APPENDIX A

DERIVATION OF THE $w(s)$ FORMULA

From the text,

$$w(s) = g[\ln(p(1+r))/\sigma - \ln(s)/\sigma - \sigma/2]/ \\ g[\ln(p(1+E))/\sigma - \ln(s)/\sigma - \sigma/2],$$

where $g(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, and $E = \mathrm{E}[R]$, the expected return on the security. Then,

$$w(s) = (2\pi)^{-1/2}\exp(-b^2/2)/(2\pi)^{-1/2}\exp(-c^2/2),$$

where b and c are the respective arguments of g in the $w(s)$ formula above:

$$b = [\ln(p) + \ln(1+r) - \ln(s) - \sigma^2/2]/\sigma, \qquad \text{and}$$

$$c = [\ln(p) + \ln(1+E) - \ln(s) - \sigma^2/2]/\sigma.$$

Simplifying:

$$w(s) = \exp[(c^2 - b^2)/2], \qquad \text{and}$$

$$w(s) = \exp[(c-b)(c+b)/2].$$

Evaluating,

$$c - b = [\ln(1+E) - \ln(1+r)]/\sigma = \Delta E/\sigma,$$

where $\Delta E = \ln(1+E) - \ln(1+r)$ is a measure of the risk premium in the security's expected return. Next,

$$(c+b)/2$$

$$= [\ln(p) + (1/2)\ln(1+E) + (1/2)\ln(1+r) - \ln(s) - \sigma^2/2]/\sigma.$$

Define $s_n = p[(1+E)(1+r)]^{1/2}/\exp(\sigma^2/2)$. Then,

$$(c+b)/2 = [\ln(s_n) - \ln(s)]/\sigma.$$

Substituting yields:

$$w(s) = \exp\{(\Delta E/\sigma)[\ln(s_n) - \ln(s)]/\sigma\}.$$

Note that $w(s_n) = 1$. Finally,

$$w(s) = \exp\{(-\Delta E/\sigma^2)[\ln(s/s_n)]\};$$

$$w(s) = (s/s_n)^{(-\Delta E/\sigma^2)}.$$

EXHIBIT 1

CALL OPTION VALUE

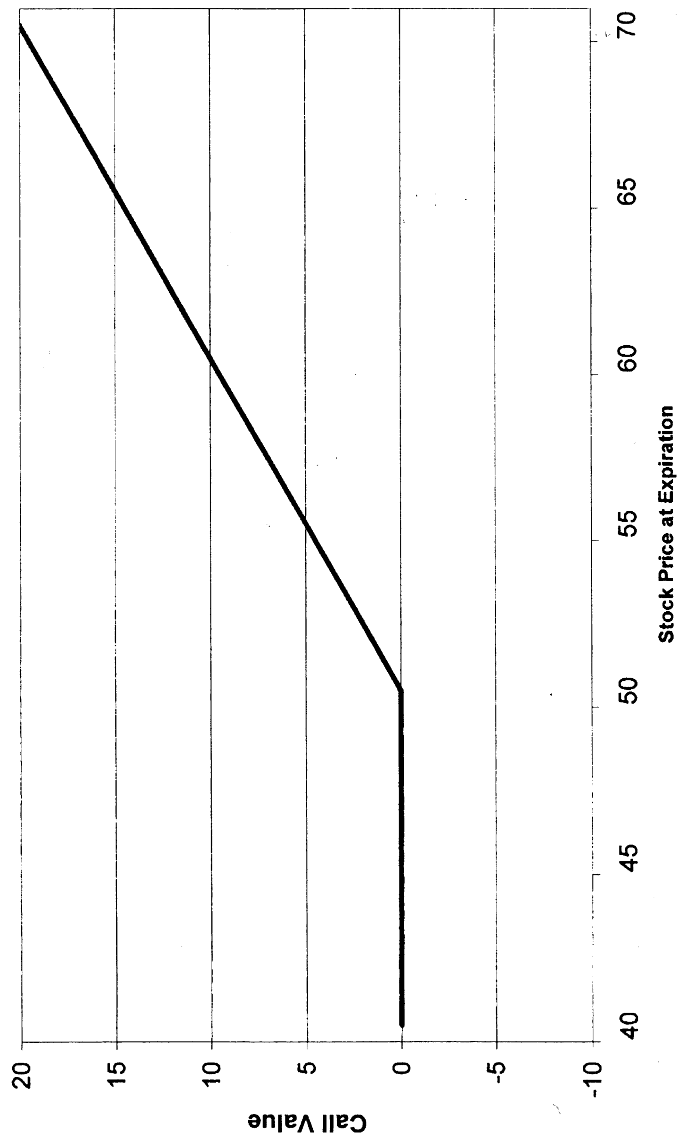

EXHIBIT 2

SPREAD VALUE

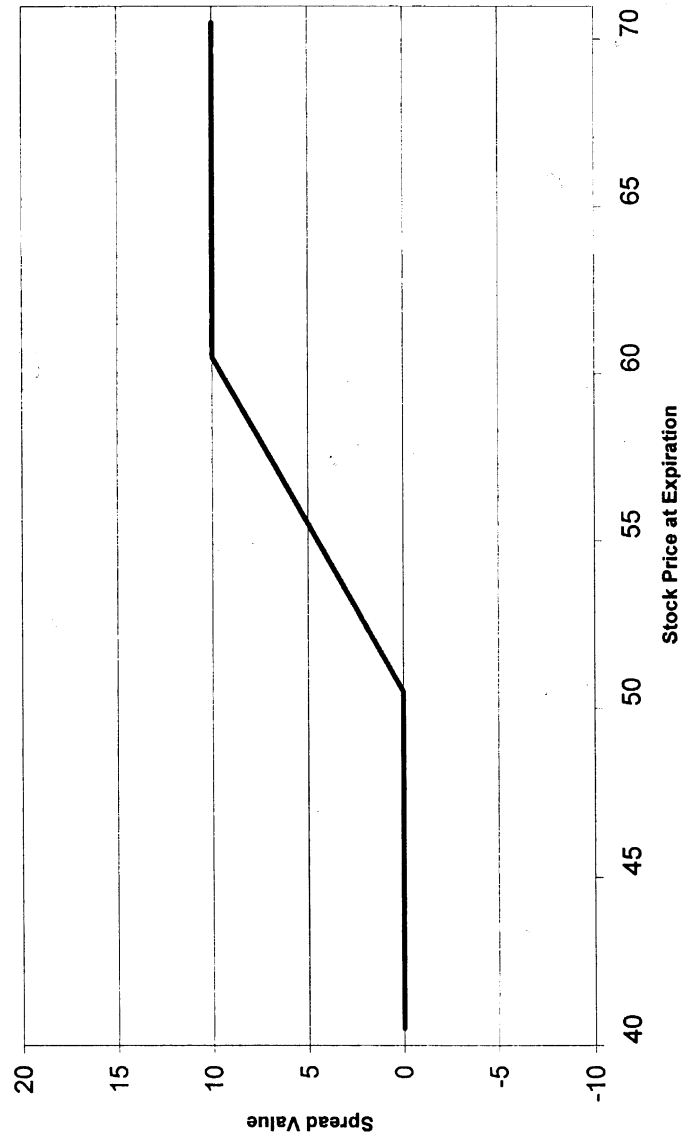

EXHIBIT 3

DERIVATIVES CORRESPONDING TO ROULETTE WHEEL SPACES

Roulette Wheel Space	Derivative Type	Lower Strike (s)	Upper Strike (t)	Payoff Probability	Price	Expected Return	Risk Factor	Roulette Risk Surcharge	Ratio to Roulette Payoff
00	Segment	0.00	58.80	0.0263	0.0384	−31.54%	151.91%	105.56%	69%
0	Segment	58.80	64.68	0.0263	0.0346	−23.91%	136.68%	105.56%	77%
1	Segment	64.68	68.84	0.0263	0.0330	−20.21%	130.35%	105.56%	81%
2	Segment	68.84	72.23	0.0263	0.0319	−17.46%	126.00%	105.56%	84%
3	Segment	72.23	75.17	0.0263	0.0310	−15.18%	122.61%	105.56%	86%
4	Segment	75.17	77.83	0.0263	0.0303	−13.19%	119.81%	105.56%	88%
5	Segment	77.83	80.29	0.0263	0.0297	−11.40%	117.38%	105.56%	90%
6	Segment	80.29	82.61	0.0263	0.0292	−9.74%	115.23%	105.56%	92%
7	Segment	82.61	84.82	0.0263	0.0287	−8.19%	113.27%	105.56%	93%
8	Segment	84.82	86.95	0.0263	0.0282	−6.71%	111.48%	105.56%	95%
9	Segment	86.95	89.03	0.0263	0.0278	−5.29%	109.81%	105.56%	96%
10	Segment	89.03	91.07	0.0263	0.0274	−3.92%	108.24%	105.56%	98%
11	Segment	91.07	93.08	0.0263	0.0270	−2.58%	106.75%	105.56%	99%
12	Segment	93.08	95.08	0.0263	0.0267	−1.26%	105.33%	105.56%	100%
13	Segment	95.08	97.07	0.0263	0.0263	0.03%	103.96%	105.56%	102%
14	Segment	97.07	99.06	0.0263	0.0260	1.32%	102.64%	105.56%	103%
15	Segment	99.06	101.07	0.0263	0.0256	2.60%	101.36%	105.56%	104%
16	Segment	101.07	103.10	0.0263	0.0253	3.89%	100.11%	105.56%	105%
17	Segment	103.10	105.16	0.0263	0.0250	5.18%	98.87%	105.56%	107%
18	Segment	105.16	107.26	0.0263	0.0247	6.49%	97.66%	105.56%	108%
19	Segment	107.26	109.42	0.0263	0.0244	7.81%	96.47%	105.56%	109%
20	Segment	109.42	111.64	0.0263	0.0241	9.16%	95.27%	105.56%	111%
21	Segment	111.64	113.93	0.0263	0.0238	10.55%	94.08%	105.56%	112%
22	Segment	113.93	116.31	0.0263	0.0235	11.97%	92.88%	105.56%	114%
23	Segment	116.31	118.81	0.0263	0.0232	13.44%	91.68%	105.56%	115%
24	Segment	118.81	121.43	0.0263	0.0229	14.97%	90.46%	105.56%	117%
25	Segment	121.43	124.21	0.0263	0.0226	16.57%	89.21%	105.56%	118%
26	Segment	124.21	127.18	0.0263	0.0223	18.26%	87.94%	105.56%	120%
27	Segment	127.18	130.38	0.0263	0.0219	20.06%	86.62%	105.56%	122%
28	Segment	130.38	133.87	0.0263	0.0216	21.99%	85.25%	105.56%	124%
29	Segment	133.87	137.73	0.0263	0.0212	24.09%	83.81%	105.56%	126%
30	Segment	137.73	142.08	0.0263	0.0208	26.41%	82.27%	105.56%	128%
31	Segment	142.08	147.11	0.0263	0.0204	29.03%	80.60%	105.56%	131%
32	Segment	147.11	153.10	0.0263	0.0199	32.05%	78.76%	105.56%	134%
33	Segment	153.10	160.64	0.0263	0.0194	35.69%	76.65%	105.56%	138%
34	Segment	160.64	170.96	0.0263	0.0187	40.36%	74.09%	105.56%	142%
35	Segment	170.96	188.08	0.0263	0.0179	47.16%	70.67%	105.56%	149%
36	Ray	188.08	Infinity	0.0263	0.0162	62.93%	63.83%	105.56%	165%

EXHIBIT 4

RISK DISCOUNT AS A FUNCTION OF STRIKE PRICE